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On Mazur rotations problem and its multidimensional versions

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Abstract

The article is a survey related to a classical unsolved problem in Banach space theory, appearing in Banach's famous book in 1932, and known as the Mazur rotations problem. Although the problem seems very difficult and rather abstract, its study sheds new light on the importance of norm symmetries of a Banach space, demonstrating sometimes unexpected connections with renorming theory and differentiability in functional analysis, with topological group theory and the theory of representations, with the area of amenability, with Fraïssé theory and Ramsey theory, and led to development of concepts of interest independent of Mazur problem. This survey focuses on results that have been published after 2000, stressing two lines of research which were developed in the last 10 years. The first one is the study of approximate versions of Mazur rotations problem in its various aspects, most specifically in the case of the Lebesgue spaces L_p. The second one concerns recent developments of multidimensional formulations of Mazur rotations problem and associated results. Some new results are also included.

Keywords Mazur rotations problem · Transitivity · Almost transitivity · Fraïssé Banach space

Mathematics Subject Classification Primary 46-02 · Secondary 46A22 · 46B04 · 46B08 · 46C15 · 54H20

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1 Introduction and first results on Mazur problem

1.1 Mazur rotations problem

Hilbert spaces have the following rotations property:

> Given two points x, y on the unit sphere there exists an isometry T (defined on the whole space) such that $y = Tx$.

Here and throughout the paper *isometry* means *linear surjective isometry*. This clearly follows from the existence of orthogonal complements and can be rephrased by saying that the isometry group acts transitively on the unit sphere.

Mazur problem, which can be found in Banach's *Théorie des Opérations Linéaires*, asks whether every separable Banach space enjoying the above *rotations* property has to be isometric to *the* separable Hilbert space; see [16, la remarque à la section 5 du chapitre XI]. This question is perhaps best understood as two separate problems, both of which remain open to this day.

Problem 1.1 (Mazur rotations problem, the isomorphic part) Assume X is a separable Banach space whose isometry group acts transitively on its unit sphere. Is X linearly isomorphic to the separable Hilbert space $\mathcal{H}$?

As we shall see very soon both the separability and the completeness conditions are necessary since otherwise there are easy counterexamples based on the Lebesgue spaces L_p. The other part of the problem, where neither completeness or separability seems to be essential, reads as follows:

Problem 1.2 (Mazur rotations problem, the isometric part) Assume $\|\cdot\|$ is an equivalent norm on a Hibert space $\mathcal{H}$ whose isometry group acts transtitively on the unit sphere. Is $\|\cdot\|$ necessarily euclidean, that is, induced by an inner product on $\mathcal{H}$?

1.2 Notation, conventions

We tend to use $X, Y, Z, U \dots$ for infinite dimensional Banach spaces and $A, B, E, F \dots$ for finite dimensional ones. The unit sphere of X is the set $S_X = \{x \in X : \|x\| = 1\}$.

The space of operators from X to Y is denoted by $\mathcal{L}(X, Y)$. Operators are invariably assumed linear and continuous. The identity operator on X is denoted by $\mathbf{I}_X$. We use $\mathrm{GL}(X)$ for the group of linear automorphisms of X and $\mathrm{Isom}(X)$ for its group of isometries. Recall that throughout, isometries are assumed to be linear and surjective (it is worth recalling here that by Mazur's theorem, onto isometries fixing 0 are necessarily linear). An operator $T : X \to Y$ which preserves the norm ($\|Tx\| = \|x\|$ for all $x \in X$) is called an isometric embedding and we denote the subset of such operators in $\mathcal{L}(X, Y)$ by $\mathrm{Emb}(X, Y)$. An operator $T : X \to Y$, not necessarily surjective, that satisfies the estimate $(1+\varepsilon)^{-1}\|x\| \leqslant \|Tx\| \leqslant (1+\varepsilon)\|x\|$ is called an ε-isometry. We denote by $\mathrm{Emb}_\varepsilon(X,Y)$ set of all ε-isometries from X to Y.

The (multiplicative) Banach-Mazur distance between two Banach spaces X, Y is defined by

$$d_{\mathrm{BM}}(X,Y) = \inf\left\{\|T\|\|T^{-1}\| : T \text{ is an isomorphism between } X \text{ and } Y\right\},$$

where the infimum of the empty set is treated as ∞.

If G is a group acting on a set X, meaning that we have a homomorphism π from G to the group of bijections of X, then the orbit of x under the action of G is the set $\{\pi(g)(x) : g \in G\}$. If no confusion can arise, we often identify g with $\pi(g)$ and we use the notation $G \curvearrowright X$ to indicate that G acts on X. If both G and X carry topologies we say that an action $G \curvearrowright X$ is continuous if the obvious map $G \times X \to X$ sending (g, x) to $\pi(g)(x)$ is continuous. A topological group G is said to be amenable if every continuous *affine* action of G on a compact *convex* set *of a locally convex space* has a fixed point. By deleting all the words set in italics one obtains the notion of an extremely amenable group.

General references about classical but sometimes specific concepts in Banach space theory (convexity, type, cotype, asymptotic structure, finite representability, Orlicz spaces, Tsirelson space, etc...) are, for example, [4, 52, 99] or the chapter by Johnson and Lindenstrauss [83] opening *the* Handbook.

1.3 Topologies

Two topologies will be relevant for us on the spaces of operators $\mathcal{L}(X,Y)$, namely the norm topology, and the strong operator topology (SOT, the topology of pointwise convergence on X). Their restrictions provide topologies on $\mathrm{GL}(X)$, $\mathrm{Isom}(X)$ and $\mathrm{Emb}(E,X)$. We recall some well-known useful facts:

Fact 1.3 *Both* $\mathrm{GL}(X)$ *and* $\mathrm{Isom}(X)$ *are topological groups in the norm topology.*

The norm topology is somehow too strong to be used on isometry groups and actually it has a strong tendency to discretize them (see Comment 3 in Sect. 2.4 for examples of this on Lebesgue spaces).

In general the SOT is not a group topology on $\mathrm{GL}(X)$, but things get better if one looks at bounded subgroups. In particular:

Fact 1.4 *The SOT makes* $\mathrm{Isom}(X)$ *into a topological group which is Polish (separable and completely metrizable) when X is separable.*

(We refer the reader to [91, Chapter I, § 9] for an introduction to Polish groups as well as the basic examples.) These facts compel us to consider the isometry groups in the SOT topology unless otherwise stated.

We shall usually equip $\mathrm{Emb}_\varepsilon(F,X)$ and in particular $\mathrm{Emb}(F,X)$ with the distance induced by the norm on $\mathcal{L}(F,X)$. But note that here the SOT and the norm topology are equivalent when F is finite-dimensional.

1.4 Transitivity and its relatives

A Banach space is *almost transitive* (AT) if given $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\varepsilon > 0$ there exists a surjective isometry T of X such that $\|y - Tx\| \leqslant \varepsilon$. If this can be achieved for $\varepsilon = 0$ we say that X is *transitive*.

Establishing a vocabulary to study these properties, Pełczyński and Rolewicz [114] (see also Rolewicz's book [126, Chapter 9]) defined the norm $\|\cdot\|$ of a Banach space X to be *maximal* if no equivalent norm can give a strictly larger group of isometries. If, in addition, every equivalent norm with the same isometry group as $\|\cdot\|$ is a multiple of $\|\cdot\|$ the norm is called *uniquely maximal*. This happens if and only if $\|\cdot\|$ is *convex transitive*, namely for every norm one x the closed convex hull of the orbit of x under the action of the isometry group is the unit ball. One has the implications

$$\boxed{\text{Hilbert}} \Rightarrow \boxed{\text{Transitive}} \Rightarrow \boxed{\text{AT}} \Rightarrow \boxed{\text{Convex transitive}} \Rightarrow \boxed{\text{Maximal}}$$

Most of what was known on Mazur problem and its more or less natural variations until the year 2000 can be seen in the survey papers [19, 26]. Here we only recall that every Banach space is isometric to a 1-complemented subspace of an AT space; see Lusky [103] for the separable case and [19, Theorem 2.14] for the general case and some consequences. Thus, (almost) transitivity alone does not imply any Banach space property that passes to complemented subspaces (for example there exist transitive spaces without the Approximation Property, see [19, Section 2]. We shall however focus on more natural and important examples, some of which have stronger properties than AT or transitivity.

1.5 Classical isometry groups and examples of AT spaces

We now present the examples upholding the paper focusing primarily on AT spaces. Some of them will be revisited in Sect. 4 in the multidimensional setting. As we shall see, there is a wide variety of AT spaces arising in very different contexts. With the sole possible exception of Hilbert spaces, which may be seen from so many different points of view, these spaces are "large" in some sense which is difficult to make precise. Actually it is not easy to distinguish the spaces that can be given an equivalent AT norm from those that cannot; see Sect. 2 for more explanations and the basics on maximal norms. General references for the isometries of classical function spaces and many related topics are [60, 61, 96].

1.5.1 Hilbert spaces

If $\mathcal{H}$ is a Hilbert space, then Isom($\mathcal{H}$) is the unitary group. It acts transitively on the unit sphere. Moreover, if x, y are normalized then there is an isometry T sending x to y such that $\|T - \mathbf{I}_{\mathcal{H}}\| = \|y - x\|$ (optimal) with $T - \mathbf{I}_{\mathcal{H}}$ of rank 2. There is another isometry L sending x to y with $L - \mathbf{I}_{\mathcal{H}}$ of rank 1 (optimal), but $\|L - \mathbf{I}_{\mathcal{H}}\| = 2$.

1.5.2 Lebesgue spaces

Given a measure μ defined on a set S and $1 \le p < \infty$ we denote by $L_p(\mu)$ the usual Lebesgue space of p-integrable functions on S, with the usual convention about identifying functions that agree almost everywhere. If $\mu = \lambda$ is the Lebesgue measure on the unit interval we just write L_p. If $\phi : [0, 1] \to [0, 1]$ is a Borel automorphism (a bijection preserving Borel sets in both directions) which preserves null sets in both directions and h is a measurable function such that $|h|^p = d(\lambda\circ\phi)/d\lambda)$, that is $\lambda(\phi(B)) = \int_B |h|^p d\lambda$ for every Borel $B \subset [0, 1]$, then the operator

$$(Tf)(t) = h(t)f(\phi(t))$$

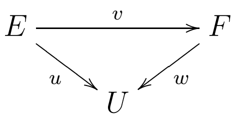
is a correctly defined isometry of L_p. If $p \neq 2$ the converse is also true and every $T \in \mathrm{Isom}(L_p)$ arises in this way (the Banach–Lamperti theorem [60, Theorem 3.2.5], although in this formulation we need a little help from von Neumann [141]). This has the following consequences for finite $p \neq 2$:

- L_p is AT, but not transitive: there are exactly two (dense) orbits on the unit sphere namely, the "full support" one, i.e. the orbit $\{f \in S_{L_p} : \lambda(f^{-1}(0)) = 0\}$ and the complement $\{f \in S_{L_p} : \lambda(f^{-1}(0)) > 0\}$.
- The dense subspace $L_p(0, 1^-) = \bigcup_{b<1}\{f \in L_p : \mathrm{supp}(f) \subset [0, b]\}$ is a *transitive* normed space (with the obvious definition).
- If $\aleph$ is an uncountable cardinal, then the Banach space $\ell_p(\aleph, L_p)$ (which can be regarded as $L_p(\mu)$, where μ is "Lebesgue measure" on $\aleph$-many disjoint copies of the unit interval) is transitive. Note that this space has density character $\aleph$, while nontrivial ultraproducts (see Sect. 1.8 below) have density character at least the continuum.

The case $p = \infty$ was excluded in the preceding discussion because the space L_∞, being a $C(K)$ in disguise, cannot be AT. The isometries of $C(K)$ are described by the Banach-Stone theorem (1932): all have the form $Tf(x) = u(x)f(\phi(x))$, where ϕ is a homeomorphism of K and $u : K \to \mathbb{K}$ is continuous and unimodular. In particular the orbit of the unit 1 cannot be dense in the sphere unless K is a singleton. And what happens with other spaces of type $\mathcal{L}_\infty$? Keep reading.

1.5.3 The Gurariy space

A Banach space U is said to be of *almost universal disposition* (AUD) if, given a finite dimensional space F, isometric embeddings $v : E \to F, u : E \to U$ and $\varepsilon > 0$, there exists an ε-isometry $w : F \to U$ such that $u = wv$. Diagramatically,



This notion was coined by Gurariy in [75], where he constructed the space that bears his name as a separable space of AUD. Gurariy also established that two separable Banach spaces of AUD are "almost isometric" (that is, the Banach-Mazur distance between them is equal to 1) and that for every $\varepsilon > 0$ the surjective ε-isometries act transitively on the unit sphere of any separable space of AUD. Although this is not completely evident from the definition, any space of AUD must be a Lindenstrauss space (i.e. a predual of an L_1-space) because AUD implies the following extension property of X: given a subspace E of a finite dimensional space F and $\varepsilon > 0$ every operator $\tau : E \to X$ has an extension $\tilde{\tau} : F \to X$ with $\|\tilde{\tau}\| \leqslant (1 + \varepsilon)\|\tau\|$.

The isometric uniqueness of the Gurariy space $\mathcal{G}$ was finally established by Lusky in a fine paper [101] where he also showed that the isometry group acts transitively on the set of smooth points of the sphere of $\mathcal{G}$. See [62, Section 4] for more general results concerning finite-dimensional subspaces of $\mathcal{G}$.

A new proof of the uniqueness of the Gurariy space was later provided by Kubiś and Solecki in [94]: they basically proved that the Gurariy space is the (approximate) Fraïssé limit of the class of finite dimensional Banach spaces and isometric embeddings. This remarkable feature of the Gurariy space inspired the study of the interactions between Fraïssé structures and Banach spaces; see [100] and the references therein. We shall pursue this approach in Sect. 4. From another point of view, see also the recent description by Cúth, Doležal, Doucha and Kurka, of the Gurariy space as the "generic" separable space [41].

1.5.4 The Garbulińska space

One should speak, more accurately, of the Garbulińska-Węgrzyn renorming of the Kadec/Pełczyński/Wojtaszczyk space, see below. The Garbulińska space plays the same role as the Gurariy space in a different category, where one takes into account 1-complemented subspaces only. Let us say that a Banach space X has the property [$\Game$] if given isometries with 1-complemented range $u : E \to X$ and $v : E \to F$, where F is finite-dimensional, and $\varepsilon > 0$ there is an ε-isometry $w : F \to X$ with $(1 + \varepsilon)$-complemented range such that $u = wv$.

Garbulińska shows in [64] that there exists a unique, up to isometries, Banach space $\mathcal{K}$ with a skeleton and property [$\Game$]. Recall that a skeleton of X is a chain of finite dimensional subpaces $(E_n)_{n\geqslant 1}$ whose union is dense in X and such that E_n is

1-complemented in E_{n+1}. This condition is a clear analogue of separability in the 1-complemented category and is just a transcription of 1-FDD. Most isometric properties of $\mathcal{K}$ depend, one way or another, on the following fact ([64, Theorem 6.3]):

> Let K and K' be Banach spaces with skeletons, satisfying the property [$\mathfrak{O}$], and let $h : A \to B$ be an isometry between 1-complemented finite-dimensional subspaces of K and K', respectively. Then for every $\varepsilon > 0$ there exists an isometry $H : K \to K'$ such that $\|H(x) - h(x)\| \leqslant \varepsilon\|x\|$ for all $x \in A$. In particular, K and K' are isometric.

Since all lines in a Banach space are 1-complemented and isometric to each other, it follows that $\mathcal{K}$ is AT.

Another important feature of $\mathcal{K}$, that is going to play its role in Sect. 2, is that $\mathcal{K}$ contains a 1-complemented copy of every space with a skeleton. This makes $\mathcal{K}$ isomorphic to some old acquaintances in the theory of complementably universal spaces. Let $\mathcal{C}$ be a class of Banach spaces. We say that a Banach space is (complementably) universal for $\mathcal{C}$ if it belongs to $\mathcal{C}$ and it contains a (complemented) isomorph of each space in $\mathcal{C}$. This concept emerged in the paper [112], where Pełczyński constructed his celebrated (space with a) universal basis (call it P_B) which is a complementably universal space for the class of Banach spaces with bases and a similar space with an unconditional basis which we shall denote by U.

Later on M.Ĭ. Kadec [84] exhibited a complementably universal space for the bounded approximation property (BAP); let us denote that specimen by K and observe that an obvious application of the Pełczyński decomposition method shows that any two complementably universal spaces for the BAP are isomorphic. In the same issue of *Studia* where Kadec' space first appeared, Pełczyński [113] showed that each separable space with the BAP is complemented in a space with a basis: the inexorable consequence is that the spaces P_B and K are isomorphic. But since the Garbulińska space has the BAP (obvious) and each Banach space with a basis can be renormed to get a skeleton (even more obvious) we can apply again the Pełczyński decomposition method to conclude that the Garbulińska space $\mathcal{K}$ is isomorphic to P_B and K, which are also isomorphic to a space complementably universal for FDDs constructed by Pełczyński and Wojtaszczyk [115] in the very same volume of *Studia*.

