ON UNIFORMLY FINITELY EXTENSIBLE BANACH SPACES

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Abstract. We continue the study of Uniformly Finitely Extensible Banach spaces (in short, UFO) initiated in Moreno-Plichko, *On automorphic Banach spaces*, Israel J. Math. 169 (2009) 29–45 and Castillo-Plichko, *Banach spaces in various positions.* J. Funct. Anal. 259 (2010) 2098-2138. We show that they have the Uniform Approximation Property of Pełczyński and Rosenthal and are compactly extensible. We will also consider their connection with the automorphic space problem of Lindenstrauss and Rosenthal –do there exist automorphic spaces other than $c_0(I)$ and $\ell_2(I)$?– showing that a space all whose subspaces are UFO must be automorphic when it is Hereditarily Indecomposable (HI), and a Hilbert space when it is either locally minimal or isomorphic to its square. We will finally show that most HI –among them, the super-reflexive HI space constructed by Ferenczi– and asymptotically ℓ_2 spaces in the literature cannot be automorphic.

1. Introduction and preliminaries

For all the unexplained notation and terms see the second half of this section. In this paper we continue the study of Uniformly Finitely Extensible Banach spaces (in short, UFO) initiated in [18, 41] and their connection with the automorphic space problem [17, 38, 11]. Following [17], a Banach space is said to be *automorphic* if every isomorphism between two subspaces such that the corresponding quotients have the same density character can be extended to an automorphism of the whole space. Equivalently, if for every closed subspace $E \subset X$ and every into isomorphism $\tau : E \to X$ for which X/E and $X/\tau E$ have the same density character, there is an automorphism T of X such that $T|_E = \tau$. The motivation for such definition is in the Lindenstrauss-Rosenthal theorem [38] asserting that c_0 is automorphic and the extension presented in [41] – for every Γ the space $c_0(\Gamma)$ is automorphic–. This leads to the generalization of the still open problem set by Lindenstrauss and Rosenthal [38]

Automorphic space problem: Does there exist an automorphic space different from $c_0(I)$ or $\ell_2(I)$?

The papers [11, 41] and [18] considered different aspects of the automorphic space problem. In particular, the following two groups of notions were isolated:

Definition 1. *A couple (Y, X) of Banach spaces is said to be (compactly) extensible if for every subspace $E \subset Y$ every (compact) operator $\tau : E \to X$ can be extended*

46B03, 46B07, 46B08, 46M18, 46B25, 46B42.

This research has been supported in part by project MTM2010-20190-C02-01 and the program Junta de Extremadura GR10113 IV Plan Regional I+D+i, Ayudas a Grupos de Investigación, and also by CNPQ projeto 455687/2011-0 .

to an operator $T : Y \to X$. If there is a $\lambda > 0$ such that some extension exists verifying $\|T\| \leq \lambda\|\tau\|$ then we will say that (Y, X) is λ-(compactly) extensible. The space X is said to be (compactly) extensible if (X, X) is (compactly) extensible and uniformly (compactly) extensible if it is λ-(compactly) extensible for some λ.

It is not known whether there exist separable extensible spaces different from c_0 and ℓ_2. Neither it is known if an extensible space must be uniformly extensible, although some partial results have already been obtained in [41] and [18]; precisely, that an extensible space isomorphic to its square is uniformly extensible.

Definition 2. *A couple (Y, X) of Banach spaces is said to be a λ-UFO pair if for every finite dimensional subspace E of Y and every linear operator $\tau : E \to X$, there exists a linear continuous extension $T : Y \to X$ with $\|T\| \leq \lambda\|\tau\|$. A couple (Y, X) of Banach spaces is said to be an UFO pair if it is a λ-UFO pair for some λ. A Banach space X is said to be Uniformly Finitely Extensible (an UFO, in short) if (X, X) is an UFO pair. It is said to be a λ-UFO if (X, X) is a λ-UFO pair.*

It is clear that every $\mathcal{L}_{\infty,\lambda}$-space is a λ-UFO. Recall that a subspace Y of a Banach space X is said to be locally complemented if $Y^{**} = Y^{\perp\perp}$ is complemented in X^{**}. Some acquaintance with ultraproduct theory will be required: Let I be a set, $\mathcal{U}$ be an ultrafilter on I, and (X_i) a family of Banach spaces. Then $\ell_\infty(X_i)$ endowed with the supremum norm, is a Banach space, and $c_0^{\mathcal{U}}(X_i) = \{(x_i) \in \ell_\infty(X_i) : \lim_{\mathcal{U}(i)} \|x_i\| = 0\}$ is a closed subspace of $\ell_\infty(X_i)$. The ultraproduct of the spaces X_i following $\mathcal{U}$ is defined as the quotient

$$[X_i]_{\mathcal{U}} = \ell_\infty(X_i)/c_0^{\mathcal{U}}(X_i).$$

We denote by $[(x_i)]$ the element of $[X_i]_{\mathcal{U}}$ which has the family (x_i) as a representative. It is not difficult to show that $\|[(x_i)]\| = \lim_{\mathcal{U}(i)} \|x_i\|$. In the case $X_i = X$ for all i, we denote the ultraproduct by $X_{\mathcal{U}}$, and call it the ultrapower of X following $\mathcal{U}$. The following lemma gathers the basic results on UFO spaces from [41] (results (i, ii)) and [18] (results (iii, iv).

Lemma 1.1.

- **i):** *Every compactly extensible space is an UFO.*
- **ii):** *Every λ-UFO that is μ-complemented in its bidual is $\lambda\mu$-extensible.*
- **iii):** *A locally complemented subspace of an UFO is an UFO.*
- **iv):** *Ultrapowers of λ-UFO are λ-UFO; consequently, biduals of λ-UFO are λ-extensible.*

The spaces Y for which (Y, ℓ_p) is an UFO pair were investigated in [14] under the name *M_p-spaces*, and Maurey's extension theorem (see e.g. [49]) can be reformulated in this language as: *Each type 2 space is M_2*. Related to this is the so-called *Maurey extension property* (in short MEP): a Banach space X has MEP if every operator $t : E \to \ell_2$ from any subspace E of X admits an extension to $T : X \to \ell_2$. The equivalence between M_2 and MEP should be known but we have been unable to find any explicit reference. The result follows from the following generalization(s) of Lemma 1.1 (ii):

Lemma 1.2. *If the pair (Y, X) is λ-UFO then (Y, X^{**}) is λ-extensible. Therefore, if X is complemented in its bidual, the pair (Y, X) is an UFO if and only if (Y, X) is extensible.*

Sketch of proof. Assume that (Y, X) is λ-UFO. Let E be an infinite dimensional subspace of Y and $\phi : E \to X$ an operator. For each finite dimensional subspace $F \subset E$ let ϕ_F be the restriction of ϕ to F and then let Φ_F be its extension to Y verifying $\|\Phi_F\| \leq \lambda\|\phi_F\|$, which exists by hypothesis. Let FIN the partially ordered set of all finite dimensional subspaces of E and let $\mathcal{U}$ be an ultrafilter refining the Fréchet filter. Let $\psi : Y \to X^{**}$ be defined as

$$\psi(x) = \text{weak}^* - \lim_{\mathcal{U}(F)} \Phi_F(x)).$$

The proof that if (Y, X) is extensible then (Y, X) is an UFO pair is similar to that of Lemma 1.1 (i). □

Corollary 1.3. *Properties M_2 and MEP are equivalent.*

The contents of the paper are as follows.