1.5.5 Spaces of continuous functions on the pseudoarc

Although regarding AT spaces of type $\mathcal{L}_\infty$ the Gurariy space is the guy to work with, there are other natural examples. One of them is the separable "M-Gurariy" space from [54] and a closely related, but non-separable creature is obtained in [28] taking ultraproducts of the spaces L_p with variable $p \to \infty$; cf. Comment 4 in Sect. 4.9.

Here we discuss spaces of continuous functions based on the pseudoarc, a continuum constructed by Knaster [93] in the 1920s which became a celebrity in *certain* circles because of the Bing's characterization: it is the only hereditarily indecomposable chainable continuum; let us denote it by P. An impressive wealth of well organized information on the pseudoarc is contained in Lewis' survey [98].

Kawamura [90] and Rambla [122], independently and almost simultaneously, proved that if P_* is the pseudoarc with one point removed, then the complex space $C_0(P_*)$ is AT in the sup norm, thus refuting a long standing conjecture of Wood [142, Section 3]. The group of homeomorphims acts transitively on P and so the homeomorphic type of P_* does not depend of which point is removed.

Curiously enough the pseudoarc can be considered as the (inverse) Fraïsé limit of a suitable class as shown by Irwin and Solecki in [81] which is simply delighful, given the approach of this survey, cf. Sect. 4.

Taking ultrapowers leads to $C_0(L)$-spaces which are transitive in the sup norm.

Naive observations in [30] suggest that if L is a locally compact space with more than one point, the complex space $C_0(L)$ is separable and AT, then the one-point compactification of L should be (homeomorphic to) the pseudoarc.

As for real spaces, Greim and Rajalopagan proved in [71] that no $C_0(L)$, can be AT in the sup norm if L has more than one point. However, curiously enough, there exists a quite natural norm under which a real $C_0(L)$ can be AT and even transitive. Indeed, if $f : L \to \mathbb{R}$ is any function, we set

$$\operatorname{diam}(f) = \sup\{|f(x) - f(y)| : x, y \in L\}.$$

If L is locally compact but not compact, then diam is a norm on $C_0(L)$, clearly equivalent to the sup norm. If K is compact then diam vanishes on the constant functions and so it defines a true norm on $C(K)/\mathbb{K}$ which agrees with the quotient norm (up to a factor) in case of real scalars. It is shown in [31, Lemma 3.1] that $C(P)/\mathbb{R}$ is AT and thus the real space $C_0(P_*)$ equipped with the diameter norm is AT. It is perhaps worth noticing that both the isometry group of the complex space $C_0(P_*)$ and that of $C(P)/\mathbb{R}$ fail to be amenable in the SOT ([31, Example 3.2]).

1.5.6 Noncommutative L_p-spaces

Other families of AT spaces come from the noncommutative generalizations of L_p. We shall not even give the definition and we refer the reader to the *official* sources [76, 138] instead, but let us mention that there is a classical construction in operator algebras, due to Haagerup, that associates to each von Neumann algebra $\mathcal{M}$ a family of spaces $L_p(\mathcal{M})$ for $p \in (0, \infty]$ in such a way that $L_1(\mathcal{M}) = \mathcal{M}_*$ is the predual of $\mathcal{M}$ and $L_\infty(\mathcal{M}) = \mathcal{M}$. The Haagerup $L_p(\mathcal{M})$-spaces consist of certain unbounded operators acting on a Hilbert space which is related to $\mathcal{M}$ in a highly nontrivial way.

By a celebrated result of Connes and Størmer [40, Theorem 4], if $\mathcal{M}$ is a factor of type III$_1$, then, given states $\phi, \psi \in \mathcal{M}_*$ and $\varepsilon > 0$, there is a unitary $u \in \mathcal{M}$ such that $\|u^*\phi u - \psi\|_{\mathcal{M}_*} < \varepsilon$, where $u^*\phi u$ is defined by $\langle u^*\phi u, x\rangle = \langle \phi, uxu^*\rangle$ for $x \in \mathcal{M}$. It follows from the generalized Power-Størmer inequality (see [80, Appendix]) that the spaces $L_p(\mathcal{M})$ for finite p have a similar homogeneity property: given positive $f, g \in L_p(\mathcal{M})$ with $\|f\|_p = \|g\|_p = 1$ and $\varepsilon > 0$ there is a unitary $u \in \mathcal{M}$ such that $\|u^* f u - g\|_p < \varepsilon$. It follows readily that for arbitrary $f, g \in L_p(\mathcal{M})$ with $\|f\|_p = \|g\|_p = 1$ and $\varepsilon > 0$ there exist unitaries $u, v \in \mathcal{M}$ such that $\|vfu - g\|_p < \varepsilon$ and so $L_p(\mathcal{M})$ is AT.

By remarks on ultraproducts presented in Sect. 1.8 below, the countable ultrapowers of $L_p(\mathcal{M})$ are transitive and, by results of Raynaud [125], the ultrapowers $L_p(\mathcal{M})_{\mathcal{U}}$ can be represented as the Haagerup spaces $L_p(\mathcal{N})$, for some large von Neumann algebra $\mathcal{N}$.

1.6 Microtransitivity

In a desperate attempt to break the impasse on Mazur problem the authors of [34] and [20] consider the following stronger form of transitivity, which has very little to do with the subject of this survey: a Banach space is called microtransitive (MT) if for every $\varepsilon > 0$ there is $\delta > 0$ so that if $x, y \in S_X$ satisfy $\|y - x\| < \delta$ there is $T \in \mathrm{Isom}X$ such that $y = Tx$ *and* $\|T - \mathbf{I}_X\| < \varepsilon$. As one may guess the only known examples of MT spaces are the Hilbert spaces, which satisfy the definition with $\delta = \varepsilon$. The issue of separability (and completeness), which is central in Mazur rotations problem, is irrelevant for MT: X is MT if and only if for every separable $Y \subset X$ there exist a further separable $Z \subset X$ which is MT and contains Y. Moreover, MT passes to the dual and implies both uniform convexity and uniform smoothness of the norm; see [20, Theorems 3.11 and 3.14] for the strongest available results in this line. So, the following is a seemingly cheap, but still open, substitute for the Mazur problem:

Problem 1.5 Are the Hilbert spaces the only microtransitive Banach spaces?

Comments:

The Effros Microtransitivity Theorem [49, Theorem 2.1] states that a Polish group acting transitively on a Polish space must act microtransitively. See also van Mill's work on this topic [140]. This implies that if X is a separable transitive Banach space, then the action of the isometry group on the sphere is *SOT*-microtransitive: i.e. for any $x \in S_X$, the map assigning to an isometry T its value in x is open for the SOT on Isom(X). The notion of microtransitivity (MT) defined above is much stronger and corresponds to the map being open in the norm topology on Isom(X). The papers [20, 34] also consider "open actions" of the semigroup of contractive automorphisms.

1.7 Strict convexity and transitivity

Though much information has been obtained on almost transitive Banach spaces under additional geometric assumptions such as reflexivity [19, 26], very few conditions that are necessary for the actual transitivity in the separable case are known.

Related to the present study, let us mention that, if X is a separable transitive real Banach space, then X is strictly convex and smooth, and thus X^* is AT; see [56, Theorem 28] and [19, Corollary 2.9]. This result fails if X is only assumed to be almost transitive (resp. if X is non-separable), as can be seen by considering L_1 (resp.

an ultrapower of L_1, see the next section on ultraproducts). The paper [1] contains some related results.

1.8 Ultraproducts

The Banach space ultraproduct construction is a quite useful technique that allows one to construct large spaces with upgraded transitivity properties. We refer the reader to [77] (or Sims' booklet [132]) for two very readable expositions which suffice for our modest purposes. A more complete one, which emphasizes the model-theoretic pedigree of the ultraproduct construction is [78]. Here we only recall the definition, just to fix the notation.

Let (X_i) be a family of Banach spaces indexed by I and let $\mathcal{U}$ be an ultrafilter on I.

Consider the space of bounded families $\ell_\infty(I, X_i)$ equipped with the sup norm and the closed subspace $c_0^{\mathcal{U}}(X_i) = \{(x_i) : \lim_{\mathcal{U}} \|x_i\| = 0\}$. The Banach space $\ell_\infty(I, X_i)/c_0^{\mathcal{U}}(X_i)$, with the quotient norm, is called the ultraproduct of the family $(X_i)_{i\in I}$ along $\mathcal{U}$ and it is denoted by $[X_i]_{\mathcal{U}}$. When all $X_i = X$ for some fixed X the ultraproducts are called ultrapowers and are denoted by $X_{\mathcal{U}}$ instead.

An ultrafilter is called free if it contains no finite set; otherwise there is exactly one point $i \in I$ such that $U \in \mathcal{U} \iff i \in U$ and $\mathcal{U}$ is called principal. An ultrafilter $\mathcal{U}$ is said to be countably incomplete (CI, for short) if there exists a countable family of members of $\mathcal{U}$ whose intersection does not belong to $\mathcal{U}$; we can require the intersection to be empty without altering the definition. It is very easy to see that all free ultrafilters on a countable set are CI and that $\mathcal{U}$ is CI if and only if there is a strictly positive function $f : I \to (0, 1)$ such that $f(i) \to 0$ along $\mathcal{U}$. Ultraproducts are relevant in our business because of the following observation (see [19, Proposition 2.19] for this formulation and [70, Remark on p. 479] or [28, Lemma 1.4] for two slightly weaker forerunners):

Fact 1.6 *An ultraproduct of a family of AT spaces along a CI ultrafilter is transitive.*

Comments:

(1) It is clear that the conclusion of Fact 1.6 subsists under much weaker hypotheses. For a fixed $\varepsilon > 0$, say that X is ε-transitive if given $x, y \in S_X$ there is $T \in \operatorname{Isom} X$ such that $\|y - Tx\| \leqslant \varepsilon$. Call it δ-asymptotically transitive if, given $x, y \in S_X$ there is a surjective δ-isometry T such that $y = Tx$; this is inspired by Talponen's [136, Definition 2.1]. An easy argument on series shows that an ε-transitive Banach space is also 2ε-asymptotically transitive provided $\varepsilon \leqslant \frac{1}{2}$. It is straightforward that if (X_n) is a sequence of Banach spaces such that X_n is δ_n-asymptotically transitive and $\delta_n \to 0$ as $n \to \infty$ and $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$, then the ultraproduct $[X_n]_{\mathcal{U}}$ is transitive.

(2) Perhaps the most interesting question concerning transitivity properties of ultraproducts is whether the transitivity of the ultrapower $X_{\mathcal{U}}$ implies *anything* about the isometry group of the base space X. Of course one can ask whether X must be AT, which is quite natural from the point of view of model theory, but

actually at this point it is even open whether there exists a Banach space with only trivial isometries whose ultrapowers are transitive.

2 Maximality of norms, Wood's problems, Deville–Godefroy–Zizler problem

Recall from Sect. 1.4 that every transitive or even almost transitive norm is maximal. This follows easily from the observation that if a group of isomorphisms acts as an isometry for two norms, then these norms must be proportional on any orbit of the action of the group. This led many people to investigate which spaces have maximal norms.

In 1933-34 Auerbach [9–11] proved that for every finite dimensional real Banach space $(X, \|\cdot\|)$, there exists a norm $\|\cdot\|_2$ on X induced by an inner product and such that the isometry group of $(X, \|\cdot\|_2)$ contains the isometry group of $(X, \|\cdot\|)$. Thus the isometry group of every real finite dimensional space is contained in that of a maximal norm. Rolewicz [126, §9.8] showed that the norm of any space with a 1-symmetric basis (real or complex) is maximal. This includes norms on the classical spaces ℓ_p, whose isometries act as "signed" permutations of the vectors of the unit basis - and therefore those norms are maximal but not AT. Norms of the spaces $L_p, 1 \leqslant p < \infty$, being AT, are in particular maximal. For $C(K)$- and specially for $C_0(L)$-spaces, the situation is more involved, depending on whether the scalars are real or complex. See the survey paper by J. Becerra Guerrero and Á. Rodríguez-Palacios [19] for general information on maximal norms and [31] and the references therein for maximality in $C_0(L)$ and $C(K)$-spaces.

Note that if G is a bounded subgroup of $\mathrm{GL}(X)$, then G is a subgroup of $\mathrm{Isom}(X, \|\cdot\|_G)$, where $\|\cdot\|_G$ is an equivalent norm on X defined by $\|x\|_G = \sup_{g\in G} \|gx\|$. Thus a norm is maximal if and only if the corresponding isometry group is a maximal bounded subgroup of $\mathrm{GL}(X)$. Citing the introduction of [55], "[it seemed] natural to suspect that a judicious choice of smoothing procedures on a space X could eventually lead to a most symmetric norm, which then would be maximal on X". However the following fundamental questions on maximal norms remained open until 2013.

Problem 2.1 (1982, Wood [142]) Does every Banach space admit an equivalent maximal norm, that is, does $\mathrm{GL}(X)$ always have maximal bounded subgroups?

Problem 2.2 (1993, Deville, Godefroy, Zizler) [43, Problem IV.2 and the remark following it] Does every super-reflexive space admit an equivalent almost transitive norm?

Problem 2.3 (2006, Wood [143]) Is it true that for every Banach space, there exists an equivalent maximal renorming whose isometry group contains the original isometry group, i.e., is every bounded subgroup of $\mathrm{GL}(X)$ contained in a maximal bounded subgroup of $\mathrm{GL}(X)$?

In 2013 Ferenczi and Rosendal [55] answered these three problems negatively by exhibiting a complex super-reflexive space and a real reflexive space, both without a maximal bounded subgroup of the isomorphism group. In 2015 Dilworth and Randrianantoanina [44] studied Problems 2.2 and 2.3 further. They showed multiple examples of super-reflexive spaces (both complex and real) which provide a negative answer to Problems 2.2 and 2.3, despite the fact that they have an equivalent maximal renorming. Among others, the classical spaces ℓ_p, $1 \leq p < \infty$, $p \neq 2$, are such examples. In [44] the authors also showed that for some spaces X, the group GL(X) may contain even continuum different maximal bounded subgroups. It is open whether there exists a Banach space X with a unique, up to conjugacy, maximal bounded subgroup of GL(X), or whether Hilbert space has this property.

2.1 Almost trivial isometry groups

In this section we describe the main result of Ferenczi and Rosendal from [55].

Theorem 2.4 *There exists a complex, separable, super-reflexive Banach space X, and a real, separable, reflexive space Y, both without maximal bounded groups of isomorphisms, i.e., X and Y have no equivalent maximal norms.*

We choose to present a sketch of the result corresponding to the complex case, and to present a simplified version of the results. This allows us to give much simpler versions of the proofs of [55].

A second motivation and a source of tools for the work [55] comes from the theory of spaces with "few operators", initiated by the construction of W.T. Gowers and B. Maurey [69] of a hereditarily indecomposable (or HI) space (meaning that it contains no subspace decomposable as a direct sum of infinite dimensional subspaces). Gowers and Maurey proved that such spaces have small spaces of operators, namely, in the complex case any operator is a strictly singular perturbation of a scalar multiple of the identity map. The currently strongest result in this direction, due to S. A. Argyros and R. G. Haydon [8], is the construction of a Banach space on which every operator is a compact perturbation of a scalar multiple of the identity.

One can ask the same question for isometries. An isometry is called trivial if its a scalar multiple of the identity. Does every Banach space admit a non-trivial surjective isometry? After partial answers by P. Semenev and A. Skorik [131], and an answer in the real separable case by S. Bellenot [22], the question was settled by K. Jarosz [82], who proved that any real or complex Banach space admits an equivalent norm with only trivial isometries.

Thus, no isomorphic property of a space can force the existence of a non-trivial surjective linear isometry. On the other hand it is immediate, through renormings where some prescribed finite dimensional subspace becomes euclidean, that an infinite dimensional space always admits an equivalent norm whose isometry group contains a copy of the unitary group of the n-dimensional euclidean space.

In this line, Ferenczi and Rosendal investigate results relating the size of the isometry group Isom$(X, \|\cdot\|)$, for any equivalent norm $\|\cdot\|$, with the isomorphic structure of X, through the next definition.

Definition 2.5 A bounded subgroup $G \leqslant \mathrm{GL}(X)$ acts *nearly trivially* on X if there is a G-invariant decomposition $X = F \oplus H$, where F is finite-dimensional and G acts by trivial isometries on H.

The relation of this concept with questions of maximality is based on the following easy but powerful lemma:

Lemma 2.6 *If the isometry group of an infinite dimensional space acts nearly trivially then the norm is not maximal.*

Proof If $X = F \oplus H$ is the decomposition associated to the near triviality of Isom(X), and if H is decomposed as $R \oplus Y$, where R is 1-dimensional, then the equivalent norm defined by the formula $\|f\| = \|r\| + \|y\|$, $f \in F$, $r \in R, y \in Y$, admits an isometry group which strictly contains the original one. □

In particular, if every bounded subgroup of GL(X) (equivalently, every isometry group) acts nearly trivially, then X admits no maximal renorming.

As an initial step towards Theorem 2.4, Ferenczi and Rosendal, improving on some earlier work of F. Räbiger and W. J. Ricker [120, 121], show that in a certain class of spaces, each individual isometry acts nearly trivially.

Theorem 2.7 *Let X be a Banach space containing no unconditional basic sequence. Then each individual isometry which is of the form $\lambda \mathbf{I}_X + S$, for S strictly singular, acts nearly trivially on X (and in particular S is a finite range operator). In particular each isometry on a complex HI space acts nearly trivially.*

Proof The spectrum of an isometry of the form $\mathbf{I}_X + S$ is formed either of a finite sequence of eigenvalues, or an infinite converging sequence of eigenvalues together with their limit 1. In the latter case the authors of [55] prove that a sequence of eigenvectors associated to eigenvalues converging fast enough to 1 would form an unconditional basic sequence (with constant arbitrarily close to 1). In the former case, classical spectral decomposition results imply that the operator S has finite dimensional range, or equivalently, that $\mathbf{I}_X + S$ acts nearly trivially on X. □

For future reference the decomposition of X associated to the fact that an operator T acts nearly trivially may be written as $X = F_T \oplus H_T$ where H_T is the kernel of $\mathbf{I}_X - T$ and F_T its image. This notation will be used in what follows.

The next step is to proceed from single isometries acting nearly trivially to an understanding of the global structure of the isometry group Isom(X). Using a renorming result of Lancien [97] for separable reflexive X, the authors of [55] prove a version of Alaoglu-Birkhoff [3] ergodic decomposition theorem:

Proposition 2.8 *Assume X is separable reflexive and G is a bounded group of automorphisms of X. Let H_G be the subspace of points fixed by every $T \in G$, and H_{G^*} be the subspace of functionals fixed by every element of G under its natural action on X^*. Let $\mathcal{S}$ be a family generating a SOT-dense subgroup of G.*

Then X admits the G-invariant decomposition $F_G \oplus H_G$, where

$$F_G = H_{G^*}^{\perp} = \overline{\text{span}} \bigcup_{T \in \mathcal{S}} F_T,$$

and the associated projection onto H_G has norm at most $\|G\|^2$ (where $\|G\| := \sup_{g \in G} \|g\|$).

Denote by $\text{Isom}_f(X)$, the subgroup of isometries of the form $\mathbf{I}_X + A$, where A is a finite-rank operator on X. Note that when G is a subgroup of $\text{Isom}_f(X)$, each subspace F_T is finite dimensional. This leads the authors of [55] to consider possible FDDs of X:

Proposition 2.9 *Let X be separable and reflexive. Then either $\text{Isom}_f(X)$ acts nearly trivially on X, or X admits a complemented subspace with a finite dimensional decomposition.*

Proof This is [55, Theorem 4.16]; the proof goes as follows. Picking an SOT-dense sequence (T_n) of isometries in $\text{Isom}_f(X)$, one considers each of the Alaoglu-Birkhoff decompositions associated to the subgroups G_n generated by $T_1, \dots, T_n$, i.e.

$$X = F_n \oplus H_n,$$

where F_n is the linear span of the finite dimensional subspaces $\text{Im}(\mathbf{I}_X - T_j)$, $1 \leqslant j \leqslant n$, and H_n the set of points fixed by $T_1, \dots, T_n$. Consider the decomposition

$$X = F_G \oplus H_G,$$

associated to $G = \text{Isom}_f(X)$. It can be seen that F_G identifies with the closure of $\bigcup_n F_n$ and therefore either is finite dimensional or admits an FDD. In the former case G acts trivially on the finite codimensional space H_G and therefore nearly trivially on X. ☐

Combining Theorem 2.7 and Lemma 2.6, with the decomposition from Proposition 2.9, along with the indecomposability property of HI spaces, one deduces:

Theorem 2.10 *Let X be a separable, reflexive, hereditarily indecomposable, complex Banach space without a FDD. Then for any equivalent norm on X, the group of isometries acts nearly trivially on X. In particular X does not admit a maximal norm.*

The existence of a uniformly convex example satisfying these conditions follows from an earlier construction of a super-reflexive HI space due to Ferenczi [53], as well as conditions by Szankowski [135] for the existence of subspaces failing the Approximation Property and therefore failing to have an FDD.

Comments:

(1) The construction of [55] does not seem to provide a uniformly convex space on which no subspace admits an AT norm. Indeed on subspaces admitting a Schauder basis, the authors also obtain isometry invariant decompositions of the form $F \oplus H$ where F is finite dimensional, but are only able to prove that the group of isometries which are finite range perturbations of the identity acts as an SOT-discrete group for on H. We are unaware of a general argument suggesting that this would prevent the existence of dense orbits for the action of the isometry group on the sphere (on this subject, one can consult [7] where an SOT-discrete bounded group of automorphisms is constructed on c_0 whitout discrete orbits). Uniformly convex spaces where no subspaces admit an AT renorming will be encountered in Sect. 2.2.
(2) The paper [55] has a wider scope than presented above. First of all, by renorming, bounded groups of automorphisms may be seen as groups of isometries, to which the above results apply. The setting of several results may also be extended from the case of spaces with few operators, to a more general case of bounded actions of groups of operators of the form $\mathbf{I}_X + S$ on arbitrary Banach spaces. Through a finer analysis of the group structure of the isometries, FDD may be replaced by Schauder bases in most occurences. Finally methods of complexification allow to extend most results to the real case.
(3) Weaker forms of rigidity than in the exotic spaces considered in this section may also induce restrictions on the actions of bounded groups. For considerations in this line regarding bounded groups acting on interpolation scales and (almost) transitivity, see Section 3.4 of the recent paper [38].
(4) Before leaving the topic of "nearly trivial isometries", it is perhaps worth noticing the following result from [29]: *If* $\mathrm{Isom}_f(X)$ *acts transitively on the unit sphere of a normed space X then the norm of X is Euclidean.*

2.2 More on the Deville–Godefroy–Zizler problem

In this section we describe the results and methods of Dilworth and Randrianantoanina [44] providing additional counterexamples for Problem 2.2. They proved the following.

Theorem 2.11 *The following classes of Banach spaces do not admit an equivalent almost transitive renorming.*

(a) *subspaces of classical sequence spaces* ℓ_p *for* $1 \le p < \infty$ *different from* 2, *or* c_0,
(b) *subspaces of an* ℓ_p*-sum of finite-dimensional normed spaces, for* $1 < p < \infty$ $p \neq 2$, *and, in particular, subspaces of quotient spaces of* ℓ_p, for $1 < p < \infty$, $p \neq 2$,
(c) *subspaces of asymptotic-*ℓ_p *spaces*, $1 \le p \le \infty, p \neq 2$,
(d) *subspaces of Asymptotic-*ℓ_p *spaces*, $1 \le p \le \infty, p \neq 2$, *in the sense of* [105],
(e) *subspaces of any Orlicz sequence space* ℓ_M (*where M is an Orlicz function*) *such that* ℓ_M *does not contain a subspace isomorphic to* ℓ_2.
(f) *subspaces of* $L_p, 2 < p < \infty$, *that do not contain a subspace isomorphic to* ℓ_2.

Their method relies on an application of the classical Dvoretzky theorem, see e.g. [65], which says that in every infinite dimensional Banach space for every natural number m and every $\varepsilon > 0$, there exists a sequence $\{x_i\}_{i=1}^m$ in X that is $(1+\varepsilon)$-equivalent to the standard normalized basis of ℓ_2^m. It is a very simple but key observation, that when X is AT, then the first element x_1 in the above sequence can be chosen arbitrarily close to any element of the sphere of X. Moreover, using compactness, given $x_1 \in S_X$, there exists $x_2 \in S_X$ that is almost disjoint with x_1 (with respect to a given Schauder basis), and $\{x_1, x_2\}$ is $(1+\varepsilon)$-equivalent to the standard basis of two dimensional ℓ_2^2.

It is not known whether this can be generalized to an arbitrary dimension n, that is, whether for every $n \in \mathbb{N}$ every AT space X with a Schauder basis contains n vectors that are mutually almost disjoint and $(1+\varepsilon)$-equivalent to the standard normalized basis of ℓ_2^n. However using induction the authors of [44] prove existence of block bases in AT spaces that behave like the normalized basis of ℓ_2^n for an arbitrary but (sic!) fixed sequence of scalars.

Recall that a sequence $(x_i)_i$ of vectors is a *normalized block basis* if $\|x_i\| = 1$ $(i \ge 1)$, each vector x_i is finitely supported, and $x_1 < x_2 < x_3 < \ldots$, that is, for all $i \ge 2$, $\max \operatorname{supp} x_{i-1} < \min \operatorname{supp} x_i$.

Theorem 2.12 *Suppose that X has a Schauder basis and contains an infinite-dimensional subspace Y which is almost transitive. Then, for any* $\varepsilon > 0$ *and any sequence* $(a_i)_i$ *of nonzero scalars, there exists a normalized block basis* $(x_i)_i$ *in X such that, for all* $m \ge 1$, *we have*

$$(1-\varepsilon)\|(a_i)_{i=1}^m\|_{\ell_2} \le \left\| \sum_{k=1}^m a_k x_k \right\| \le (1+\varepsilon)\|(a_i)_{i=1}^m\|_{\ell_2}. \quad (1)$$

We stress that in Theorem 2.12 the block basis that satisfies (1) depends not only on $\varepsilon > 0$, but also on the selected scalar sequence $(a_i)_i$. It turns out that this is powerful enough to imply several results on nonexistence of AT renormings. As an illustration we show how it can be used to prove that no subspace of ℓ_p, $1 \le p < \infty$, $p \neq 2$, admits an equivalent AT renorming.

The argument is as follows: suppose that a subspace Y of X admits an equivalent AT norm $|||\cdot|||$. It is well-known that any equivalent norm on a subspace may be extended to an equivalent norm on the whole space, see e.g. [52, p. 55]. Then, by Theorem 2.12 applied to $(X, |||\cdot|||)$ with the constant sequence $(a_i = 1)_{i=1}^\infty$, there exists a disjointly supported sequence $(x_k)_k$ in X such that for all $n \in \mathbb{N}$, the norm $|||\sum_{k=1}^n x_k|||$ is $(1+\varepsilon)$-equivalent to $n^{1/2}$. However, $|||\cdot|||$ is C-equivalent to $\|\cdot\|_{\ell_p}$, and, as is well known, every block basis of ℓ_p is isometrically equivalent to the standard basis of ℓ_p, see e.g. [99], i.e. $|||\sum_{k=1}^n x_k|||$ is C-equivalent to $n^{1/p}$, which gives the contradiction when $p \neq 2$ and n is large enough.

Essentially the same argument works for spaces X with a Schauder basis (e_i) that satisfy (p, q)-estimates, where $1 < q \leq p < \infty$, that is, such that there exists $C > 0$ with

$$\frac{1}{C}\Big(\sum_{k=1}^n \|x_k\|^p\Big)^{1/p} \leq \Big\|\sum_{k=1}^n x_k\Big\| \leq C\Big(\sum_{k=1}^n \|x_k\|^q\Big)^{1/q},$$

whenever $x_1 < x_2 < \cdots < x_n$. Thus we have:

Corollary 2.13 *Suppose that a Banach space X with a Schauder basis (e_i) contains a subspace Y which admits an equivalent almost transitive norm. If (e_i) satisfies (p, q)-estimates, then $q \leq 2 \leq p$.*

Using similar reasoning and known properties of Banach spaces the authors of [44] obtain a list of classes of Banach spaces such that none of their subspaces admits an equivalent AT renorming, stated in Theorem 2.11.

Remark 2.14 For a Banach space X, let

$$\mathrm{FR}(X) := \{1 \leq r \leq \infty : \ell_r \text{ is finitely representable in } X\}.$$

The proof of Theorem 2.12, is valid not only for the exponent 2, as stated, but (after the obvious modifications) for any exponent $r \in \mathrm{FR}(Y)$, where Y is an infinite dimensional AT subspace of X (and X has a Schauder basis).

By the Maurey-Pisier theorem [106], this holds for all $r \in [p_Y, 2] \cup \{q_Y\}$, where $p_Y := \sup\{1 \leq p \leq 2 : Y \text{ has type } p\}$ and $q_Y := \inf\{2 \leq q < \infty : Y \text{ has cotype } q\}$. For spaces with an unconditional basis, using results of Sari [129], the authors of [44] then obtain a stronger version of Theorem 2.12 for certain values of r.

Theorem 2.15 *Suppose that X has an unconditional basis $(e_i)_i$. If X has an equivalent almost transitive renorming $|||\cdot|||$, then for $r = p_X$ and $r = q_X$ and for all $n \geq 1$ and $\varepsilon > 0$, there exist disjointly supported vectors $(x_i)_{i=1}^n \subset X$ such that $(x_i)_{i=1}^n$ in the norm $|||\cdot|||$ is $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_r^n.*

Comments:

(1) The "extreme" cases in Theorem 2.11(a) namely subspaces of ℓ_1 and c_0 are much easier and were proven earlier by Cabello Sánchez [27, Theorem 2.1]: *every space which is either Asplund or has the Radon-Nikodým property with*

an AT renorming must be super-reflexive. Actually these spaces and also those in entries (a), (b), (f) lack convex transitive norms, by [19, Corollary 6.9.].

(2) It is instructive to observe that Theorem 2.12, when applied to the Haar basis of L_p, does not contradict the fact that L_p is AT. This is because the unit vector basis of ℓ_2^m is $(1+\varepsilon)$-equivalent to a block basis of the Haar basis.

(3) A natural question, in the light of the third and fourth item of Theorem 2.11, is whether every super-reflexive space which does not admit a subspace with an AT norm must contain an asymptotic-ℓ_p subspace. The authors of [44] answer this question negatively using as an example a space constructed in [36].

(4) This comment is in response to questions by the referee and Gilles Godefroy. A Banach space X has property (M) (respectively, property (m_p) for some $1 \le p \le \infty$) if for each weak-null sequence (x_n) and all $x, y \in X$ with $\|x\| = \|y\|$, one has $\limsup_n \|x + x_n\| = \limsup_n \|y + x_n\|$ (respectively, $\limsup_n \|x + x_n\|^p = \|x\|^p + \limsup_n \|x_n\|^p$). These asymptotic properties of the norm were introduced by Kalton in his study of M-ideals [85] and by Kalton and Werner in [87], respectively. The papers [85] and [87] contain a wealth of examples and counterexamples as well as the ultimate connections between properties (M), (m_p), and (M^*) and M-ideals of compact operators.

The presence of an equivalent norm with property (M) has a considerable impact on the isomorphic structure of the underlying Banach space and has been used in [85] and [87] in particular to obtain important characterizations of subspaces and quotients of L_p, the Schatten class c_p, and of ℓ_p-sums of finite dimensional spaces.

Dutta and Godard [47, Proposition 2.3] showed that norms with property (M) have "optimal" moduli of asymptotic uniform smoothness and asymptotic uniform convexity among all other equivalent renormings of the space. In [47, Theorem 2.6] they also identified an equivalent condition in terms of the "optimal" growth of the Szlenk indices of X and X^* for the separable reflexive space X with property (M) to be isomorphic to a subspace of an ℓ_p-sum of finite dimensional normed spaces, where $p = \inf\{r : \sup_{\varepsilon>0} \varepsilon^r Sz(X^*, \varepsilon) < \infty\}$. By Theorem 2.11(b), in this situation if $p \neq 2$, no subspace of X can be renormed to be AT.

Theorem 3.3 of [87] says, in particular, that if a Banach space X is separable, does not contain an isomorphic copy of ℓ_1, and has property (m_p) for some $1 < p < \infty$, then X is isomorphic to a subspace of an ℓ_p-sum of finite dimensional normed spaces. Thus, by Theorem 2.11(b), if a Banach space X is separable, does not contain an isomorphic copy of ℓ_1, and can be equivalently renormed to satisfy property (m_p) for some $1 < p < \infty$, $p \neq 2$, then X does not contain any infinite dimensional subspace that can be renormed to be AT. However we do not know if the same holds true only under the assumption that X is non-hilbertian and has property (M), or even whether a non-hilbertian norm with property (M) can be AT. We note that by [85, Remark after Proposition 4.1] every reflexive Orlicz space which is not isomorphic to any ℓ_p can be renormed to satisfy property (M) and simultaneously fail property (m_p) for every p. By Theorem 2.11(e) we know that if an Orlicz space ℓ_F does not contain a subspace isomorphic to ℓ_2, then no subspace of ℓ_F can be renormed to be AT.