In section 2 we show that UFO spaces have the Uniform Approximation Property, as a consequence of the following general principle: If there is a constant C so that all X-valued norm one finite-rank operators defined on subspaces of X admit extensions to X with norm at most C then for every $\varepsilon > 0$ they admit finite-rank extensions with norm at most $(C + \varepsilon)$. As a consequence we show that when X is an UFO, X-valued compact operators defined on its subspaces can be uniformly extended to the whole space, and the extension operator can even be chosen to be compact. This solves in the affirmative the question posed in [41] of whether UFO and compactly extensible spaces coincide. The corresponding question for all operators (i.e., whether extensible implies uniformly extensible) is open. The following diagram displays the basic implications:

$$\begin{array}{r} \text{automorphic} \Rightarrow unif.\ \text{extensible} \Rightarrow \text{extensible} \Rightarrow \text{compactly extensible} \\ \Leftrightarrow \\ unif.\ \text{compactly extensible} \\ \Leftrightarrow \\ \text{UFO.} \end{array}$$

It was obtained in [18] –see also Theorem 3.1 below– the dichotomy principle asserting that an UFO must be either i) an $\mathcal{L}_\infty$-space or ii) a B-convex near-Hilbert space (i.e., [18], a space X such that $p(X) = q(X) = 2$ where $p(X) = \sup\{p : X \text{ is of type } p\}$ and $q(X) = \inf\{q : X \text{ is of cotype } q\}$) with the Maurey Extension Property, called B-UFO from now on. In section 3 we focus on classifying B-UFO in the presence of some additional properties; mainly: to be isomorphic to its square or to be Hereditarily Indecomposable (HI from now on), which somehow are properties at the two ends of the spectrum. For instance, we will show that a space all whose subspaces are UFO must be automorphic when it is HI, and a Hilbert space when it is either isomorphic to its square or locally minimal. Among other stability properties of the class of UFO spaces, we prove that $X \oplus \ell_2$ is B-UFO when X is B-UFO, however it still remains unsolved the question whether the product of two B-UFO spaces must be be B-UFO. Actually, a positive answer to that question would imply that every hereditarily UFO is Hilbert. Although the quotient of two UFO does not have to be UFO, we prove that when Y is locally complemented in an UFO space X then X/Y is UFO. The following table could help the reader:

Type	$\mathcal{L}_\infty$	B-convex
UFO	basis	near Hilbert, MEP, UAP
UFO + HI	exist	?
UFO minimal	c_0	superreflexive
HUFO	don't exist	exist
HUFO locally minimal	don't exist	Hilbert
HUFO isomorphic to square	don't exist	Hilbert

Section 4 examines possible counterexamples to the automorphic space problem. Indeed, while the papers [41] (resp. [11]) showed that most of the currently known Banach spaces (resp. $C(K)$-spaces) cannot be automorphic, a work continued in [18], in this paper we turn our attention to HI or asymptotically ℓ_2 spaces to show that most of the currently known examples cannot be automorphic. Special attention is paid to the only HI space which is candidate to be automorphic: the super-reflexive HI space $\mathcal{F}$ constructed by the second author in [22], for which we show it is not UFO since it is not near-Hilbert (see Prop. 4.4). The concluding Section 5 contains the technical results required to prove this last assertion.

2. Compactly extensible spaces

It is known that extensible spaces are UFO (it follows from Lemma 1.1 (i)) and that UFO spaces are not necessarily extensible (say, $C[0,1]$). The question of whether UFO spaces must be compactly extensible was posed in [41] and will be affirmatively solved in Proposition 2.4 below. Moreover, we will show that compactly extensible spaces are uniformly compactly extensible. This is remarkable since, as we have already mentioned, it is an open question whether an extensible space must be uniformly extensible. The following general principle is the key:

Lemma 2.1. *If (Y,X) is a λ-UFO pair then, for every $\varepsilon>0$, every X-valued operator t defined on a finite-dimensional subspace of Y admits a finite-rank extension to the whole of Y with norm at most $(\lambda+\varepsilon)\|t\|$.*

Proof. Let F be a finite-dimensional subspace of Y and let $t: F\to X$ be an operator. Take (y_n) to be a finite ε-net of the unit sphere of F, with $\varepsilon<1$, then pick norm one functionals (f_n) so that $f_n(y_n)=\|y_n\|$, and form the finite-codimensional subspace of Y

$$H=\bigcap_n \ker f_n.$$

It is clear that $F\cap H=0$: Indeed, if all f_n vanish on some norm one element $y\in F$ then take y_k in the unit sphere of F with $\|y-y_k\|\le\varepsilon$ and thus $\|y_k\|=f_k(y_k)=f_k(y_k-y)+f_k(y)\le\varepsilon$, which is impossible. Since F is finite-dimensional, $F+H$ is closed so $F+H=F\oplus H$. Given y of norm 1 in F and $h\in H$, and if y_k is defined as above, then

$$\|y+h\|\ge f_k(y+h)=f_k(y)=f_k(y-y_k)+f_k(y_k)\ge 1-\varepsilon.$$

This means that the natural projection $p: F\oplus H\to F$ actually has norm at most $(1-\varepsilon)^{-1}$. Then $tp: F\oplus H\to X$ is a finite-rank operator with norm at most

$(1-\varepsilon)^{-1}\|t\|$. By Lemma 1.2, tp admits an extension $T : Y \to X^{**}$, with norm $(1-\varepsilon)^{-1}\lambda\|t\|$, which moreover has finite-dimensional range since $T_{|_H} = 0$. So, by the principle of local reflexivity, there is an operator $Q : T(Y) \to X$ with norm at most $(1-\varepsilon)^{-1}$ so that $Q(u) = u$ for all $u \in T(Y) \cap X$. The operator $QT : Y \to X$ has finite range and $QT(f) = Qtp(f) = Qt(f) = t(f)$ for all $f \in F$. Moreover $\|QT\| \leq (1-\varepsilon)^{-2}\lambda\|t\|$. □

Recall that a Banach space X is said to have the *Bounded Approximation Property* (BAP in short) if for some λ and each finite dimensional subspace $F \subset X$ there is a finite rank operator $T : X \to X$ such that $\|T\| \leq \lambda$ and $T(y) = y$ for each $y \in F$ (in which case it is said to have to λ-BAP). The corresponding notion in local theory is the *Uniform Approximation Property* (UAP in short) introduced by Pełczyński and Rosenthal [42] by asking the existence of a function $f : \mathbb{N} \to \mathbb{N}$ so that the choice above can be performed verifying rank$T \leq f(\dim F)$. It is easy to see that X has the UAP if and only if every ultrapower of X has the BAP.