At this time the most that we can say about general spaces with property (M) and AT subspaces is the following. It follows from [44, Lemma 2.1] and [85, Lemma 3.6 and the remark before it] that if X has a Schauder basis and property (M), and if an infinite dimensional subspace of Y of X is AT, then Y contains a subspace isomorphic to ℓ_2 and, by [85, Proposition 3.8], every infinite dimensional subspace Z of Y, for every $\varepsilon > 0$, contains a subspace $E \subseteq Z$ with $d_{\mathrm{BM}}(E, \ell_2) < 1 + \varepsilon$.

2.3 Spaces with multiple maximal bounded subgroups of GL(X)

We have seen in Sect. 2.1 that there exist Banach spaces X without maximal bounded subgroups of $\mathrm{GL}(X)$. On the other hand, in this section, following [44], we will show examples of spaces with multiple different (i.e. non-conjugate) maximal equivalent renormings.

We say that two equivalent norms $\|.\|$ and $|||\cdot|||$ on X are *conjugate* if there exists a bounded linear automorphism T of X such that $\|x\| = |||Tx|||$ for all $x \in X$. Note that in this case, T induces an bilipschitz homemorphism between the unit spheres of X under the two norms. On the other hand we say that the groups $\mathrm{Isom}(X, \|.\|)$ and $\mathrm{Isom}(X, |||\cdot|||)$ are conjugate if they are conjugate as subgroups of $\mathrm{GL}(X)$, that is, if there exists a bounded linear automorphism L of X such that

$$\mathrm{Isom}(X, \|.\|) = L^{-1}\,\mathrm{Isom}(X, |||\cdot|||)L.$$

Note that if two norms are conjugate then their respective isometry groups are conjugate, but the converse does not hold (the isometry group for the two norms could e.g. be trivial and therefore equal without the norms being conjugate). However the two notions are equivalent in the following important case, which will simplify considerably some of our proofs.

Lemma 2.16 *Two convex transitive norms on a Banach space are conjugate if (and only if) they have conjugate isometry groups.*

Proof Let $\|\cdot\|$ and $|\cdot|$ be CT norms on X and assume that $\mathrm{Isom}(\|\cdot\|)$ and $\mathrm{Isom}(|\cdot|)$ are conjugate through $L \in \mathrm{GL}(X)$ so that T is an isometry of $\|\cdot\|$ if and only if $\tilde{T} = LTL^{-1}$ is an isometry of $|\cdot|$. Pick $x_0 \in X$ so that $\|x_0\| = 1$. Multiplying L by a positive constant if necessary, we can and do assume WLOG that $|Lx_0| = 1$. Then for all $T \in \mathrm{Isom}(\|\cdot\|)$, $\tilde{T} = LTL^{-1} \in \mathrm{Isom}(|\cdot|)$ and thus $|L(Tx_0)| = |LT(L^{-1}Lx_0)| = |\tilde{T}Lx_0| = |Lx_0| = 1$. Hence, by CT, for all $x \in X$, $|Lx| \leq \|x\|$.

Let us then check that the CT of $\|\cdot\|$ entails that $L : (X, \|\cdot\|) \to (X, |\cdot|)$ is contractive. By symmetry we shall also have that $L^{-1} : (X, |\cdot|) \to (X, \|\cdot\|)$ is contractive and so $\|\cdot\|$ and $|\cdot|$ are conjugate. As the unit ball of $\|\cdot\|$ is the closed convex hull of the orbit of x under the action of $\mathrm{Isom}(\|\cdot\|)$ it suffices to see that if $y = Tx$ for some $T \in \mathrm{Isom}(\|\cdot\|)$, then $|Ly| = 1$. Which is easy: $Ly = LTx = LTL^{-1}Lx = \tilde{T}Lx$ and $\tilde{T}$ is an isometry of $|\cdot|$. □

Constructions in [44] use vector valued spaces defined as follows.

Let X be a Banach space with a 1-unconditional basis $E = (e_k)_{k\in\mathbb{N}}$ and $(Y_k)_{k\in\mathbb{N}}$ be Banach spaces. Then

$$Z = \Big(\sum_{k\in\mathbb{N}} \oplus Y_k\Big)_E$$

is the space of all sequences $(z_k)_{k\in\mathbb{N}}$ such that for all k, $z_k \in Y_k$, and

$$\|(z_k)_k\|_Z := \Big\| \sum_{k\in\mathbb{N}} \|z_k\|_{Y_k}\, e_k \Big\|_X$$

is finite. When E is a standard basis of ℓ_p, we will sometimes write $Z = (\sum_{k\in\mathbb{N}} \oplus Y_k)_{\ell_p}$ to mean the same as $Z = (\sum_{k\in\mathbb{N}} \oplus Y_k)_E$.

Rosenthal [128] characterized isometries of spaces of this form in the case when all spaces Y_k are hilbertian and X is a *pure* space with a normalized 1-unconditional basis. A Banach space X with a normalized 1-unconditional basis $\{e_\gamma\}_{\gamma\in\Gamma}$ is called *impure* if there exist $j \neq k$ in Γ so that (e_j, e_k) is isometrically equivalent to the usual basis of 2-dimensional ℓ_2^2, and for all $x, x' \in \mathrm{span}(e_j, e_k)$ with $\|x\| = \|x'\|$ and for all $y \in \mathrm{span}\{e_m : m \neq j, k\}$ we have $\|x + y\| = \|x' + y\|$ [128, Corollary 3.4]. Otherwise the space is called *pure*. The space ℓ_p, $1 \le p < \infty$, $p \neq 2$, is a natural example of a pure space.

Rosenthal proved the following result.

Theorem 2.17 [128, Theorem 3.12] *Let X be a pure space with a* 1*-unconditional basis $E = \{e_\gamma\}_{\gamma\in\Gamma}$, $(H_\gamma)_{\gamma\in\Gamma}$ be Hilbert spaces all of dimension at least* 2*, and let $Z = (\sum_\Gamma \oplus H_\gamma)_E$.*

Let $P(Z)$ denote the set of all bijections $\sigma : \Gamma \to \Gamma$ so that

(a) $\{e_{\sigma(\gamma)}\}_{\gamma\in\Gamma}$ *is isometrically equivalent to* $\{e_\gamma\}_{\gamma\in\Gamma}$, *and*
(b) $H_{\sigma(\gamma)}$ *is isometric to* H_γ *for all* $\gamma \in \Gamma$.

Then $T : Z \to Z$ is a surjective isometry if and only if there exist $\sigma \in P(Z)$ and surjective linear isometries $T_\gamma : H_\gamma \to H_{\sigma(\gamma)}$, for all $\gamma \in \Gamma$, so that for all $z = (z_\gamma)_{\gamma\in\Gamma}$ in Z, and for all $\gamma \in \Gamma$,

$$(Tz)_{\sigma(\gamma)} = T_\gamma(z_\gamma). \tag{2}$$

Theorem 2.17 is valid for both real and complex spaces. For separable complex Banach spaces it was proved earlier by Fleming and Jamison [59], cf. also [88].

Dilworth and Randrianantoanina [44], using Theorem 2.17, described a countable number of different equivalent maximal norms on every Banach space with a 1-symmetric basis, which is not isomorphic to ℓ_2.

Theorem 2.18 *Suppose $X = \ell_p$, $1 \le p < \infty$, $p \neq 2$, or, more generally, X is a pure Banach space with a 1-symmetric basis $E = \{e_k\}_{k=1}^\infty$, and X is not isomorphic* *to ℓ_2. Then X admits countably many mutually non-conjugate equivalent maximal renormings.*

Namely, for $n \in \mathbb{N}, n \geq 2$, let

$$Z_n = Z_n(X) = (\sum_{k=1}^{\infty} \oplus H_k)_E,$$

where, for all $k \in \mathbb{N}$, H_k is isometric to ℓ_2^n. Then Z_n is isomorphic to X, the isometry group of Z_n is maximal, and, if $n \neq m$, the groups $\mathrm{Isom}(Z_n)$ and $\mathrm{Isom}(Z_m)$ are not conjugate in $\mathrm{GL}(X)$.

Idea of proof It is easy to see that Z_n is isomorphic to the direct sum of n copies of X and hence isomorphic to X itself since X has a symmetric basis.

By Theorem 2.17, all isometries of Z_n have form (2), and, since the basis is 1-symmetric and all H_k are isometric to each other, the set $P(Z_n)$ is equal to the set of all bijections of $\mathbb{N}$.

We claim that the group $\mathrm{Isom}(Z_n)$ is maximal. Let's first consider the case when σ is the identity of $\mathbb{N}$ and, for all k, $T_k \in \mathrm{Isom}(H_k)$. Since $\mathrm{Isom}(H_k) = \mathrm{Isom}(\ell_2^n)$ is the largest possible group of isometries of any n-dimensional space, it is impossible to renorm each H_k to increase the isometry group of Z_n. So how can we renorm Z_n to introduce additional isometries?

A first natural idea that comes to mind is to "glue" two or more, but finitely many, fibers of Z_n and equip this new larger fiber with the norm that has the largest possible isometry group, i.e. the ℓ_2 norm. Say, if we put for all $k \in \mathbb{N}$, $\widetilde{H}_k = H_{2k-1} \oplus_2 H_{2k}$ and consider $\widetilde{Z}_n = (\sum_{k=1}^{\infty} \oplus \widetilde{H}_k)_E$. Then, for all k, $\dim \widetilde{H}_k = 2n$, and if $\sigma = \mathbf{I}_{\mathbb{N}}$, there exists an isometry $\widetilde{T}_k$ of $\widetilde{H}_k$ so that $\widetilde{T}_k(H_{2k-1})$ intersects both H_{2k-1} and H_{2k}, so the operator $\widetilde{T} : \widetilde{Z}_n \to \widetilde{Z}_n$ defined by $\widetilde{T}((\widetilde{z}_k)_k) = (\widetilde{T}(\widetilde{z}_k))_k$ is an isometry of $\widetilde{Z}_n$ but not of Z_n.

On the other hand, since the basis E is 1-symmetric, if we consider a permutation σ of $\mathbb{N}$ so that, say, $\sigma(1) = 3$ and $\sigma(2) = 5$, and arbitrary isometries $T_k : H_k \to H_{\sigma(k)}$, then the operator $T : Z_n \to Z_n$ such that for all $z = (z_k)_k \in Z_n$, $(Tz)_{\sigma(k)} = T_k(z_k)$ is an isometry of Z_n. However T is not an isometry of $\widetilde{Z}_n$, since, by Theorem 2.17, any isometry of $\widetilde{Z}_n$ maps the fiber $\widetilde{H}_1$ either to itself or onto another fiber and we have that $T(\widetilde{H}_1)$ intersects both $\widetilde{H}_2$ and $\widetilde{H}_3$.

Hence we have that $\mathrm{Isom}(\widetilde{Z}_n) \not\subseteq \mathrm{Isom}(Z_n)$ and $\mathrm{Isom}(\widetilde{Z}_n) \not\supseteq \mathrm{Isom}(Z_n)$, and thus our "gluing" of fibers failed to produce a space with a larger isometry group.

It follows from [128] that if $\widetilde{Z}_n = (Z_n, \|\!| \cdot |\!\|)$ is an equivalent renorming of Z_n so that $\mathrm{Isom}(\widetilde{Z}_n) \supseteq \mathrm{Isom}(Z_n)$, then $\widetilde{Z}_n = (\sum_{k=1}^{\infty} \oplus \widetilde{H}_k)_E$, where each new fibers $\widetilde{H}_k$ is an ℓ_2 sum of a certain finite subcollection of the original fibers. The idea of the remaining part of the proof is same as above.

The fact that isometry groups $\mathrm{Isom}(Z_n)$ are mutually non-conjugate follows from (2), since for different values of n the dimensions of hilbertian fibers are different and E is pure, see [44, Proposition 3.4] and [128, Theorem 2]. □

The construction of Theorem 2.18 can be generalized to describe a continuum of different (pairwise non-conjugate) maximal renormings of Banach spaces Z that have the form $Z = (\sum_{k=1}^{\infty} \oplus \ell_2^{n(k)})_E$, where $E = (e_k)_{k=1}^{\infty}$ is a 1-symmetric basis of a pure Banach space X that is not isomorphic to ℓ_2. It follows from the Pełczyński decomposition method that the space Z is isomorphic to X if, for example, $X = \ell_p$, with $1 < p < \infty$, or if $X = U$, Pełczyński's space with a universal unconditional basis [112] mentioned in Sect. 1.5, see [44, Theorem 3.7] for details.

Note that, as Z is a separable Banach space, the collection of all equivalent norms on Z has cardinality $\mathfrak{c}$. Hence the maximal cardinality of a collection of pairwise non-conjugate maximal bounded subgroups of $\mathrm{GL}(Z)$ is exactly equal to $\mathfrak{c}$.

Corollary 2.19 *Each of the spaces* ℓ_p, for $1 < p < \infty$, $p \neq 2$, *and the space* U *with a universal unconditional basis, admits a continuum of equivalent renormings whose isometry groups are maximal and pairwise non conjugate in the group of bounded isomorphisms.*

The above results suggest the following questions:

Problem 2.20

(a) Let $\mathcal{H}$ be a Hilbert space. Is the unitary group the unique, up to conjugacy, maximal bounded subgroup of $\mathrm{GL}(\mathcal{H})$?
(b) Does there exist a separable Banach space X with a unique, up to conjugacy, maximal bounded subgroup of $\mathrm{GL}(X)$?
(c) If yes, does X have to be isomorphic to a Hilbert space?

Comments:

(1) Problem 2.20(a) may be reformulated as asking whether the Hilbert space admits a maximal, "non-unitarizable" bounded group of automorphisms. See Sect. 3 for more about this question.
(2) Theorem 2.18 applies in particular to the space $S(T^{(2)})$, the symmetrization of the 2-convexified Tsirelson space, see [37]. Indeed, it is known that $S(T^{(2)})$ does not contain ℓ_2, and it is easy to verify that for all $k, l \in \mathbb{N}$, $\|e_k + e_l\|_{S(T^{(2)})} = 1$, and thus the standard basis of $S(T^{(2)})$ is pure. It is clear that the renormings of $S(T^{(2)})$ described in Theorem 2.18 are not AT.

It is known that any symmetric weak Hilbert space is isomorphic to a Hilbert space, but in some sense the space $S(T^{(2)})$ is very close to a weak Hilbert space, see [37, Note A.e.3 and Proposition A.b.10]. We do not know the answers to the following problems:

Problem 2.21 Does the space $S(T^{(2)})$ admit an AT renorming? Does there exist a symmetric space not isomorphic to ℓ_2 which admits an AT renorming?

2.4 Spaces with multiple almost transitive norms

In this section we consider the existence of different maximal renormings of the space L_p, for $p \in [1, \infty)$ different from 2. We show that an analogue of Theorem 2.18 holds for L_p, and in this case it gives a countable family of mutually non-conjugate equivalent almost transitive norms. All the results in this section seem to be new and we have included (more or less) full proofs.

Theorem 2.22 *For $p \in [1, \infty)$ different from 2 the space L_p has at least countably many non-conjugate almost transitive norms.*

Proof For each $n \geq 1$ the space L_p is isomorphic to $L_p(H_n)$, where H_n is the n-dimensional Hilbert space. In [70, Theorem 2.1] it was proved that the standard norm on $L_p(H_n)$ is AT. The (AT) norms in L_p induced by an isomorphism onto $L_p(H_n)$ are, however, not conjugate in $\mathrm{GL}(L_p)$ for different values of n because $L_p(H_n)$ is isometric to $L_p(H_m)$ only if $n = m$, by results of Cambern and Greim; see [61, 8.2.11. Theorem]. □

The same occurs in $C[0, 1]$. Indeed Aizpuru and Rambla proved in [2, Proposition 6.2] that $C_0(P_*, H_n)$ is AT for all $n \geqslant 2$ no matter which field of scalars one considers. While the isometric type of these spaces effectively depends on n, by a classical result of Jerison [61, 7.2.16. Theorem], they are all isomorphic to $C[0, 1]$ by Miljutin's Theorem; see [4, Section 4.4] for a polished proof. Other "individual" AT renormings of $C[0, 1]$ arise from [28, Theorem 3.4], [32, Examples 2.4 and 3.2] and [54, Corollary 6.9].

The Garbulińska space provides a more spectacular example:

Theorem 2.23 *The Garbulińska space $\mathcal{K}$ has a continuum of mutually non-conjugate almost transitive norms.*

Proof As remarked in [33, p. 1551], $\mathcal{K}$ is the peskiest Banach space there is. In particular $\mathcal{K}$ is isomorphic to each of the spaces $L_p(\mathcal{K})$ for $1 \leqslant p < \infty$. To see this we observe that $L_p(\mathcal{K})$ has the BAP for all p, $1 \leq p < \infty$, and therefore it is isomorphic to a complemented subspace of $\mathcal{K}$ since the latter is complementably universal for the BAP. On the other hand, any space X is 1-complemented (as the space of constant functions) in $L_p(X)$ for any $1 \leqslant p \leqslant \infty$ by means of the "obvious" projection $P(f) = \int_0^1 f(t)dt$, where the integral is taken in the Bochner sense. An easy application of Pełczyński decomposition method yields $\mathcal{K} \simeq L_p(\mathcal{K})$ for $1 \leqslant p < \infty$.

Next we remark that $L_p(\mathcal{K})$ is AT for $1 \leqslant p < \infty$ by the result of Greim, Jamison and Kamińska already mentioned. For $1 \leqslant p < \infty$, let $| \cdot |_p$ denote the AT renorming of $\mathcal{K}$ induced by some (fixed) isomorphism $\mathcal{K} \to L_p(\mathcal{K})$. We claim that $(\mathcal{K}, | \cdot |_p)$ and $(\mathcal{K}, | \cdot |_q)$ cannot be isometric if $p \neq q$. To see this recall that an L^p-projection on a Banach space X is a projection P such that $\|x\|^p = \|Px\|^p - \|x - Px\|^p$ for all $x \in X$. It is clear that $L_p(\mathcal{K})$ (and so $(\mathcal{K}, | \cdot |_p)$) has *non-trivial* L^p-projections (think of multiplication by characteristic functions). But the only Banach space that admits nontrivial L^p-projections for two different values of p is $\ell_1^2 \approx \ell_\infty^2$ (real case; see [21, Main theorem]) from which the claim follows. □

By taking ultrapowers of the preceding examples, and using general representation results to describe the corresponding ultrapowers if necessary, we obtain:

Corollary 2.24 *Let $\mathcal{U}$ be a free ultrafilter on the integers.*

(a) *For each $p \in [1, \infty)$ different from 2 the ultrapower $(L_p)_\mathcal{U}$ has countably many pairwise non-conjugate transitive norms.*
(b) *$(C[0,1])_\mathcal{U}$ has countably many pairwise non-conjugate transitive norms.*
(c) *$\mathcal{K}_\mathcal{U}$ has a continuum of pairwise non-conjugate transitive norms.*

Proof The case of $\mathcal{K}_\mathcal{U}$ is clear because ultrapowers of spaces with nontrivial L^p-projections have again nontrivial L^p-projections so we can use the ultrapowers of the norms in Theorem 2.23.

(b) Note that $(C[0,1])_\mathcal{U} \simeq C_0(P_*)_\mathcal{U}$ by Miljutin's Theorem. It is known that if L is a locally compact space, then $C_0(P_*)_\mathcal{U}$ is isometrically isomorphic (even as a ring) to $C_0(L^\mathcal{U})$, with $L^\mathcal{U}$ a "huge" locally compact space. Explicit descriptions are available. Now, the point is that for fixed n, the ultrapower $C_0(L, H_n)_\mathcal{U}$ is isometric with $C_0(L^\mathcal{U}, H_n)$. This can be proved in many ways. Perhaps the simplest one is to identify $C_0(L, H_n)$ with the injective tensor product of $C_0(L)$ and H_n. That said, we have that $C[0,1]_\mathcal{U}$ is isomorphic to each of the transitive spaces $C_0(P_*, H_n)_\mathcal{U} = C_0(P_*^\mathcal{U}, H_n)$ which cannot be isometric for different values of n because of Jerison's result: [61, 7.2.16 Theorem]: *If Y is a strictly convex Banach space, then (X, Y) has the Banach-Stone property for any Banach space X. If both X and Y are strictly convex, then (X, Y) has the strong Banach-Stone property* (which in particular implies that if $C_0(L_1, X)$ is isometric to $C_0(L_2, Y)$ then L_1 is homeomorphic to L_2 and X is isometric to Y.

(a) The L_p case is a bit trickier. Fix $p \in [1, \infty)$ and use that $(L_p)_\mathcal{U}$ is isometric, even as a lattice, to $L_p(\mu)$ for some "huge" measure μ; see Heinrich's [77, Theorem 3.3(ii)]. In any case one can assume μ strictly localizable, by a result of Maharam (*cf*. Lacey [96, Corollary on p. 137]). After that show that for each fixed n the space $(L_p(H_n))_\mathcal{U}$ is isometric to $L_p(\mu, H_n)$; one can use a basis of H_n or a tensor product argument. Finally, dig into the details of Section 8.2 of Fleming–Jamison to check that [61, 8.2.11 Theorem] survives if the Hilbert spaces are finite-dimensional and one considers strictly localizable (instead of σ-finite) measures. □

Comments

(1) Rather curiously, we do not know whether the space $\mathcal{K}$ "itself" (i.e. in the Garbulińska norm) has non-trivial L^p-projections for some (necessarily unique) $p \in [1, \infty]$.

(2) A separable version of Corollary 2.24 is clearly out of reach, as it would require an answer to the Mazur rotations problem in its isometric or isomorphic version. See Sect. 3 for discussion about transitive renormings of the Hilbert space.
(3) Regarding Theorem 2.22, we have been unable to decide whether the isometry groups of the spaces $L_p(H_n)$ for different n's are isomorphic either in the purely algebraic sense or when they are equipped with SOT or the norm topology. In the real case one can prove that for any $n \geq 2$, $\mathrm{Isom}(L_p)$ is not topologically isomorphic to $\mathrm{Isom}(L_p(H_n))$ in the norm topologies because if T, L are different isometries of any $L_p(\mu)$, then $\|T - L\| \geqslant 2^{1/p}$. Thus $\mathrm{Isom}(L_p)$ is discrete in the norm topology, while for each $n \geq 2$, $\mathrm{Isom}(L_p(H_n))$ is not as it contains $\mathrm{Isom}(H_n)$.
(4) It is perhaps worth noticing the following application: if X is a *real* Lindenstrauss space (that is, X^* is isometric to $L_1(\mu)$ for some measure μ), then $\mathrm{Isom}(X)$ is discrete in the norm - just use the estimate in Comment (3) together with the natural isometric embedding of $\mathrm{Isom}(X)$ into $\mathrm{Isom}(X^*)$. In this case the isometries are as far as they can be: $\|T - L\| = 2$ unless $T = L$. This applies, in particular to the Gurariy space.

2.5 Isometry groups not contained in any maximal bounded subgroup of the isomorphism group

In [44] Dilworth and Randrianantoanina showed that Problem 2.3 can have a negative answer even if $\mathrm{GL}(X)$ contains many maximal bounded subgroups. Namely they proved (constructively):

Theorem 2.25 *Each of the spaces ℓ_p, for $p \in [1, \infty)$ different from 2, and U has a continuum of pairwise non conjugate renormings none of whose isometry groups is contained in any maximal bounded subgroup of the isomorphism group of ℓ_p.*

Compare with Corollary 2.19. The idea of the proof is similar to the proof of Theorem 2.18. The essential difference is that this time the E-sums are taken of sequences of Hilbert spaces that are not of the same dimension, but are all of different dimensions and, in addition, sums of dimensions of any two finite subcollections of fibers are never equal to each other, see [44, Section 4] for details.

Problem 2.26 Does there exist a separable Banach X space so that every bounded subgroup of $\mathrm{GL}(X)$ is contained in some maximal bounded subgroup of $\mathrm{GL}(X)$? Is this true for $X = L_p$?

Comments

(1) The conclusion of Theorem 2.25 is also true, for example, for the 2-convexified Tsirelson space $T^{(2)}$ and spaces of the form $(\sum_{n=1}^{\infty} \oplus \ell_2^n)_E$, where E is symmetric, pure, and not isomorphic to a Hilbert space. We note that $T^{(2)}$ is a weak Hilbert space.

(2) We do not know whether $T^{(2)}$ or general weak Hilbert spaces, other than the Hilbert, have a maximal bounded subgroup of $\mathrm{GL}(X)$.

2.6 Almost-transitivity, subspaces, and stabilizers

In the Hilbert space case we may note that the unitary group acts transitively not only on the sphere of X, but also on spheres of all infinite dimensional subspaces. We may ask to which extent this characterizes the Hilbert space. Some results in this direction were obtained in [44] as a consequence of Theorem 2.11 and known properties of Banach spaces.

Proposition 2.27 *Let X be a subspace of $L_p, 2 < p < \infty$, so that every subspace of X admits an almost transitive renorming, then X is isomorphic to ℓ_2.*

In view of Proposition 2.27 (see also comments to this section) it is natural to ask:

Problem 2.28 Suppose that every subspace of a Banach space X admits an almost transitive renorming. Is X isomorphic to a Hilbert space?

Next we turn to some sufficient conditions on hyperplanes (i.e. 1-codimensional subspaces) which together with almost transitivity of X imply that X is isometric to a Hilbert space. The first result that we want to mention here is due to J. Talponen, who generalized an earlier result of Randrianantoanina [124] that all real AT spaces that have a 1-complemented hyperplane are isometric to a Hilbert space.

Theorem 2.29 [136, Theorem 2.3] *Suppose that X is a real almost transitive Banach space and that for each $\varepsilon > 0, X$ contains a $(1+\varepsilon)$-complemented hyperplane. Then X is isometric to a Hilbert space.*

Another type of condition that is natural to consider is that the group $\mathrm{Isom}(X)$ acts almost transitively on some hyperplane on X. This by itself is not sufficient to conclude that X a Hilbert space, since Talponen [137] showed that the isometry group of L_1 acts almost transitively on the hyperplane $M = \{f \in L_1 : \int_0^1 f = 0\}$ (and leaves it invariant). Thus some additional conditions are necessary.

The results in the remaining part of this section are new, so we include their full proofs.

If $x_0 \in S_X$ then we define

$$\mathrm{Stab}_{x_0}(X) = \{T \in \mathrm{Isom}(X) : Tx_0 = x_0\}.$$

This is a closed subgroup of the isometry group, which under some natural hypotheses, acts on the hyperplane H_{x_0} generated by the norming functional of x_0.

We investigate the case where the stabilizers act almost transitively on the appropriate hyperplane and obtain a partial answer to Mazur rotations problem.

Recall that by a theorem of Mazur [52, Theorem 8.2], the norm is Gâteaux differentiable on a dense G_δ subset of S_X when X is a separable Banach space. Thus, if X is separable and transitive, then the norm is Gâteaux differentiable at every point of S_X and also strictly convex; see Sect. 1.7.

Note that the Gâteaux differentiability of the norm at some $x \in S_X$, supported by the (unique) normalized funcional ϕ, implies that the group $\mathrm{Stab}_x(X)$ leaves invariant the hyperplane $H_0 = \mathrm{Ker}\phi$: indeed from $Tx = x$ it is immediate to deduce $T^*\phi = \phi$ and therefore that H_0 is invariant.

Conversely, strict convexity implies the following:

Lemma 2.30 *Assume X is a real Banach space and the norm in strictly convex at x. Let ϕ be a support functional for x, and $H_0 = \mathrm{Ker}\phi$. If an isometry T satisfies $T(H_0) = H_0$, then $Tx = \pm x$.*

Proof Let $T \in \mathrm{Isom}(X)$ with $T(H_0) = H_0$. Since $\phi(y) = 0$ implies $T^*\phi(y) = 0$, there exists a scalar c so that $T^*\phi = c\phi = \pm\phi$. Therefore $\phi(Tx) = \pm 1$. Strict convexity implies that $Tx = \pm x$. □

Summing up, we may note that on a separable transitive space, $Tx = \pm x$ if and only if $T(H_0) = H_0$. Transitivity is however not needed for the next result:

Theorem 2.31 *Let X be an almost transitive real Banach space. Suppose that for some $x_0 \in S_X$ supported by ϕ, $\mathrm{Stab}_{x_0}(X)$ acts almost transitively on $S_{\mathrm{Ker}\phi}$. Then X is isometric to a Hilbert space.*

Proof By Theorem 2.29, it is enough to prove that the hyperplane $\mathrm{Ker}\phi$ is 1-complemented in X, that is, that the projection $P(z) \overset{\text{def}}{=} z - \phi(z)x_0$ has norm one.

Fix any $z \in S_X$ and let $\alpha = \phi(z)$. Since $\mathrm{Stab}_{x_0}(X)$ acts almost transitively on $S_{\mathrm{Ker}\phi}$, for every $\varepsilon > 0$ there exists an isometry $T_\varepsilon \in \mathrm{Stab}_{x_0}(X)$ so that $\|T_\varepsilon(z - \alpha x_0) - (-(z - \alpha x_0))\| \le \varepsilon$. Hence

$$\|T_\varepsilon(z) - (2\alpha x_0 - z)\| \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we get that $\|z - 2\alpha x_0\| = \|z\| = 1$. Thus

$$\|P(z)\| = \|z - \alpha x_0\| = \left\|\frac{1}{2}\big(z + (z - 2\alpha x_0)\big)\right\| \le \frac{1}{2}\big(\|z\| + \|z - 2\alpha x_0\|\big) \le 1,$$

which ends the proof. □

We finish this section with an observation that Theorem 2.31 implies in particular that for any $1 \le p < \infty$, $p \ne 2$, and for all $x_0 \in S_{L_p}$, the group $\mathrm{Stab}_{x_0}(L_p)$ does not act almost transitively on $S_{\mathrm{Ker}\phi}$. This is easy to see directly in the case when $x_0(t) = 1$ for all $t \in [0, 1]$.

Lemma 2.32 *If $1 \leqslant p < \infty, p \ne 2$, then $\mathrm{Stab}_1(L_p)$ does not act almost transitively on $M = \{f : \int_0^1 f = 0\}$.*

Proof Every isometry in $\mathrm{Stab}_1(L_p)$ is of the form $T(f) = f \circ \sigma$ where σ is a measure preserving automorphism of [0, 1]. Consider $h = 1_{[0,1/2)} - 1_{[1/2,1]} \in S_M$. Then for any $T \in \mathrm{Stab}_1(L_p)$ the measure of the support of Th is also equal to 1. Thus, if f is any function in S_M whose support has measure $\mu < 1$, we have

$$\|Th - f\|_p \geq \left(\int_{[0,1]\setminus \mathrm{supp}(f)} 1 \right)^{1/p} = (1-\mu)^{1/p}.$$

Therefore $\mathrm{Stab}_1(L_p)$ does not act almost transitively on S_M. □

Comments:

(1) Proposition 2.27 is also true when X is a subspace of the Schatten class $S^p(\ell_2)$, $1 < p < \infty$, $p \neq 2$ or of the non-commutative $L_p[0,1]$, $2 < p < \infty$, and when X is a stable Banach space that admits a C^2-smooth bump. However it is open whether it remains true when X is a subspace of L_p for $1 < p < 2$. In this case we only know that every subspace of X contains isomorphic (even almost isometric) copies of ℓ_2, see [44] for details.

(2) Notice that Lemma 2.32 is true also in the case when $p = 1$, despite the fact that, as we mentioned above, Talponen [137] showed that $\mathrm{Isom}(L_1)$ acts almost transitively on $S_M \subset S_{L_1}$. Talponen also showed that in this case for all $T \in \mathrm{Isom}(L_1)$ we have $T(M) = M$, but, of course, the conclusion of Lemma 2.30 does not hold.

(3) By the way, since every hyperplane of L_1 is isomorphic to the whole space, the result of Talponen [137] mentioned above provides another AT renorming of L_1, different from those described in Theorem 2.22.

3 Maximal norms and unitarisable representations on spaces isomorphic to Hilbert spaces

In this section we focus on Mazur rotations problem on a space already known to be linearly isomorphic to the Hilbert space. The results presented in this section are mainly from [56]. A way of understanding this concept is by considering the *G-invariant norms* corresponding to a bounded subgroup $G \leqslant \mathrm{GL}(X)$ on a Banach space X. In the language of representations, the question is to investigate the *invariant norms* for a representation of a group Γ, i.e. the norms for which the representation induces an action of Γ by isometries on X. Recall from Sect. 2 that if G is bounded, then

$$|||x||| = \sup_{T \in G} \|Tx\|$$

defines an equivalent G-invariant norm on X, i.e., G may be seen as a subgroup of $\mathrm{Isom}(X, |||\cdot|||)$. Moreover, if $\|\cdot\|$ is uniformly convex, then so is $|||\cdot|||$ (see, e.g., [27, Lemma 1.1]; or used more recently from another perspective, [15, Proposition 2.3]. However, if X is a Hilbert space and $\|\cdot\|$ is hilbertian, i.e., induced by an inner product, then $|||\cdot|||$ will not, in general, be hilbertian. The *unitarizability* problem therefore asks which bounded subgroups of $GL(\mathcal{H})$ admit invariant euclidean norms. It is a classical result of representation theory dating back to the beginning of the 20th century that if G is a bounded subgroup of $GL(\mathbb{C}^n)$, then there is a G-invariant inner product, or equivalently a G-invariant euclidean norm. B. Sz.-Nagy [134] showed that any bounded representation $\pi : \mathbb{Z} \to GL(\mathcal{H})$ is *unitarizable*, i.e., $\mathcal{H}$ admits an equivalent $\pi(\mathbb{Z})$-invariant inner product. This was extended by M. Day [42] and J. Dixmier [45] to any bounded representation of an amenable topological group, via averaging over an invariant mean.