Proposition 2.2. *If X is a λ-UFO then, for any $\varepsilon > 0$, it has the $(\lambda+\varepsilon)$-UAP.*

Proof. Lemma 2.1 immediately implies that a λ-UFO must have the $(\lambda+\varepsilon)$-BAP for all $\varepsilon > 0$. Since ultrapowers of λ-UFO are λ-UFO [18], the result is clear. □

Remarks. The previous proof provides an estimate for the function $f : \mathbb{N} \to \mathbb{N}$ that defines the UAP. From the proof it follows that if $v : \mathbb{N} \times \mathbb{R}^+ \to \mathbb{N}$ is the function for which $v(n, \varepsilon)$ is the infimum of the N such that every n-dimensional subspace of an UFO space X admits an ε-net for its unit sphere with N points, then it is possible to check the UAP property in X with $f(n) = v(n, \varepsilon)$ for any choice of $\varepsilon < 1$. Volume estimates indicate that the unit ball of a real space of dimension n cannot be covered by less than $(\frac{1}{\varepsilon})^n$ balls of radius ε, and a similar estimate holds for $v(n, \epsilon)$. This is essentially optimal, since actually $v(n, \varepsilon) \leq (1 + 2/\varepsilon)^n$ (see [43], Lemma 4.10). Therefore if a real space is UFO, then the UAP property may be verified with, for example, $f(n) = 4^n$, which is clearly a bad estimate. Johnson and Pisier [33] proved that a Banach space is a weak Hilbert space if and only if $f(n) = O(n)$.

We now refine Lemma 2.1 in order to show compact extensibility and UFO are equivalent.

Lemma 2.3. *Let X be a space with BAP. The following statements are equivalent:*

1. *The pair (Y, X) is UFO.*
2. *There exists $C \geq 1$ such that for every subspace $E \subset Y$, every finite-rank operator $t : E \to X$ admits a finite-rank extension with $\|T\| \leq C\|t\|$.*
3. *The pair (Y, X) is compactly extensible.*
4. *There exists $C \geq 1$ such that for every subspace $E \subset Y$, every compact operator $t : E \to X$ admits a compact extension with $\|T\| \leq C\|t\|$.*

Proof. That (iv) implies (iii) and (ii) implies (i) are obvious. And that (iii) implies (i) is Lemma 1.1 (i) and was proved in [41] without the need of asking X to have the BAP.

(i) implies (ii): we prove that if (Y, X) is λ-UFO and X has the μ-BAP, then (ii) holds with $C = \lambda\mu + \varepsilon$, for all $\epsilon > 0$. Let $t : E \to X$ be a finite-rank operator from a subspace $E \subset Y$ and let $F = tE$. By Lemma 1.2, there exists an extension $T : Y \to X^{**}$ of t with $\|T\| \leq \lambda\|t\|$. Now, since X has μ-BAP, there is a finite range operator $\tau : X \to X$ such that $\tau(u) = u$ for every $u \in F$ and $\|\tau\| \leq \mu$. Consider the bi-adjoint operator τ^{**} and take the finite dimensional subspace $\tau^{**}T(Y) \subset X^{**}$. By the principle of local reflexivity [39], there exists an operator $Q : \tau^{**}T(Y) \to X$ of norm at most $1 + \varepsilon$ such that $Q(x) = x$ for every $x \in \tau^{**}T(Y) \cap X$. The operator $Q\tau^{**}T : Y \to X$ has finite range and for every $e \in E$, $Q\tau^{**}T(e) = Q\tau t(e) = \tau(te) = te$. Moreover $\|Q\tau^{**}T\| \leq (1 + \varepsilon)\mu\lambda\|t\|$.

(ii) implies (iv): we prove that if X has the AP and (2) holds with C, then the pair (Y, X) is $(C + \varepsilon)$-compactly extensible for any $\varepsilon > 0$, with compact extension. Consider any subspace $E \subset Y$ and a compact operator $t : E \to X$, since X has the AP, t can be uniformly approximated by finite rank operators $t_n : E \to X$, that is $t = \|\cdot\| - \lim t_n$, and we may assume $\|t_n\| = \|t\|$ for all n. Let $\varepsilon > 0$, without lost of generality (passing to a subsequence if necessary) we may assume that $\|t_1 - t\| \leq \varepsilon$ and $\|t_n - t_{n-1}\| \leq \varepsilon_n$, for $n \geq 2$, with (ε_n) a given sequence of positive numbers such that $\sum_{n=2}^{+\infty} \varepsilon_n \leq \varepsilon\|t\|/C$. Consider for every $n \geq 2$ finite-rank extensions $T_n : Y \to X$ of $\tau_n = t_n - t_{n-1}$ with $\|T_n\| \leq C\|\tau_n\|$ and let T_1 be an extension of t_1 such that $\|T_1\| \leq C\|t_1\|$. The operator $T = \sum T_n$ is a well defined compact operator which extends t and such that $\|T\| \leq (C + \varepsilon)\|t\|$. □

Proposition 2.4. *The following statements are equivalent:*

1. *X is UFO.*
2. *X is compactly extensible.*
3. *X is uniformly compactly extensible.*

Moreover the extension operators in (ii) and (iii) may be chosen compact.

When Y is a fixed subspace of the Banach space X, and E is a Banach space it is a direct consequence from the open mapping theorem that if all (compact) operators $Y \to E$ can be extended to operators $X \to E$ then they can be uniformly extended. Also, $\mathcal{L}_\infty$-spaces have the property when acting as target spaces E that all E-valued compact operators can be (uniformly) extended [36] and admit compact extensions.

3. B-convex UFO

The following dichotomy result was obtained in [18]:

Theorem 3.1 (**Dichotomy principle**). *An UFO Banach space is either an $\mathcal{L}_\infty$-space or a B-convex near-Hilbert space with MEP.*

This dichotomy provides an affirmative answer to a question of Galego [25]: Is an automorphic subspace of $c_0(\Gamma)$ isomorphic to some $c_0(I)$? Indeed, any infinite dimensional closed subspace of $c_0(\Gamma)$ contains c_0, hence every UFO subspace of $c_0(\Gamma)$ must be an $\mathcal{L}_\infty$-space; and $\mathcal{L}_\infty$-subspaces of $c_0(\Gamma)$ are isomorphic to $c_0(I)$ (cf. [26]). We will now focus our attention on UFO which are not $\mathcal{L}_\infty$-spaces.

Definition 3. *A Banach space is said to be a* B-UFO *if it is a B-convex UFO.*

Of course the first open question is: Do there exist non-Hilbert B-UFO spaces? The only candidate currently known seems to be Tsirelson's 2-convexified space $\mathcal{T}_2$ [15], which is near-Hilbert, superreflexive and, having type 2, also enjoys MEP [15, p. 127]. It is not known whether $\mathcal{T}_2$ is an UFO (the answer would be negative if $\mathcal{T}_2$

would, for instance, contain an uncomplemented copy of itself since no automorphism of the space can send a complemented subspace into an uncomplemented one. The space $\mathcal{T}_2$ moreover is *minimal* –a Banach space X is minimal when every infinite-dimensional closed subspace contains a copy of X–. Now observe that by the Gowers dichotomy [28], every Banach space contains either an Hereditarily Indecomposable subspace or a subspace with unconditional basis, and spaces with unconditional basis must be either reflexive, contain ℓ_1 or c_0 [31]. Thus, if X is a minimal UFO that contains an HI subspace, X itself must be HI; but HI spaces cannot be minimal since they are not isomorphic to their proper subspaces. If, however, X contains either ℓ_1 or c_0, it must be itself be a subspace of either ℓ_1 or c_0 and an $\mathcal{L}_\infty$-space, so it must be c_0. All this means that a minimal UFO must be either c_0 or a reflexive near-Hilbert space with MEP. This can be improved using the local version of minimality. Recall that a Banach space X is said to be *locally minimal* ([24]) if there is some $K \geq 1$ such that every finite-dimensional subspace $F \subset X$ can be K-embedded into any infinite-dimensional subspace $Y \subset X$. This notion is much weaker than classical minimality; for instance, every c_0-saturated Banach space is locally minimal.

Proposition 3.2. *A minimal UFO must be either c_0 or a superreflexive near-Hilbert space with MEP.*

Proof. By Johnson's local version of James' trichotomy for spaces with unconditional basis [32] we get that a Banach space with local unconditional structure must contain uniformly complemented ℓ_1^n, ℓ_∞^n or to be superreflexive. Since minimal implies locally minimal, a minimal UFO containing either ℓ_1^n or ℓ_∞^n must be c_0; since a reflexive space space cannot be $\mathcal{L}_\infty$, it must be superreflexive. □

Definition 4. *A Banach space is said to be* hereditarily UFO *(a HUFO in short* [18]*) if each of its closed subspaces is an UFO.*

In [18] it was shown that a HUFO spaces must be *asymptotically* ℓ_2, i.e., there is a constant $C > 0$ such that for every n there is a finite-codimensional subspace X_n all whose n-dimensional subspaces G verify $\operatorname{dist}(G, \ell_2^{\dim G}) \leq C$. Asymptotically ℓ_2 spaces are reflexive (see [43, p. 220]), and reflexive UFO are extensible. So, no $\mathcal{L}_\infty$ space can be HUFO, which means that every HUFO is (hereditarily) B-UFO.

Proposition 3.3. *A locally minimal HUFO is isomorphic to a Hilbert space.*

Proof. Let X be a locally minimal HUFO space. On one side every HUFO must be asymptotically ℓ_2; while, on the other hand, the hypothesis of local minimality means that there is some $K \geq 1$ so that for every infinite-dimensional subspace $Y \subset X$ and every finite-dimensional subspace $F \subset X$ there is a subspace $F_Y \subset Y$ such that $\operatorname{dist}(F, F_Y) \leq K$. Now, if X is not a Hilbert space, there is a sequence (H_n) of finite-dimensional subspaces such that $\lim \operatorname{dist}(H_n, \ell_2^{\dim H_n}) = \infty$ —otherwise X would be a subspace of the Hilbert space formed as the ultraproduct of all the finite-dimensional subspaces of X—. So H_n cannot K-embed into the finite-codimensional subspaces X_m of the definition of asymptotically ℓ_2 for large m. □

In [24] a dichotomy is proved opposing local minimality to *tightness with constants* (one formulation of this property is that a space X with a basis is tight with constants if no subspace of X is crudely finitely representable into all tail subspaces of X). The proof of Proposition 3.3 indicates that a HUFO space with a basis is either tight with constants or Hilbert. One also has:

Proposition 3.4. *HUFO spaces isomorphic to their square are isomorphic to Hilbert spaces.*

Proof. This follows from [18] where it was proved that if X contains a subspace of the form $A \oplus A$ with $A \neq \ell_2$ then X contains a non-UFO subspace. □

A basic open question is whether the product of two B-UFO is a B-UFO. It is well known that the product of two $\mathcal{L}_\infty$-spaces is an $\mathcal{L}_\infty$-space, and it is clear that $c_0 \oplus \ell_2$ cannot be an UFO (it is not B-convex or $\mathcal{L}_\infty$); so the question above is the only case that has to be elucidated.

Proposition 3.5. *If X is B-UFO then $\ell_2 \oplus X$ is B-UFO.*

Proof. Since X is B-convex, it contains ℓ_2^n uniformly complemented and hence, for some ultrafilter $\mathcal{U}$, the ultraproduct $X_\mathcal{U}$ contains ℓ_2 complemented. Hence $X_\mathcal{U} \simeq \ell_2 \oplus Z \simeq \ell_2 \oplus \ell_2 \oplus Z \simeq \ell_2 \oplus X_\mathcal{U}$ is an B-UFO, as well as its locally complemented subspace $\ell_2 \oplus X$. □

Corollary 3.6. *If for some sequences of scalars $p_n \neq 2$ and naturals k_n a Banach space X contains finite dimensional $l_{p_n}^{k_n}$ uniformly complemented then it is not an UFO.*

Proof. The hypothesis means that $\ell_2 \oplus X$ contains $\ell_2^n \oplus \ell_{p_n}^{k_n}$ uniformly complemented. Using [18, Lemma 6.11] we get copies of $\ell_{p_n}^{k_n}$ which are not uniformly complemented in X. An appeal to [41, Thm. 4.4] shows that X is not an UFO. □

This improves the analogous result in [41, Cor. 4.6] for fixed $p_n = p \neq 2$. Another intriguing partial result is the following.

Lemma 3.7. *If the product of two B-UFO is always an B-UFO then every HUFO is isomorphic to a Hilbert space.*

Proof. Let X be HUFO; then $X \oplus X$ is UFO. If X is not Hilbert, following the proof in [18], $X \oplus X$ must contain a non-UFO subspace with FDD of the form $\sum(G_k \oplus F_k)$, where $G_k \subset X$, $F_k \subset X$ and the F_k containing badly complemented subspaces uniformly isomorphic to G_k. The space $\sum G_k \oplus \sum F_k \subset X \oplus X$ is not UFO by [41, Th. 4.4]. But since X is HUFO, both $\sum G_k$ and $\sum F_k$ must be UFO. □

To be an UFO is by no means a 3-space property (see [16]) because non-trivial twisted Hilbert spaces –i.e., spaces Z containing an uncomplemented copy of ℓ_2 for which Z/ℓ_2 is isomorphic to ℓ_2– cannot be B-UFO since the MEP property would make every copy of ℓ_2 complemented. Such twisted sum spaces exist: see [16] for general information and examples. The quotient of two $\mathcal{L}_\infty$ spaces is an $\mathcal{L}_\infty$-space, hence UFO. Nevertheless, it was shown in [18, 10] that ℓ_∞/c_0 is not extensible, which implies that the quotient of two extensible spaces does not have to be extensible. Let us show that the quotient of two UFO is not necessarily an UFO.