In the opposite direction, the first example of a non-unitarizable bounded representation of a (necessarily non-amenable) group in the Hilbert space is due to L. Ehrenpreis and F. I. Mautner [50].

Since, by a result of A. J. Ol'šhanskiĭ [111], there are non-amenable countable groups which do not contain a copy of $\mathbb{F}_2$, it remains open whether the result of Sz.-Nagy, Day and Dixmier admits a converse.

Problem 3.1 (Dixmier's unitarizability problem) Suppose Γ is a countable group all of whose bounded representations on $\mathcal{H}$ are unitarizable. Is Γ amenable?

In [56] Ferenczi and Rosendal investigate the relation of certain non-unitarizable representations with the notions of maximality, almost transitivity, or transitivity of norms, through the following problem:

Problem 3.2 [Ferenczi-Rosendal, 2017] Find a non-unitarizable representation on the Hilbert space which admits an equivalent invariant maximal (resp. almost transitive, transitive) norm.

In the case of a positive answer, a maximal (resp. AT, transitive) non-hilbertian norm on $\mathcal{H}$ would be obtained, and the Hilbert space would admit non-conjugate maximal norms (see Problem 2.20). In the last case, there would exist a transitive, non-hilbertian norm on $\mathcal{H}$, and therefore a negative answer to the isometric version of Mazur rotations problem.

We focus here on a specific class of possibly non-unitarizable representations, which first appeared in [119]: triangular representations on a direct sum of two Hilbert spaces, where the diagonal elements of the matrix are unitary and where the upper right element is called a *derivation*.

Precisely, suppose that $\lambda : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation. A *bounded derivation* associated to λ is a uniformly bounded map $d : \Gamma \to \mathcal{B}(\mathcal{H})$ so that $d(gf) = \lambda(g)d(f) + d(g)\lambda(f)$ for all $g,f \in \Gamma$. This is simply equivalent to requiring that

$$\lambda_d(g) = \begin{pmatrix} \lambda(g) & d(g) \\ 0 & \lambda(g) \end{pmatrix}$$

defines a bounded representation of Γ on $\mathcal{H} \oplus \mathcal{H}$. The representation λ_d is unitarizable exactly when d is *inner*, i.e., $d(g) = \lambda(g)A - A\lambda(g)$ for some bounded linear operator A on $\mathcal{H}$ (a classical result whose proof may be found, e.g., in [56]).

Of course such a representation λ_d cannot be transitive or almost transitive, since it leaves the first summand invariant. Citing [56] this leads to the study of "bounded groups $G \leqslant \mathrm{GL}(\mathcal{H} \oplus \mathcal{H})$ containing $\lambda_d[\Gamma]$ for λ and d as above, which are potential examples of maximal non-unitarizable groups":

Proposition 3.3 ([56]) *Suppose that $\lambda : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of a group Γ on a separable infinite-dimensional Hilbert space $\mathcal{H}$ and let d be a bounded derivation associated to λ. Consider the assertions*

(1) *There is an almost transitive bounded subgroup G of $\mathrm{GL}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ containing $\lambda_d[\Gamma]$.*
(2) *There is a $\lambda_d[\Gamma]$-invariant norm on $\mathcal{H}_1 \oplus \mathcal{H}_2$ with moduli of convexity and smoothness of power type 2.*
(3) *There is a $\lambda_d[\Gamma]$-invariant norm on $\mathcal{H}_1 \oplus \mathcal{H}_2$ such that the $\mathcal{H}_1$-nearest point map $\mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1$ is well-defined and Lipschitz.*
(4) *There is a homogeneous Lipschitz map $\psi : \mathcal{H}_2 \to \mathcal{H}_1$ such that $d(a) = \lambda(a)\psi - \psi\lambda(a)$.*
(5) *The group $\lambda_d[\Gamma_z]$ is unitarizable for z outside of a Gauss null subset of $\mathcal{H}_2$, where $\Gamma_z = \{a \in \Gamma : \lambda(a)(z) = z\}$.*

Then $(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (4) \Longrightarrow (5)$.

Proof The idea of the proof is as follows. From (1) one deduces that the G-invariant norm $\sup_{g\in G} \|gx\|_2$ (which has modulus of convexity of power type 2) is a multiple of any given G-invariant norm on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. The same holds for the dual norm to the G-invariant norm $\sup_{g\in G} \|g^*\phi\|_2$, defined on the dual, and this norm has modulus of smoothness of power type 2. So there is a G-invariant (and in particular $\lambda_d[\Gamma]$-invariant) norm with both moduli of power type 2. The implication $(2) \Longrightarrow (3)$ follows from classical estimates relating the modulus of continuity of the nearest point to the moduli of convexity and smoothness, which appear in [23] as Theorem 2.8. Since the $\mathcal{H}_1$-nearest point map $n : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1$ is equivariant under translation by any vector in $\mathcal{H}_1$ and under isometries in $\lambda_d[\Gamma]$, it is given by the formula: $n(x, y) = x + \psi(y)$. The map $\psi(y) = n(0, y)$ is Lipschitz and the identity $d(a)(x) = \lambda(a)\psi(x) - \psi(\lambda(a)x)$ follows from the relation $n(T(x, y)) = T(n(x, y))$ for any $T = \lambda_d(a)$.

$(4) \Longrightarrow (5)$ Outside of a Gauss null set the map ψ is Gâteaux differentiable [23, Theorem 6.42]. Derivating the relation above for $a \in \Gamma_z$, $\psi'(z)$ witnesses that $d(a)$ is a linear derivation for the group $\lambda_d[\Gamma_z]$. □

It may be interesting to note here that geometric properties of general Banach spaces (such as uniform convexity or smoothness) are relevant even to the seemingly trivial case of a Hilbert space. For example, choosing to see a bounded group on $\mathcal{H}$ as an isometric group on some non-hilbertian renorming X of $\mathcal{H}$ allows to use relations of the nearest point map with convexity or smoothness of the norm of X.

Corollary 3.4 *Suppose that $\lambda : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of a group Γ on a separable Hilbert space $\mathcal{H}$ and $d : \Gamma \to \mathcal{B}(\mathcal{H})$ is an associated non-inner bounded derivation. Suppose that $G \leqslant \mathrm{GL}(\mathcal{H} \oplus \mathcal{H})$ is a bounded almost transitive subgroup containing $\lambda_d[\Gamma]$. Then there is a homogeneous Lipschitz non-linear map $\psi : \mathcal{H} \to \mathcal{H}$ defining the derivation by $d(a) = \lambda(a)\psi - \psi\lambda(a)$.*

The authors of [56] call *Lipschitz inner* a bounded derivation of the form $d(g) = [L, g]$, for L Lipschitz homogeneous on $\mathcal{H}$, and ask the following natural question [56, end of Section 3]:

Problem 3.5 Does there exist a Lipschitz inner derivation on $\mathcal{H}$ which is not inner?

It is unclear whether differentiability techniques may be used to obtain that every Lipschitz inner derivation is inner. Those techniques usually do not have any kind of invariance or equivariance with respect to the action of the isometry group and this seems to be an unsurmountable problem.

Comments:

(1) The survey [118] by G. Pisier and also [51, 109] contain the present state of affairs on Dixmier's problem.
(2) F. Cabello Sánchez [25] gives some partial restrictions on transitive renormings of the separable Hilbert space. Such renormings must be twice Gâteaux differentiable everywhere apart from zero, and the duality mapping must be Gâteaux differentiable everywhere apart from zero. For recent results regarding AT or transitivity of certain "Schatten restricted" renormings of the Hilbert space, see [107].
(3) It is a classical geometric problem in Banach space theory whether a (necessarily superreflexive) space admitting an equivalent norm with modulus of convexity of power type p, and another with modulus of smoothness of power type q, must admit an equivalent norm with both properties. Although such results hold for the LUR property, through Baire category methods on the set of equivalent norms, [43, Section II.4], the same method does not apply to uniformly convex norms. It was noticed by C. Finet [58] that this would hold if every superreflexive space admitted an equivalent almost transitive norm (indeed every almost transitive norm on a superreflexive space must have modulus of convexity of optimal power type). But this hope was shattered by the example of [55] and later by those of [44]. When $p = q = 2$ the question becomes trivial because of Kwapień's theorem [95]. However, given a bounded group G of automorphisms on the space, the version of this problem for G-invariant norms remains open, even for $p = q = 2$ (note that on the Hilbert space it is only relevant for non unitarizable groups):

Problem 3.6 Assume G is a bounded group of automorphisms on the Hilbert space $\mathcal{H}$, and that there exist G-invariant norms on $\mathcal{H}$ with modulus of convexity (resp. modulus of smoothness) of power type 2. Must there exist a G-invariant norm on $\mathcal{H}$ with these two properties?

4 Multidimensional Mazur problem

As a general principle, we wish to identify properties of Banach spaces which are stronger than transitivity, satisfied by Hilbert spaces, and for which however there exist non-hilbertian non-separable examples. Any positive solution to the Mazur rotations problem would need to solve the rotations problem associated to this stronger version of transitivity as a first step. The direction explored for this in this section is *multidimensionality*, and its results are mainly from the recent paper [54].

4.1 Ultrahomogeneity

Ultrahomogeneity (or *ultratransitivity*) of a Banach space is the multidimensional version of the transitivity property. The term "ultrahomogeneity" is closer to tradition in the Fraïssé theory and this explains the choice of this term in the paper [54], to which we adhere.

It is undisputable that all 1-dimensional spaces are mutually isometric. In higher dimensions, however, a global isometry on a space can only send a finite dimensional subspace onto another if those were isometric to begin with.

Definition 4.1 (Ultrahomogeneity) A Banach space X is said to be ultrahomogeneous (UH) when for every finite dimensional subspace $E \subset X$ every isometric embedding $E \to X$ extends to a global (surjective) isometry of X.

Less clearly, X is UH if for every finite dimensional $E \subset X$ the canonical action $\mathrm{Isom}(X) \curvearrowright \mathrm{Emb}(E, X)$ is transitive (see Sect. 1.2 for the definition of $\mathrm{Emb}(E, X)$).

Note that any ultrahomogeneous space (or norm) is in particular transitive. As a consequence of the existence of orthogonal complements in Hilbert spaces:

Fact 4.2 *Hilbert spaces are ultrahomogeneous.*

Problem 4.3 (Multidimensional Mazur problem) Is every separable infinite dimensional ultrahomogeneous Banach space isometric (or isomorphic) to the Hilbert space?

Similarly as for the one-dimensional problem, this leads quite naturally to two separate questions namely: Is every separable UH Banach space isomorphic to a Hilbert space? Is every UH renorming of a Hilbert space Euclidean?

Is there any other (nonseparable) UH space in sight? Yes, ultrapowers of the Gurariy space or of L_p-spaces for appropriate values of p, with respect to CI ultrafilters. See below.

Comments:

The two-dimensional part of the UH property (that any isometric embedding of a two-dimensional subspace extends to a surjective isometry) is not to be confused with the notion of 2-transitivity (whenever x, y, x', y' belong to the sphere, and $d(x, y) = d(x', y')$, then there exists a surjective isometry sending x to x' and y to y'). The second one is much stronger and already implies that the space is isometrically hilbertian (no separability needed): Ficken [57] proved that if for all $x, y \in S_X$ there exists $T \in \mathrm{Isom}(X)$ with $T(x) = y$ and $T(y) = x$, then X is isometric to a Hilbert space, see also [6, Condition 2.8].

4.2 Approximate ultrahomogeneity

Let us introduce the following "approximate" version of UH, taken from [54]:

Definition 4.4 A Banach space X is called approximately ultrahomogeneous (AUH) when for every finite dimensional subspace E of X, every isometric embedding $u : E \to X$ and every $\varepsilon > 0$ there is an isometry U of X such that $\|u|_E - U\| < \varepsilon$.

Thus X being AUH exactly means that the canonical action of $\mathrm{Isom}(X)$ on $\mathrm{Emb}(E, X)$ is *almost transitive* (i.e. has dense orbits), where $\mathrm{Emb}(E, X)$ is equipped with the metric induced by the operator norm; informally, this means that any partial isometry between finite dimensional subspaces can be well approximated by a global isometry.

The following sums up the known examples of separable, non hilbertian, AUH spaces.

Theorem 4.5 *The following spaces are AUH, but not UH:*

(a) *The Gurarij space* $\mathcal{G}$.
(b) L_p, *for* $p \neq 2, 4, 6, 8, \ldots$

The AUH character of $\mathcal{G}$ is a relatively recent result by Kubiś and Solecki [94]. The Gurariy space is the only universal, separable AUH Banach space. The part concerning L_p spaces was essentially established by Lusky in the late 1970s [102] elaborating on the *Plotkin/Rudin equimeasurability theorem*. It is clear that these spaces cannot be UH because they are not even transitive. It is a remarkable fact that the L_p spaces, for $p = 4, 6, \ldots$ fail to be AUH. This follows from work of Randrianantoanina [123] who, as part of an answer to a question of H.P. Rosenthal

[127], showed that those spaces contain isometric copies of certain finite dimensional spaces with very different projection constants; see details in [54]. This is quite surprising since for $p \in (1, \infty)$ different from 2, the groups $\mathrm{Isom}(L_p)$ are all topologically isomorphic to each other, including p even, both in the SOT and in the norm topology. However, their canonical actions on $\mathrm{Emb}(E, L_p)$ turn out to have very different properties, depending on whether or not p is an even integer.

The Garbulińska space $\mathcal{K}$ described in Sect. 1.5 provides a more "canonical" example of an AT space which is not AUH. This can be seen as follows. Every Banach space with a skeleton (in particular, a finite dimensional one) is isometric to a 1-complemented subspace of $\mathcal{K}$. This applies to $\ell_\infty^{2^n}$ and ℓ_1^n. But $\ell_\infty^{2^n}$ contains an isometric copy of ℓ_1^n whose projection constant is large (let us be foolhardy: it is exactly $\frac{2m+1}{2^{2m}}\binom{2m}{m}$, where m is the integer part of $\frac{1}{2}(n-1)$, proved by Grünbaum in 1960, [74]). Thus $\mathcal{K}$ contains well- and bad-complemented subspaces isometric to ℓ_1^n so that it canot be AUH, and neither can its ultrapowers.

As we already mentioned, there exist non-separable ultrahomogeneous spaces. A method of finding them used in [13, Chapters 3 and 4], and then in [54], has been to investigate weaker forms of transitivity of separable spaces, with the objective of then taking ultrapowers. What catches us off-guard is that the AUH of a Banach space does not automatically imply UH of its ultrapowers. The reason for this is that, in general, an isometric embedding $u : E \to [X_i]_{\mathcal{U}}$ can arise from a family of ε_i-isometric embeddings $u_i : E \to X_i$ with $\varepsilon_i \to 0$ along $\mathcal{U}$.

Nevertheless, the Gurariy space, being separable and of almost universal disposition, has the following "perturbed" version of UH that is much easier to establish than AUH and was known to Gurariy himself:

Lemma 4.6 *Let $u : E \to F$ be a δ-isometry acting between two finite dimensional subspaces of $\mathcal{G}$. Then, for every $\varepsilon > \delta$ there is a surjective ε-isometry U of $\mathcal{G}$ extending u.*

Curiously enough, no isometry *sensu stricto* is involved in the preceding statement. As a consequence we have the following result [13, Proposition 4.16]:

Proposition 4.7 *Ultrapowers of the Gurariy space built on countably incomplete ultrafilters are ultrahomogeneous.*

The density character of any such space is at least the continuum; we do not know if there are examples whose density character is $\aleph_1$; see Sect. 1.5 *and* the comments around [35, Proposition 4.2].

And what about ultrapowers of L_p? Keep reading.

4.3 Fraïssé Banach spaces

One of the main technicalities of the definition of a Fraïssé Banach space from [54] is that it is expressed in terms of the canonical actions of the linear isometry group, not only on the spaces $\mathrm{Emb}(E, X)$ of isometric embeddings, but also on $\mathrm{Emb}_\delta(E, X)$, the class of δ-isometric embeddings, which is equipped with the distance induced by the norm. As the reader may guess, the canonical action $\mathrm{Isom}(X) \curvearrowright \mathrm{Emb}_\delta(E, X)$ is defined by $(g, T) \mapsto g \circ T$. Also, the action of a subgroup G of $\mathrm{Isom}(X)$ on $\mathrm{Emb}_\delta(E, X)$ is said to be *ε-transitive* if for any $T, U \in \mathrm{Emb}_\delta(E, X)$, there exists $g \in G$ such that $\|g \circ T - U\| \leqslant \varepsilon$.

Following a terminology inspired by the Fraïssé theory (but without using the abstract setting of model theory which is common in the general Fraïssé theory), given a Banach space X, we denote by $\mathrm{Age}\,(X)$ the set of all finite dimensional subspaces of X, and by $\mathrm{Age}_k(X)$ the set of its k-dimensional subspaces. Our presentation of the results of [54] is slightly modified to point out the role of the dimension.