Lemma 3.8. *The space ℓ_∞/ℓ_2 is not UFO:*

Proof. Indeed, since B-convexity is a 3-space property [16], it cannot be B-convex. Let $p : \ell_\infty^{**} \to \ell_\infty$ be a projection through the canonical embedding $\ell_\infty \to \ell_\infty^{**}$. The commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_2 & \longrightarrow & \ell_\infty & \longrightarrow & \ell_\infty/\ell_2 & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle u} & & \\ 0 & \longrightarrow & \ell_2 & \longrightarrow & \ell_\infty^{**} & \longrightarrow & \ell_\infty^{**}/\ell_2 & \longrightarrow & 0 \end{array}$$

immediately implies ℓ_∞/ℓ_2 is complemented in its bidual. Thus, if ℓ_∞/ℓ_2 was an $\mathcal{L}_\infty$-space then it would be injective. Let us show it is not: Take an embedding $j : \ell_2 \to L_1(0,1)$ and consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_2 & \longrightarrow & \ell_\infty & \longrightarrow & \ell_\infty/\ell_2 & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle u} & & \\ 0 & \longrightarrow & \ell_2 & \longrightarrow & Z & \longrightarrow & \ell_2 & \longrightarrow & 0. \end{array}$$

in which the lower sequence is any nontrivial twisted sum of ℓ_2. If there is an operator $U : L_1(0,1) \to \ell_\infty/\ell_2$ such that $Uj = u$ then one would get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_2 & \longrightarrow & \ell_\infty & \longrightarrow & \ell_\infty/\ell_2 & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle U} & & \\ 0 & \longrightarrow & \ell_2 & \longrightarrow & X & \longrightarrow & L_1(0,1) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle j} & & \\ 0 & \longrightarrow & \ell_2 & \longrightarrow & Z & \longrightarrow & \ell_2 & \longrightarrow & 0, \end{array}$$

which is impossible since the ℓ_2 subspace in the middle exact sequence must be complemented by the Lindenstrauss lifting principle [37] while in the lower sequence it is not. □

Nevertheless, when the subspace is locally complemented the quotient must be UFO:

Proposition 3.9. *Let X be an UFO and Y a locally complemented subspace of some ultrapower $X_{\mathcal{U}}$ of X. Then $X_{\mathcal{U}}/Y$, $Y \oplus X$ and $X \oplus (X_{\mathcal{U}}/Y)$ are all UFO.*

Proof. The space $X_{\mathcal{U}}/Y$ is UFO since it is locally complemented in $(X_{\mathcal{U}}/Y)^{**}$, which in turn is complemented in $X_{\mathcal{U}}^{**}$. The space $Y \oplus X$ is locally complemented in $X_{\mathcal{U}} \oplus X_{\mathcal{U}}$, which is an UFO, while $X \oplus (X_{\mathcal{U}}/Y)$ is locally complemented in $X_{\mathcal{U}} \oplus X_{\mathcal{U}}^{**}$, which is also UFO. □

A couple of questions about B-UFO for which we conjecture an affirmative answer are: Is every B-UFO reflexive? Is every B-UFO isomorphic to its square? It can be observed that if every B-UFO contain a subspace of the form $A \oplus A$ for infinite-dimensional A then every HUFO must be Hilbert.

Turning our attention to the other end of the spectrum, spaces which do not contain subspaces isomorphic to its square, we find the Hereditarily Indecomposable (HI, in short) spaces. Recall that a Banach space is said to be HI if no subspace admits a decomposition in the direct product of two infinite dimensional subspaces. Hereditarily indecomposable UFO exist after the constructions of HI $\mathcal{L}_\infty$-spaces obtained by Argyros and Haydon [7], and later by Tarbard [47]. Let us show that, in some sense, spaces which are simultaneously HI and UFO are close to automorphic.

Lemma 3.10. *For HI spaces, extensible and automorphic are equivalent notions.*

Proof. Operators on HI spaces are either strictly singular of Fredholm with index 0. The extension of an embedding cannot be strictly singular, so it is an isomorphism between two subspaces with the same finite codimension, which means that some other extension of the embedding is an automorphism of the whole space. □

Hence, taking into account Lemma 1.1 (ii) one gets:

Proposition 3.11. *A reflexive space which is HI and UFO is automorphic.*

In particular, HI spaces which also are HUFO must be automorphic. A variation in the argument shows:

Proposition 3.12. *Assume that X is an $\mathcal{L}_\infty$ space with the property that every operator $Y \to X$ from a subspace $i : Y \to X$ has the form $\lambda i + K$ with K compact. Then X is automorphic.*

Proof. Since compact operators on an $\mathcal{L}_\infty$-space can be extended to larger superspaces, the hypothesis implies that the space is extensible. The hypothesis also yields that X must be HI. □

These two propositions suggest a way to obtain a counterexample for the automorphic space problem. We however conjecture that an HI space cannot be automorphic.

4. Counterexamples

As we said before, there are not many examples known of either near-Hilbert spaces or spaces with MEP. For a moment, let us focus on B-convex spaces. A good place to look for them is among weak Hilbert spaces. Recall that a Banach space X is said to be weak Hilbert if there exist constants δ, C such that every finite dimensional subspace F contains a subspace G with $\dim G \geq \delta \dim F$ so that $\mathrm{dist}(G, \ell_2^{\dim G}) \leq C$. It is well-known that weak Hilbert spaces are B-convex and near-Hilbert, although it is not known if they must have MEP. Tsirelson's 2-convexified space $\mathcal{T}_2$ [15] is a weak Hilbert type 2 space, but we do not know whether it is an UFO. Nevertheless, since $\mathcal{T}_2$ is isomorphic to its square we can apply Proposition 3.4 to conclude it is not HUFO. Since subspaces of weak Hilbert spaces are weak Hilbert, we obtain that not all weak Hilbert spaces are UFO. This answers the question left open in [18, p.2131] of whether weak Hilbert spaces must be UFO. Argyros, Beanland and Raikoftsalis [4] have recently constructed a weak Hilbert space X_{abr} with an unconditional basis in which no disjointly supported subspaces are isomorphic (such spaces are called tight by support in [24]). Clearly this space does not contain a copy of ℓ_2. By the criterion of Casazza used by Gowers in its solution to Banach's hyperplane problem [27], tightness by support implies that X_{abr} is not isomorphic to its proper subspaces, and in particular is not isomorphic to its square. It remains open whether this space is an UFO or even a HUFO.

"Less Hilbert" than weak Hilbert spaces are the asymptotically ℓ_2 spaces, still to be considered since HUFO spaces are of this type. See also [3] for related properties. Asymptotically ℓ_2 HI spaces have been constructed by different people. The space of Deliyanni and Manoussakis [21] cannot be HUFO since it has the property that c_0 is finitely represented in every subspace, so it is locally minimal. Apply now Proposition 3.3. We do not know however if this space or if the asymptotically ℓ_2 HI space constructed by Androulakis and Beanland [1] are UFO. In [12] Casazza, García and Johnson construct an asymptotically ℓ_2 space without BAP; which, therefore, cannot be UFO. Actually, the role of the BAP in these UFO affairs is another point not yet understood. Johnson and Szankowski introduce in [34] HAPpy spaces as those Banach spaces all whose subspaces have the approximation property. Szankowski had already shown in [46] that HAPpy spaces are near-Hilbert, while Reinov [44] showed the existence of a near-Hilbert non-HAPpy space. This motivates the following question: Is every B-UFO space HAPpy? Another construction of Johnson and Szankowski in [34] yields a HAPpy asymptotically ℓ_2 space with the property that every subspace is isomorphic to a complemented subspace. This is a truly wonderful form of not being extensible; and since the space is reflexive, of not being UFO.

Passing to more general HI spaces, the Argyros-Deliyanni asymptotically ℓ_1 space [5] was shown in [41] not to be UFO. Argyros and Tollias [9, Thm. 11.7] produce an HI space X so that both X^* and X^{**} are HI and $X^{**}/X = c_0(I)$; they also produce [9, Thm. 14.5] for every Banach space Z with basis not containing ℓ_1 an asymptotically ℓ_1 HI space X_Z for which Z is a quotient of X_Z. Argyros and Tollias extend they result [9, Thm. 14.9] to show that for every separable Banach space Z not containing ℓ_1 there is an HI space X_Z so that Z is a quotient of X_Z. The space X_Z can be obtained applying a classical result of Lindenstrauss (see [39]) asserting that every separable Z is a quotient E^{**}/E of a space E^{**} with basis. Apply the previous result to E^{**} to obtain an asymptotically ℓ_1 HI space X_Z such that $X_Z^{**}/X_Z = E^{**}$ which makes also Z a quotient of X_Z. None of them can be UFO:

Lemma 4.1. *Asymptotically ℓ_1 spaces cannot be UFO.*

Proof. Asymptotically ℓ_1 spaces contain ℓ_1^n uniformly and UFO spaces containing ℓ_1^n are $\mathcal{L}_\infty$-spaces; which cannot be asymptotically ℓ_1. □

Passing to $\mathcal{L}_\infty$-spaces, we show that the Argyros-Haydon $\mathcal{AH}$ space [7] is not automorphic.

Proposition 4.2. *The space $\mathcal{AH}$ is not extensible; hence it cannot be automorphic.*

Proof. Indeed, each operator in $\mathcal{AH}$ is a sum of scalar and compact operators, but there is a subspace $Y \subset \mathcal{AH}$ and an operator $\tau : Y \to Y$ which is not a sum of scalar and compact operators. This operator cannot be extended onto the whole space $\mathcal{AH}$. □

Argyros and Raikoftsalis [8] constructed another separable $\mathcal{L}_\infty$ counterexample to the scalar-plus-compact problem. However it contains ℓ_1 and

Lemma 4.3. *No separable Banach space containing ℓ_1 can be extensible.*

Proof. Indeed, the proof of Theorem 3.1 in [18] actually shows that an extensible space containing ℓ_1 must be separably injective, and Zippin's theorem [48] yields that c_0 is the only separable separably injective space. □

Related constructions are those of a different HI $\mathcal{L}_\infty$-space of Tarbard [47] and that of Argyros, Freeman, Haydon, Odell, Raikoftsalis, Schlumprecht and Zisimopoulou, who show in [6] that every uniformly convex separable Banach space can be embedded into an $\mathcal{L}_\infty$-space with the scalar-plus-compact property. We do not know whether these spaces can be automorphic.

On the other side of the dichotomy, essentially the only known example of a uniformly convex (hence B-convex) HI space was given by the second author in 1997 [22]. Other examples include, of course, its subspaces, and also a variation of this example, all whose subspaces fail the Gordon-Lewis Property, due to the second author and P. Habala [23]. From now on the space of [22] will be denoted by $\mathcal{F}$. The space $\mathcal{F}$ is defined as the interpolation space in $\theta \in]0,1[$ of a family of spaces similar to Gowers-Maurey's space and of a family of copies of ℓ_q for some $q \in]1,+\infty[$. We shall prove the following result concerning the type and cotype of $\mathcal{F}$. Here $p \in]1,q[$ is defined by the relation $1/p = 1-\theta+\theta/q$, and we recall that $p(\mathcal{F}) = \sup\{t : \mathcal{F} \text{ is of type } t\}$ and $q(\mathcal{F}) = \inf\{c : \mathcal{F} \text{ is of cotype } c\}$.

Proposition 4.4. *We have the following estimates*

(1) *if* $q \geq 2$ *then*
  (a) $\mathcal{F}$ *has type* $[1-(\theta/2)]^{-1}$ *and* $p(F) \leq p$,
  (b) $\mathcal{F}$ *has cotype* q/θ *and* $q(\mathcal{F}) = q/\theta$,

(2) *if* $q \leq 2$ *then*
  (a) $\mathcal{F}$ *has type* p *and* $p(\mathcal{F}) = p$,
  (b) $\mathcal{F}$ *has cotype* $2/\theta$ *and* $q(\mathcal{F}) \geq q/\theta$.

In particular $\mathcal{F}$ *is not near-Hilbert and therefore it is not UFO.*

It may be interesting to observe that if we choose $q = 2$ and θ sufficiently close to 1, then the above estimates imply that for any $\epsilon > 0$, $\mathcal{F}$ may be chosen to be of type $2-\epsilon$ and cotype $2+\epsilon$. We leave the technical proof of Proposition 4.4 for the last section of this paper.

5. Determination of type and cotype of $\mathcal{F}$

This section is devoted to the proof of Proposition 4.4. Recall that the space $\mathcal{F}$ is a complex space defined as the interpolation space in θ of a family of spaces $X_t, t \in \mathbb{R}$, on the left of the border of the strip $\mathcal{S} = \{z \in \mathbb{C} : 0 \leq Re(z) \leq 1\}$, and of a family of copies of the space ℓ_q on the right of the border of $\mathcal{S}$, based on the theory of interpolation of a family of complex norms on $\mathbb{C}^n$ developed in [19, 20]. Here $1 < q < +\infty$ and $0 < \theta < 1$. So it is more adequate to say that in [22] is produced a family of uniformly convex and hereditarily indecomposable examples depending on the parameters θ and q. Each space X_t is quite similar to the HI Gowers-Maurey space GM [30], and this occurs in a uniform way associated to a coding of analytic functions. Since those spaces satisfy approximate lower ℓ_1-estimates, it follows that $\mathcal{F}$ satisfy approximate lower ℓ_p-estimates, where p is defined by the classical interpolation formula $\ell_p \simeq (\ell_1, \ell_q)_\theta$, that is $\frac{1}{p} = (1-\theta)+\theta/q$; this is the main tool used in [22] to prove that $\mathcal{F}$ is uniformly convex.

On the other hand combined results of Androulakis - Schlumprecht [2] and Kutzarova - Lin [35] imply that the space GM contains ℓ_∞^n's uniformly. In what follows we shall prove that this result extends to each space X_t, and furthermore that this happens uniformly in t. We shall then deduce by interpolation methods that the space $\mathcal{F}$ contains arbitrary long sequences satisfying upper ℓ_r estimates, where $\frac{1}{r} = \theta/q$. From this we shall deduce that $\mathcal{F}$ cannot have cotype less than r, whereas by the approximate lower ℓ_p-estimates, it does not have type more than p. Therefore $\mathcal{F}$ may not be near-Hilbert and neither may it be UFO.