Definition 4.8 (Ferenczi, López-Abad, Mbombo, Todorcevic [54]) Let $k \in \mathbb{N}$. A Banach space X is k-Fraïssé if and only if for every $\varepsilon > 0$ there is $\delta = \delta_k(\varepsilon) > 0$ such that for every $E \in \mathrm{Age}_k(X)$, the action $\mathrm{Isom}(X) \curvearrowright \mathrm{Emb}_\delta(E, X)$ is ε-transitive. A Banach space X is Fraïssé if and only if it is k-Fraïssé for every $k \in \mathbb{N}$.

Since isometric embeddings are δ-isometric for any $\delta > 0$, Fraïssé $\Longrightarrow$ (AUH). We pass to an important characterization of the Fraïssé property indicating that the possibility of choosing δ uniformly on subspaces of dimension k is related to the closedness of $\mathrm{Age}_k(X)$ in the Banach-Mazur compactum.

Definition 4.9 A space X is weak k-Fraïssé if and only if for every $E \in \mathrm{Age}_k(X)$ and every $\varepsilon > 0$, there is $\delta = \delta_E(\varepsilon) > 0$ such that the action $\mathrm{Isom}(X) \curvearrowright \mathrm{Emb}_\delta(E, X)$ is ε-transitive.

The following is proved in [54, proof of Theorem 2.12]:

Lemma 4.10 *The following are equivalent for X Banach and* $k \in \mathbb{N}$:

(1) *X is* k-Fraïssé,
(2) *X is weak k-Fraïssé and* $\mathrm{Age}_k(X)$ *is compact in the Banach–Mazur (pseudo) distance.*

And therefore

Proposition 4.11 *For a Banach space X the following are equivalent*:

(1) *X is Fraïssé*,
(2) *X is weak Fraïssé and for all* $k \in \mathbb{N}$, $\mathrm{Age}_k(X)$ *is compact in the Banach–Mazur (pseudo) distance*.

Comments:

Given a (hereditary) class $\mathscr{F}$ of finite (or sometimes finitely generated) structures, the *Fraïssé theory* (Fraïssé 1954, [63]) investigates the existence of a countable structure $\mathcal{A}$, universal for $\mathscr{F}$ and *ultrahomogeneous* (any isomorphism between finite substructures extends to a global automorphism of $\mathcal{A}$). The "Fraïssé correspondence" shows that this is equivalent to certain "amalgamation properties" of $\mathscr{F}$. In this case $\mathcal{A}$ is unique up to an isomorphism and called the Fraïssé limit of $\mathscr{F}$. Analogies of this situation with the ultrahomogeneity properties of Banach spaces considered in their paper led to the use of the Fraïssé terminology in [54].

4.4 Examples of Fraïssé Banach spaces

As expected, the list of usual suspects provides examples of Fraïssé spaces:

Theorem 4.12 *The following Banach spaces are Fraïssé*:

(a) *Hilbert spaces* (*with* $\varepsilon = \delta$),
(b) *the Gurariy space* (*with* $\varepsilon = 2\delta$),
(c) *The spaces* L_p *for finite* $p \neq 4, 6, 8, \dots$

However L_p *is not Fraïssé for* $p = 4, 6, 8, \dots$ *since is not AUH.*

Part (a) is very easy: it consists in showing that every δ-isometric embedding between finite dimensional Hilbert spaces is at distance δ from a true isometric embedding, see [54, Example 2.4] for details. Part (b) is due to Kubiś and Solecki [94, Theorem 1.1]. Part (c) is a recent result by Ferenczi, López-Abad, Mbombo and Todorcevic [54, Theorem 4.1].

Comments:

(1) Citing Lusky [102], *"We show that a certain homogeneity property holds for* $L_p; p \neq 4, 6, 8, \dots$ *which is similar to a corresponding property of the Gurariy space..."* The Fraïssé Banach space definition gives a more precise meaning to this similarity.
(2) The proof of Theorem 4.12(c) is quite technical and will not be presented here. It is based on proving an *approximate equimeasurability principle*, a continuous statement extending the classical equimeasurability principle of Plotkin and Rudin, see [54, Section 4.2]. This result implies a local statement about extension of almost isometric embeddings which is equivalent to the weak Fraïssé property for L_p. The other ingredient is the classical fact from the theory of L_p

-spaces, that $\mathrm{Age}_k(L_p)$ is compact in the Banach-Mazur distance. One then concludes the proof by Proposition 4.11.

(3) An optimal estimative of the values of $\delta_k(\epsilon)$ appearing in the Fraïssé property for the space L_p remains to be computed. In particular, it is unclear whether δ could be chosen uniformly in k, see the next item.

(4) The estimates obtained on δ in the cases of the Hilbert space and the Gurariy space witness that δ may be chosen independently of the dimension of the subspace E. This leads to the following definition, see [54, p. 5], as well as [100] in a much more general context: a Banach space X is *stable Fraïssé* if for every $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that for every $E \in \mathrm{Age}\,(X)$, the action $\mathrm{Isom}(X) \curvearrowright \mathrm{Emb}_\delta(E, X)$ is ε-transitive. Please note that the meaning of the adjective "stable" here has nothing to do with stable spaces in the sense of Krivine-Maurey. The following natural question is open:

Problem 4.13 Are the spaces L_p, for finite $p \neq 4, 6, 8 \ldots$, stable Fraïssé?

4.5 Embeddings and isometries between Fraïssé spaces

The following properties of Fraïssé Banach spaces may be thought of as natural counterparts to their "exact" equivalent statements in the Fraïssé theory, relating an ultrahomogeneous countable structure to its finite parts.

Recall that a space Y is finitely representable in X if for any finite dimensional subspace E of Y and any $\varepsilon > 0$, there exists a finite dimensional subspace F of X such that $d_{\mathrm{BM}}(E, F) < 1 + \varepsilon$. This is a basic notion of local theory of Banach spaces, which aims to compare the finite dimensional structures of spaces "up to arbitrarily small perturbation".

Proposition 4.14 *Assume X is Fraïssé, and that Y is separable. Then the following are equivalent*:

(1) *Y is finitely representable in X.*
(2) *Every finite dimensional subspace of Y embeds isometrically in X.*
(3) *Y embeds isometrically in X.*

Therefore embeddings into Fraïssé spaces are exactly prescribed by the natural order relation between the respective local structures. In particular, by the Dvoretzky's theorem about finite representability of the Hilbert space in infinite dimensional Banach spaces (cf. [65]), all Fraïssé spaces must contain an isometric copy of the Hilbert space $\mathcal{H}$:

Proposition 4.15 *The Hilbert space is the minimal separable Fraïssé space.*

Let for $\mathscr{F}, \mathscr{G}$ classes of finite dimensional spaces, $\mathscr{F} \equiv \mathscr{G}$ mean that any element of $\mathscr{F}$ has an isometric copy in $\mathscr{G}$ and conversely. By means of a back-and-forth argument it is proven in [54, Proposition 2.22 and Theorem 2.19] that separable AUH (resp. Fraïssé) spaces are uniquely isometrically determined (among spaces with the same property) by their age modulo $\equiv$ (resp. by their local structure). Precisely:

Proposition 4.16 *Assume X and Y are separable AUH spaces. Then the following are equivalent:*

(1) $\mathrm{Age}(X) \equiv \mathrm{Age}(Y)$,
(2) *X and Y are isometric.*

If furthermore X and Y are assumed to be Fraïssé spaces, then (1)&(2) are also equivalent to

(3) *X is finitely representable in Y and vice-versa.*

A consequence of Proposition 4.16 is that any separable Fraïssé space which does not have non-trivial cotype must be isometric to the Gurariy space. Indeed, ℓ_∞ is finitely representable in such a space, and therefore condition (3) of Proposition 4.16 may be applied.

4.6 Internal characterizations: amalgamation

In [54] are also obtained internal characterizations of those classes of finite dimensional spaces corresponding to the age of some Fraïssé space ("amalgamation properties").

Definition 4.17 A class $\mathscr{F}$ of finite dimensional spaces has the Fraïssé amalgamation property if whenever $E, F, G \in \mathscr{F}$ with $\dim E = k$, and $\gamma \in \mathrm{Emb}_\delta(E, F)$, $\eta \in \mathrm{Emb}_\delta(E, G)$, there are $K \in \mathscr{F}$ and isometric embeddings $i : F \to K$ and $j : G \to K$ such that $\|i \circ \gamma - j \circ \eta\| \leq \varepsilon$.

It is not hard to check that the age of a Fraïssé Banach space must satisfy the Fraïssé amalgamation property. Conversely and more importantly, the amalgamation property is equivalent to the existence of an associated Fraïssé space X, which, by Proposition 4.16, in the separable case, is uniquely determined.

Definition 4.18 For a class $\mathscr{F}$ with the Fraïssé amalgamation property, there exists an isometrically unique separable Fraïssé space X such that $\mathrm{Age}(X) \equiv \mathcal{F}$. In this case it is said that X is the *Fraïssé limit of* $\mathscr{F}$.

Question 4.19 Are there other examples of amalgamation classes, apart from the classes of finite dimensional subspaces of L_p for $p \neq 4, 6, 8 \ldots$, or the class of all finite dimensional normed spaces?

It may be that a hypothetical new separable Fraïssé space will appear not as a "preexisting space" such as the L_p's but rather as a "new space" defined as the limit of a new amalgamation class.

4.7 Fraïssé is an ultraproperty

In the same spirit as the relation between AT and transitivity in Sect. 1, there exist characterizations of the Fraïssé property through ultrapowers. This point of view allows for formulations of the Fraïssé property without use of epsilontics.

Given an ultrafilter $\mathcal{U}$, denote by $(\mathrm{Isom}(X))_{\mathcal{U}}$ the subgroup of isometries of $X_{\mathcal{U}}$ of the form $(T_i)_{\mathcal{U}}$, where each $T_i \in \mathrm{Isom}(X)$. Note that $(T_i)_{\mathcal{U}}$ is (correctly) defined on $X_{\mathcal{U}}$ by $(T_i)_{\mathcal{U}}[(x_i)] = [(Tx_i)]$.

We state here k-dimensional versions of some general properties proved in [54].

Proposition 4.20 *Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. For a Banach space X and $k \geqslant 1$ the following are equivalent:*

(1) *X is weak k-Fraïssé (respectively, X is k-Fraïssé).*
(2) *For every $E \in \mathrm{Age}_k(X)$ (respectively, for every $E \in \mathrm{Age}_k(X_{\mathcal{U}})$), the action $(\mathrm{Isom}(X))_{\mathcal{U}} \curvearrowright \mathrm{Emb}(E, X_{\mathcal{U}})$ is almost transitive.*
(3) *For every $E \in \mathrm{Age}_k(X)$ (respectively, for every $E \in \mathrm{Age}_k(X_{\mathcal{U}})$), the action $(\mathrm{Isom}(X))_{\mathcal{U}} \curvearrowright \mathrm{Emb}(E, X_{\mathcal{U}})$ is transitive.*

Note the difference between finite dimensional subspaces of X and finite dimensional subspaces of $X_{\mathcal{U}}$. By classical results of local theory and ultraproducts, the elements of $\mathrm{Age}\,(X_{\mathcal{U}})$ are exactly those belonging to the closure of the set $\mathrm{Age}\,(X)$ with respect to the Banach-Mazur distance. As an illustration, every finite dimensional subspace of L_p is a limit (in the Banach-Mazur distance) of a sequence of finite dimensional subspaces of ℓ_p; $\mathrm{Age}\,(L_p)$ is closed but $\mathrm{Age}\,(\ell_p)$ is not.

Several equivalent characterizations of the Fraïssé property appear in [54] and follow essentially from Proposition 4.20. Informally and under some restrictions, we see that UH, AUH and the Fraïssé property induced by isometries on the space, become indistinguishable in its ultrapowers.

Proposition 4.21 *The following are equivalent for a Banach space X:*

(1) *X is Fraïssé.*
(2) *The action $(\mathrm{Isom}(X))_{\mathcal{U}} \curvearrowright \mathrm{Emb}(E, X_{\mathcal{U}})$ is almost transitive for every $E \in \mathrm{Age}\,(X_{\mathcal{U}})$.*

(3) *The action* $(\mathrm{Isom}(X))_{\mathcal{U}} \curvearrowright \mathrm{Emb}(E, X_{\mathcal{U}})$ *is transitive for every* $E \in \mathrm{Age}(X_{\mathcal{U}})$.
(4) *The action* $(\mathrm{Isom}(X))_{\mathcal{U}} \curvearrowright \mathrm{Emb}(Z, X_{\mathcal{U}})$ *is transitive for every separable* $Z \subset X_{\mathcal{U}}$.
(5) $X_{\mathcal{U}}$ *is Fraïssé and* $(\mathrm{Isom}(X))_{\mathcal{U}}$ *is SOT-dense in* $\mathrm{Isom}(X_{\mathcal{U}})$

Corollary 4.22 *The non-separable* L_p-space $(L_p)_{\mathcal{U}}$ *is ultrahomogeneous for each* $p \in [1, \infty)$ *different from* $4, 6, 8 \ldots$

Comments:

(1) The proof of Proposition 4.21 in [54] essentially follows the argument given by Avilés, Cabello Sánchez, Castillo, González and Moreno in [13, Section 4.3.3]. A natural version of Corollary 4.22 for the Gurariy space was also obtained by these authors [13, Proposition 4.16]. The spaces in Corollary 4.22 seem to be the only known super-reflexive ultrahomogeneous examples (if $p > 1$). The existence of *separable* ultrahomogeneous spaces other than the Hilbert space still remains unknown, *cf.* Problem 4.3.
(2) In Corollary 2.24(a) we observed that when $p \neq 2$, $(L_p)_{\mathcal{U}}$ admits infinitely many non-isometric transitive renormings. However, while $(L_p)_{\mathcal{U}}$ is ultrahomogeneous in its natural norm by Corollary 4.22, it is not with respect to the norms transferred from $(L_p(\ell_2^n))_{\mathcal{U}}$ for $n \geqslant 2$. Indeed these spaces admit both a 1-complemented isometric copy of ℓ_2^2 and another which is not 1-complemented (the copy induced by an isometric embedding of ℓ_2^2 into L_p, whose best constant of complementation is computed in [67, 68]: relapsing into bad habits let us add that it is exactly $\sqrt{\Gamma(1/2)\Gamma(p/2+1)/\Gamma(p/2+1/2)}$). It is not known whether or not there exist spaces with two or more UH renormings.

4.8 Local versions of the Fraïssé property

If one wants to deduce some properties of the isometry group $\mathrm{Isom}(L_p)$ from combinatorial properties of embeddings between subspaces of L_p, general subspaces of L_p do not seem easy to handle. Auspiciously, and not unexpectedly, a lot can be said on the general structure of the space L_p just from its finite-dimensional ℓ_p-subspaces. In this direction we recall a result of Schechtman [130] (and Dor [46] for $p = 1$) - as observed by Alspach [5].

Theorem 4.23 (Dor–Schechtman) *Let* $1 \leqslant p < \infty$ *be fixed. For every* $\varepsilon > 0$ *there exists* $\delta > 0$, *depending only on* ε *and* p, *so that for every* n *and every* δ-*isometry* $u : \ell_p^n \to L_p$, *there is an isometric embedding* $\tilde{u} : \ell_p^n \to L_p$ *such that* $\|u - \tilde{u}\| < \varepsilon$.

In other words, a form of the Fraïssé property in L_p is satisfied when “restricted” to subspaces of L_p isometric to an ℓ_p^n (including $p = 4, 6, 8 \ldots$). As commented earlier there is no hope to extend this to general finite dimensional subspaces when $p = 4, 6, 8 \ldots$

With this example in mind, it is possible and useful to develop a Fraïssé theory with respect to restricted classes of finite dimensional subspaces, which are not the age of any X, because they are not hereditary. In this sense this may be called a "local version" of the Fraïssé theory for Banach spaces. Informally, given a class $\mathscr{F}$ of finite dimensional Banach spaces, the $\mathscr{F}$-Fraïssé spaces are those for which the natural actions on δ-embeddings are ε-transitive, provided that the embeddings have as domain an element of $\mathscr{F}$. For L_p the authors of [54] use Theorem 4.23 to give meaning to the affirmation:

Theorem 4.24 *For any* $p \in [1, \infty)$, *even or not*, L_p *is the Fraïssé limit of the class* $(\ell_p^n)_n$.

4.9 Fraïssé and extreme amenability

Recall from Sect. 1.2 that a topological group G is called extremely amenable (EA) when every continuous action $G \curvearrowright K$ on a compact K has a fixed point. The Fraïssé theory is related to this notion through the celebrated *KPT correspondence* (Kechris/Pestov/Todorcevic 2005 [92]), a combinatorial characterization of the extreme amenability of an automorphism group in terms of a Ramsey property of Age: as a beautiful example, Pestov's result that the group of order preserving automorphisms of the rationals is extremely amenable [116] may be seen as combination of "$(\mathbb{Q}, <)$ is the Fraïssé limit of finite ordered sets" and of the classical finite Ramsey theorem on $\mathbb{N}$. The authors of [54] use a form of the KPT correspondence for metric structures which applies without difficulty to the isometry group of a Fraïssé, or even AUH, Banach space X.