As it is also known from the arguments of [35] and [2] that any subspace of GM contains ℓ_∞^n's uniformly, it is probable that our proof would also apply to deduce that no subspace of $\mathcal{F}$ is near-Hilbert and therefore UFO. The same probably holds for the Ferenczi-Habala space which is constructed by the same interpolation method as above, using a variation of the HI space GM.

We shall call (e_n) the standard vector basis of c_{00}, the space of eventually null sequences of scalars. We use the standard notation about successive vectors in c_{00}. In particular the *support* of a vector $x = \sum_i x_i e_i$ in c_{00} is supp $x = \{i \in \mathbb{N} : x_i \neq 0\}$ and the *range* of x is the interval of integers ran $x = [\min(\text{supp } x), \max(\text{supp } x)]$, or $\emptyset$ if $x = 0$. Also, if $x = \sum_i x_i e_i \in c_{00}$ and $E = [m, n]$ is an interval of integers, then Ex denotes the vector $\sum_{i=m}^n x_i e_i$. We recall that Schlumprecht's space S [45] is defined by the implicit equation on c_{00}:

$$\|x\|_S = \|x\|_\infty \vee \sup_{n \geq 2, E_1 < \cdots < E_n} \frac{1}{f(n)} \sum_{k=1}^n \|E_k x\|_S,$$

where $f(x) = \log_2(x+1)$ and $E_1, \ldots, E_n$ are successive intervals of integers. Therefore every finitely supported vector in S is normed either by the sup norm, or by a functional of the form $\frac{1}{f(j)} \sum_{s=1}^j x_s^* : x_1^* < \cdots < x_j^*$ where $j \geq 2$ and each x_s^* belongs to the unit ball $B(S^*)$ of S^*. For $l \geq 2$, we define the equivalent norm $\|\cdot\|_l$ on S by

$$\|x\|_l = \sup_{E_1 < \cdots < E_l} \frac{1}{f(l)} \sum_{j=1}^l \|E_j x\|_S.$$

This norm corresponds to the supremum of the actions of functionals of the form $\frac{1}{l} \sum_{s=1}^j x_s^* : x_1^* < \cdots < x_l^*$ where each x_i^* belongs to $B(S^*)$. It will be useful to observe that if x is a single vector of the unit vector basis of S, then $\|x\|_l = \frac{1}{f(l)}$.

In Gowers-Maurey's type constructions a third term associated to the action of so-called "special functionals" is added in the implicit equation. We proceed to see how this is done in the case of $\mathcal{F}$. For the rest of this paper, $q \in]1, +\infty[$ and $\theta \in]0, 1[$ are fixed, and $p \in]1, q[$ is given by the formula

$$1/p = 1 - \theta + \theta/q.$$

As was already mentioned, the space $\mathcal{F}$ is defined as the interpolation space of two vertical lines of spaces, a line of spaces X_t, $t \in \mathbb{R}$, on the left side of the strip $\mathcal{S}$ and a line of copies of ℓ_q on the right side of it. In [19, 20] the spaces X_t need only be defined for t in a set of measure 1, and for technical reasons in the construction of $\mathcal{F}$ in [22] the X_t's are of interest only for almost every t real. So in what follows, t

will always be taken in some set S_0^∞ of measure 1 which is defined in [22].

Our interest here is on the spaces X_t and their norm $\|\cdot\|_t$. As written in [22, p. 214] we have the following implicit equation for $x \in c_{00}$:

$$\|x\|_t = \|x\|_\infty \vee \sup_{n\geq 2, E_1<\dots<E_n} \frac{1}{f(n)} \sum_{k=1}^{n} \|E_k x\|_t \vee \sup_{G\ special, E} |EG(it)(x)|.$$

The first two terms are similar to Schlumprecht's definition. Now *special analytic functions* G on the strip $\mathcal{S}$ have a very specific form, and special functionals in X_t^*, for $t \in S_0^\infty$, are produced by taking the value $G(it)$ in it of special analytic functions. A special analytic function is of the following form:

$$G = \frac{1}{\sqrt{f(k)}^{1-z} k^{z-z/q}} (G_1 + \ldots + G_k),$$

where each G_j is of the form

$$G_j = \frac{1}{f(n_j)^{1-z} n_j^{z-z/q}} (G_{j,1} + \ldots + G_{j,n_j}),$$

with each $G_{j,m}$ an analytic function on the strip such that $G_{j,m}(it)$ belongs to the unit ball of X_t^* for each t and $G_{j,m}(1+it)$ belongs to the unit ball of $(\ell_q)^*$ for each t. In the above, all analytic functions are finitely supported, i.e. for any analytic function G, there exists an interval of integers E such that supp $G(z) \subset E$ for all z, and therefore it makes sense to talk about successive analytic functions; moreover in the formulas above it is assumed that $G_1 < \dots < G_k$, respectively $G_{j,1} < \dots < G_{j,n_j}$. Furthermore it is required that $n_1 = j_{2k}$ and $n_j = \sigma(G_1, \ldots, G_{j-1})$ for $j \geq 2$, where $j_1, j_2, \ldots$ is the increasing enumeration of a sufficiently lacunary subset J of $\mathbb{N}$ and σ is some injection of some set of finite sequences of analytic functions into J. We refer to [22] for more details. This means that for $t \in S_0^\infty$, any special functional z^* in X_t^*, obtained by the formula $z^* = G(it)$ will have the form

$$z^* = \lambda \frac{1}{\sqrt{f(k)}} (z_1^* + \ldots + z_k^*),$$

where $|\lambda| = 1$ and each $z_j^* \in B(X_t^*)$ has the form

$$z_j^* = \frac{1}{f(n_j)} (z_{j,1}^* + \ldots + z_{j,n_j}^*),$$

with each $z_{j,m}^*$ in the unit ball of X_t^*, with $n_1 = j_{2k}$, and for $j \geq 2$, $n_j = \sigma(G_1, \ldots, G_{j-1})$, where $G_1, \ldots, G_k$ is a sequence of analytic functions on $\mathcal{S}$ such that $G_j(it) = z_j^*$ for each j. Observe that if two initial segments of two sequences $z_1^*, \ldots, z_i^*$ and $z_1^{*\prime}, \ldots, z_j^{*\prime}$ defining two special functionals are different, then the associated initial segments of sequences of analytic functions must be different as well, and therefore by the injectivity of σ the integers n_{i+1} and n'_{j+1} associated to z_{i+1}^* and $z_{j+1}^{*\prime}$ will be different. So the part of classical Gowers-Maurey procedure using coding may be applied in spaces X_t. G. Androulakis and Th. Schlumprecht proved that the spreading model of the unit vector basis of GM is isometric to the unit vector basis of S [2]. Similarly, we shall prove that the unit vector basis of S is "uniformly" the spreading model of the unit vector basis of any X_t. We shall base the proof on a technical lemma, which is essentially Lemma 3.3 from [2], and that we state now but whose proof will be postponed.