Definition 4.25 A collection $\mathscr{F}$ of finite dimensional normed spaces has the Approximate Ramsey Property (ARP) when for every $F, G \in \mathscr{F}$ and $r \in \mathbb{N}, \varepsilon > 0$ there exists $H \in \mathscr{F}$ such that every coloring c of $\mathrm{Emb}(F, H)$ into r colors admits an embedding $\rho \in \mathrm{Emb}(G, H)$ which is ε-monochromatic for c: there exists a color i so that for all $u \in \mathrm{Emb}(F, G)$ there is $v \in \mathrm{Emb}(F, H)$ such that $c(v) = i$ and $\|v - \rho u\| \leqslant \varepsilon$.

Theorem 4.26 (KPT correspondence for Banach spaces) *For an AUH Banach space X the following are equivalent*:

(1) $\mathrm{Isom}(X)$ i*s extremely amenable in the SOT*.
(2) Age (X) *has the approximate Ramsey property*.

The KPT correspondence turns out to extend to the setting of ℓ_p^n-subspaces of L_p. This means that one can expect to prove the extreme amenability of $\mathrm{Isom}(L_p)$ through internal properties, i.e. through an approximate Ramsey property of the class of isometric embeddings between ℓ_p^n's. This expectation was fulfilled for $p = \infty$ in [18], and then for $1 \leqslant p < \infty, p \neq 2$, in [54]:

Theorem 4.27 (Ramsey theorem for embeddings between ℓ_p^n's) *Given* $1 \leqslant p \leq \infty, p \neq 2$, *integers* d, m, r, *and* $\varepsilon > 0$ *there exists* $n = n_p(d, m, r, \varepsilon)$ *such that whenever c is a coloring of* $\mathrm{Emb}(\ell_p^d, \ell_p^n)$ *into r colors, there exists some isometric embedding* $\gamma : \ell_p^m \to \ell_p^n$ *which is* ε*-monochromatic.*

The proofs of these two results are quite complex and beyond the scope of this survey. The proof obtained in [54] for $p \neq 2$, as well as the estimates on $n_p(d, m, r, \varepsilon)$ that would follow from it (but are not computed by the authors), do not extend to the Hilbert space, due to the different nature of isometric embeddings between finite dimensional subspaces in this case. Theorem 4.27 is still valid for $p = 2$, but as a consequence of Theorem 4.26 and of Gromov–Milman's result [65] about extreme amenability of the unitary group, see the following comments.

The Fraïssé spaces encountered in [17, 18, 54] were known to have extremely amenable isometry groups when equipped with the strong operator topology:

Example 4.28 The isometry group of L_p, for any $1 \leqslant p < \infty, p \neq 2$, and the isometry group of $\mathcal{G}$ are extremely amenable in the SOT.

The extreme amenability of $\mathrm{Isom}(L_p)$ for $p \in [1, \infty), p \neq 2$ was proved in 2006 by Giordano and Pestov [66], and the methods of [54] allow to recover this result through Theorem 4.27 and the KPT correspondence for Banach spaces. In any case this statement refers to one group only because Choksi and Kakutani proved long time ago that the groups $\mathrm{Isom}(L_p)$ are all topologically isomorphic in the SOT for $p \in [1, \infty)$, see [39, Theorem 8].

The extreme amenability of $\mathrm{Isom}(\mathcal{G})$ is a recent result by Bartosová, López-Abad, Lupini, and Mbombo [17, 18], and may be seen as a corollary of the combination of the KPT correspondence and Theorem 4.27 for $p = \infty$.

When $p = 2$ the isometry group of L_p is the unitary group whose extreme amenability was established in 1983 by Gromov and Milman [73].

The KPT correspondence for Banach spaces also implies new results for some non-separable versions of those spaces. As a consequence of Gromov–Milman's result and of Theorem 4.26, the unitary group of any infinite dimensional Hilbert space is extremely amenable in the SOT, regardless of its density character. From Theorem 4.26, the result that L_p, $1 \leqslant p < \infty$ is Fraïssé, and Proposition 4.21, we also have:

Example 4.29 For $1 \leqslant p < \infty, p \neq 2$ the isometry group of any ultrapower of L_p with respect to a free ultrafilter on the integers is extremely amenable in the SOT.

Comments:

(1) The Gromov–Milman [73] result of extreme amenability of $\mathcal{U}(\mathcal{H})$ is based on the concentration of measure phenomenon. The result of Giordano-Pestov [66] for L_p also uses concentration of measure and a general description of $\mathrm{Isom}(L_p)$

as a topological group. In [54] and for p not even this may be recovered by the above considerations through the fact that L_p is AUH and through the Ramsey property; for p even, the local version of the Ramsey property, Theorem 4.27, needs to be used. The extreme amenability of the isometry group of the Gurariy space relies on its AUH property and the Ramsey property for embeddings between finite dimensional spaces (or equivalently, between ℓ_∞^n-spaces).

(2) There are some precursors of the Ramsey result Theorem 4.27. In [110], Odell-Rosenthal-Schlumprecht (1993) proved that that for every $1 \le p \le \infty$, every $m, r \in \mathbb{N}$ and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that for every coloring c of the unit sphere of ℓ_p^n into r colors there is $Y \subset \ell_p^n$ isometric to ℓ_p^m so that S_Y is ε -monochromatic. Note that Odell-Rosenthal-Schlumprecht is the case $d = 1$ in Theorem 4.27. Matoušek-Rödl (1995) [104] gave a combinatorial proof of the [110] result for $1 \le p < \infty$.

(3) The authors of [54] also develop a Fraïssé theory by restricting the type of embeddings, for example by analyzing lattices, where now isometries and embeddings (resp. δ-embeddings) must respect (resp. maybe up to ε) the lattice structure. In this manner Fraïssé Banach lattices, i.e. certain unique universal objects for classes of finite dimensional lattices with an approximate lattice ultrahomogeneity property, are defined [54, Definition 6.1].

For example, L_p is a Fraïssé Banach lattice for $p \in [1, \infty)$ which is the "lattice Fraïssé" limit of its finitely generated sublattices, clearly isometric to the corresponding ℓ_p^n. For $p = \infty$ they define a new object which they call the "Gurariy M-space", proving that there exists a renorming of $C[0, 1]$ as an M-space which is the lattice Fraïssé limit of the class of ℓ_∞^n's finite lattices.

(4) The "Gurariy M-space" cited in the previous item is inspired by a couple of constructions from [28]; namely an M-space, which is transitive and easier to define, albeit non-separable (the ultraproduct of the lattices L_p for p tending to ∞) and a family of separable AT M-spaces some of which (all?) might be isometric to the "Gurariy M-space" ... or not. Avilés and Tradacete [14] and M.A. Tursi [139] recently and independently studied amalgamation properties for Banach lattices: Avilés and Tradacete constructed a (necessarily non-separable) Banach lattice of universal disposition for separable lattices. Tursi's paper contains, among other things, the construction of a separable approximately ultrahomogeneous Banach lattice. In the even more recent [89], Kawach and López-Abad study amalgamation and Fraïssé properties for Fréchet spaces.

5 Questions and problems

In this final section we gather and discuss a number of problems that arise naturally from the contents of the survey. We have classified them according to the topics covered in the preceding sections, although the borders are quite permeable.

Local questions, ultraproducts, finite dimensional objects It is clear (use the $\sqrt{\dim}$ estimate for the ellipsoid of minimal volume or an ultraproduct argument) that for each finite n there is a function $f_n : [0,2] \to [1,\infty)$ with $f_n(\delta) \to 1$ as $\delta \to 0$ so that if E is n-dimensional and δ-transitive, then $d_{\mathrm{BM}}(E, \ell_2^n) \leqslant f_n(\delta)$. See the comments closing Sect. 1.5.

5.1 Can the hypothesis on the dimension be removed? That is, is it true that for every $\varepsilon > 0$ there exist $\delta > 0$ so that every finite dimensional δ-transitive (or δ -asymptotically transitive) space is $(1+\varepsilon)$-close to the Hilbert space of the corresponding dimension?

An obvious ultraproduct argument in combination with Theorem 2.29 shows that the answer is affirmative if we moreover require the existence of a $(1+\delta)$-complemented hyperplane.

Banach spaces that arise as ultraproducts of families of finite dimensional ones are called *hyperfinite* in nonstandard ambients, see for instance [79]. A couple of closely related questions are:

5.2 Is every hyperfinite transitive (or even ultrahomogeneous) space (isometric or isomorphic to) a Hilbert space?

5.3 (Henson and Moore [79, Problem 5], Plichko) Do hyperfinite spaces of universal disposition exist?

5.4 (F. Cabello Sánchez) Let X be a space with an (almost) transitive norm and which admits a non trivial finite-dimensional isometry. Must X be hilbertian?

The answer is affirmative if the hypothesized finite-dimensional isometry is a rank-one perturbation of the identity (see [19, Section 3]). Also, by [55, Corollary 4.14], if X is separable, reflexive, and satisfies the hypotheses in Problem 5.4, then X must have a Schauder basis. With an eye on Theorem 2.29 we can ask:

5.5 (F. Cabello Sánchez) Let X be a space with an (almost) transitive norm and which admits a 1-complemented subspace of finite codimension greater than 1. Must X be hilbertian?

Let $\mathrm{GL}_f(X)$ denote the group of automorphisms of X that have the form $\mathbf{I}_X + F$ where F is a finite-rank operator. [55, Problem 8.11] asked to find a separable space X and a bounded subgroup of $\mathrm{GL}_f(X)$ which is infinite and discrete in the SOT. This was solved in [7] with an example on c_0. The question remains in reflexive spaces:

5.6 Find a separable reflexive space X and a bounded subgroup of $\mathrm{GL}_f(X)$ which is infinite and discrete in the SOT.

In the same vein we ask:

5.7 If X is separable reflexive and a bounded subgroup G of $\mathrm{GL}_f(X)$ is discrete in the SOT, does it imply that all orbits of the action of G on the sphere are discrete? Or at least, not dense in S_X?

Maximality of the norm, renormings Not surprisingly, the hottest issue in this line is about norms on Hilbert spaces:

5.8 (Sect. 2.4, Problem 2.20)

(a) Does the Hilbert space have a unique, up to conjugacy, maximal bounded subgroup of automorphisms?
(b) Does there exist a separable Banach space X with a unique, up to conjugacy, maximal bounded subgroup of $\mathrm{GL}(X)$?
(c) If yes, does X have to be isomorphic to a Hilbert space?

Note that, while the isometric part of Mazur problem asks whether every (almost) transitive renorming of a Hilbert space is Euclidean, Part (a) is asking if this is true even for maximal renormings. Concerning the possible impact that the existence of AT norms can have regarding the isomorphic structure of the underlying space:

5.9 ([55, Problem 8.14]) Let X be a separable, reflexive, Banach space with an AT norm. Does it follow that X has a Schauder basis?

This was originally asked for CT norms. However, as we already mentioned, CT and AT are equivalent notions for reflexive spaces and imply uniform convexity and uniform smoothness of the norm; cf. [19, Corollary 6.9]. Without the hypothesis of reflexivity the answer is no in view of Lusky's [103]. By [55, Corollary 4.14], the answer is affirmative when there exists a power bounded operator in $\mathrm{GL}_f(X)$.

5.10 Assume that X is a (complex) HI space. Show that X does no admit an almost transitive norm, or even, that the isometry group acts almost trivially on X.

By [55, Corollary 6.7] the answer to this problem is affirmative when X is a separable reflexive HI space without a Schauder basis.

All the examples appearing in Theorem 2.11 are, in some sense, "far from being Hilbert". One may wonder if there exist counterexamples within the most popular classes of spaces that are "close to being Hilbert":

5.11 Find a weak Hilbert space, an asymptotically hilbertian space, or even a near Hilbert space that does not admit an AT renorming.

Please note that *asymptotically hilbertian* is not the same as asymptotic (or Asymptotic) ℓ_p space for $p = 2$; see Theorem 2.11. The definition of weak-Hilbert and asymptotically hilbertian spaces, as well as various characterizations, can be seen in Pisier [117]; a near Hilbert space is just a Banach space having type $2 - \varepsilon$ and cotype $2 + \varepsilon$ for every $\varepsilon > 0$. These include all "twisted Hilbert spaces", in particular the Kalton-Peck spaces [86]. Going in the opposite direction (see the comments closing Sect. 2.2):

5.12 Does there exist any symmetric space not isomorphic to ℓ_2 which admits an almost transitive renorming?

5.13 Are the Hilbert spaces the only (almost) transitive spaces with property (M)?

5.14 Does there exist a separable Banach space so that every bounded subgroup of $\mathrm{GL}(X)$ is contained in some maximal bounded subgroup of $\mathrm{GL}(X)$? Is this true for L_p or Kadec' complementably universal space?

In view of Theorem 2.10 this problem could have different answers for L_p and $\mathcal{K}$ since the latter contains a complemented copy of each separable HI space with the BAP.

5.15 Does $T^{(2)}$, the 2-convexified Tsirelson space, or do more general weak Hilbert spaces, other than the Hilbert space, have a maximal bounded subgroup of $\mathrm{GL}(X)$?

5.16 (Dilworth and Randrianantoanina [44, Problem 1.1]) Suppose that every subspace of a Banach space X admits an equivalent almost transitive renorming. Is X isomorphic to a Hilbert space?

Going back to the genuine Mazur affairs we find the following question, especially the case $p = 1$, most itching:

5.17 Does L_p admit a *transitive* renorming for some $p \neq 2$?

Problems relative to Fraïssé or homogeneous spaces Here, the fundamental question seems to be Problem 4.3, namely

5.18 (Multidimensional Mazur problem) Is every separable ultrahomogeneous Banach space isometric (or isomorphic) to the Hilbert space?

Even in this setting the gap between an eventual affirmative answer and the existing knowledge is sideral.

5.19 ([54, Problem 2.9]) Are the Gurariy space and the spaces L_p for $p \neq 4, 6, 8, \ldots$ the only separable Fraïssé spaces? or even AUH spaces?

Variants of this problem were suggested to us by G. Godefroy, based on the well-known fact that the norm on L_p is a C^∞-smooth norm exactly when p is even (see [43, Chapter V] for much more information on this, in particular for a proof that the canonical norm of L_p is optimal regarding smoothness for $1 < p < \infty$). For example:

5.20 Show that the Hilbert space is the only separable Fraïssé (or even AUH) space with a C^∞-smooth norm.

5.21 Show that a C^∞-smooth norm which is Fraïssé (or even AUH) is necessarily ultrahomogeneous.

Note that any Fraïssé renorming of the Gurariy space must be isometric to the Gurariy space itself. Indeed, cotype considerations imply that ℓ_∞ is finitely representable in such space, and then we may apply the observation after Proposition 4.16. The question seems to remain open for L_p:

5.22 Let $1 \leq p < \infty$. Is any Fraïssé norm on L_p conjugate to the usual norm?

The multidimensional version of Problem 5.17 is:

5.23 Show that L_p does not admit an ultrahomogeneous renorming.

5.24 ([54, Problem 2.6]) Are the Gurariy space and the Hilbert space the only separable stable Fraïssé Banach spaces?

In particular,

5.25 (Problem 4.13) Are the spaces L_p, $p \neq 2, 4, 6, \ldots$ stable Fraïssé?

In relation to [54, Proposition 2.14] we may ask:

5.26 Is every (separable) AUH space necessarily Fraïssé? Is every ultrahomogeneous space Fraïssé? Is every space having an ultrahomogeneous ("countable") ultrapower Fraïssé?

The Banach–Gromov "conjecture" Following a suggestion of the referee, we close with a few remarks on another problem in Banach's book (remarques au Chapitre XII, p. 215) concerning isometric characterizations of Hilbert spaces.

5.27 (Banach) Let X be a Banach space such that, for some $2 \leq n < \infty$, all n-dimensional subspaces of X are isometric. Must X be a Hilbert space?

The hypothesis on X in this problem is somehow "dual" to that appearing in Mazur's: note that all hyperplanes in a reflexive transitive space are mutually isometric.

An affirmative answer for $n = 2$ (real case) was quickly provided by Auerbach, Mazur, and Ulam in [12] and then for infinite-dimensional X and any n by Dvoretzky [48] (the complex version of Dvoretzky's theorem was established later by Milman [108]) making it clear that Banach's question reduced to considering hyperplanes in finite-dimensional spaces. In 1967, Gromov [72] solved the problem in the affirmative for even n and all X (real or complex), for odd n and real X with $\dim(X) \geqslant n + 2$, and for odd n and complex X with $\dim(X) \geq 2n$, which also settled the problem in any infinite dimensional X. Thus, the first integer for which Banach's problem remains open is 3. We refer ther reader to Soltan [133, Section 6] for more information on this topic and to [24] for a recent result.

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