For an interval $I \subset \mathbb{N}$ we define

$J(I) = \{\sigma(G_1, \ldots, G_n) : n \in \mathbb{N},\ G_1 < \cdots < G_n,\ \min I \leq \max \mathrm{supp}\ G_n < \max I\}.$

For $z^* \in c_{00}$, we define

$$J(z^*) = J(\mathrm{ran}\ z^*).$$

It will be useful to observe that whenever ran $z^* \subset$ ran w^*, then $J(z^*) \subset J(w^*)$.

Lemma 5.1. *Let $t \in S_0^\infty$. There exists a norming subset B_t of the unit ball of X_t^* such that for any $z^* \in B_t$ there exists $T_0(z^*) \in B(S^*)$, and a family $(T_j(z^*))_{j\in J(z^*)} \subset B(S^*)$ such that*

(1) *for $j \in \{0\} \cup J(z^*)$,* $\mathrm{ran}\ (T_j(z^*)) \subset \mathrm{ran}\ (z^*)$,
(2) *for $j \in J(z^*)$,*

$$T_j(z^*) \in aco\{\frac{1}{f(j)} \sum_{s=1}^{j} x_s^* : x_1^* < \cdots < x_j^* \in B(S^*)\},$$

where "aco" denotes the absolute convex hull,

(3)

$$z^* = T_0(z^*) + \sum_{j\in J(z^*)} T_j(z^*).$$

The proof of Lemma 5.1 is given at the end of the article.

Proposition 5.2. *Let $\epsilon > 0$ and $k \in \mathbb{N}$. Then there exists $N \in \mathbb{N}$ such that for any $t \in S_0^\infty$ and for any $N \leq n_1 < \cdots < n_k$, for any scalars $(\lambda_i)_i$,*

$$\|\sum_{i=1}^{k} \lambda_i e_i\|_S \leq \|\sum_{i=1}^{k} \lambda_i e_{n_i}\|_t \leq (1+\epsilon)\|\sum_{i=1}^{k} \lambda_i e_i\|_S.$$

Proof. It is based on the similar lemma of [2] relating GM to S. The left-hand side inequality is always true by the respective definitions of X_t and S, so we concentrate on the right-hand side. Fix $\epsilon > 0$. We may assume that $\max_i |\lambda_i| = 1$. Since J is lacunary enough we can find M sufficiently large, such that

$$k \sum_{l\in J, l\geq M} \frac{1}{f(l)} < \epsilon.$$

Since σ is injective, there exists an N such that the condition $N \leq \max \mathrm{supp}(G_n)$ guarantees that $\sigma(G_1, \ldots, G_n)$ is at least M. In other words, whenever $l \in J([N, +\infty))$, then $l \geq M$, and therefore

$$k \max_i |\lambda_i| \sum_{I\in J([N,+\infty))} \frac{1}{f(l)} < \epsilon. \ (1)$$

Thus if $N \leq n_1 < \ldots < n_k$, and t is arbitrary in S_0^∞, then by Lemma 5.1,

$$\|\sum_{i=1}^{k} \lambda_i e_{n_i}\|_t \leq \|\sum_{i=1}^{k} \lambda_i e_i\|_S + \sum_{l\in J([N,\infty))} \|\sum_{i=1}^{k} \lambda_i e_{n_i}\|_l$$

$$\leq \|\sum_{i=1}^{k} \lambda_i e_i\|_S + k \max_i |\lambda_i| \sum_{I\in J([N,+\infty))} \frac{1}{f(l)} \leq \|\sum_{i=1}^{k} \lambda_i e_i\|_S + \epsilon.$$

Together with the fact that the basis of S is bimonotone, this concludes the proof. □

Definition 5.3. *A vector in* c_{00} *of the form*
$$\frac{f(m)}{m}\sum_{i\in K} e_i,$$
where $m = |K|$, *will be said to be an* S*-normalized constant coefficients vector.*

It is observed in [35] that such a vector has norm 1 in Schlumprecht's space. In each X_t any sequence of successive vectors satisfies the inequality
$$\frac{1}{f(n)}\sum_{i=1}^{n}\|x_i\|_t \leq \|\sum_{i=1}^{n} x_i\|_t,$$
from which we deduce immediately that any S-normalized constant coefficients vector must have norm at least 1 in each X_t.

Proposition 5.4. *Let* $n \in \mathbb{N}$, $\epsilon > 0$. *There exists a sequence of vectors* $u_1, \ldots, u_n$ *in* c_{00} *such that*

(a) *the support of the* u_j*'s are pairwise disjoint,*
(b) *each* u_j *is an* S*-normalized constant coefficients vector,*
(c) *for any* $t \in S_0^\infty$, *for each* j, $1 \leq \|u_j\|_t \leq 1 + \epsilon$,
(d) *for any* $t \in S_0^\infty$, *any* $a_1, \ldots, a_n$ *of modulus* 1, $\|a_1u_1 + \ldots + a_nu_n\|_t \leq 1 + \epsilon$.

Proof. D. Kutzarova and P. K. Lin [35] have proved that for any $n \geq 1$ there exists a sequence of S-normalized constant coefficient vectors $u_1, \ldots, u_n$ in c_{00} which are disjointly supported and such that $\|u_1 + \ldots + u_n\|_S \leq 1 + \epsilon$. Recall that the basis of S is 1-unconditional, so as well $\|a_1u_1 + \ldots + a_nu_n\|_S \leq 1 + \epsilon$ whenever the a_j's have modulus 1. Since the basis of S is 1-subsymmetric, by Proposition 5.2 we may assume that $u_1, \ldots, u_n$ were taken far enough on the basis to guarantee that $\|u_j\|_t \leq (1+\epsilon)^2$ and $\|a_1u_1 + \ldots + a_nu_n\|_t \leq (1+\epsilon)^2$ for any $t \in S_0^\infty$ and any choice of $a_1, \ldots, a_n$. Since ϵ was arbitrary this proves the result. □

Definition 5.5. *A vector in* c_{00} *of the form*
$$\frac{f(m)^{1-\theta}}{m^{1/p}}\sum_{i\in K} e_i,$$
where $m = |K|$, *will be said to be an* S_θ*-normalized constant coefficients vector.*

Such a vector would have norm 1 in the θ-interpolation space of S and ℓ_q; see [13] where such spaces are studied. For our purposes we shall only use the following fact:

Fact 5.6. *Any* S_θ*-normalized vector* $x \in c_{00}$ *satisfies* $\|x\|_F \geq 1$.

Proof. By [22] Proposition 1, for any successive vectors $x_1 < \cdots < x_m$ in c_{00},
$$\frac{1}{f(m)^{1-\theta}}\big(\sum_{k=1}^{m}\|x_k\|_F^p\big)^{1/p} \leq \|\sum_{k=1}^{m} x_k\|_F \leq \big(\sum_{k=1}^{m}\|x_k\|_F^p\big)^{1/p}.$$
In the case of an S_θ-normalized constant coefficients vector the left-hand side gives the result. □

Proposition 5.7. *Let $n \in \mathbb{N}$, $\epsilon > 0$. There exists a sequence of vectors $v_1, \dots, v_n$ in c_{00} such that*

(a) *the supports of the v_j's are pairwise disjoint,*
(b) *each v_j is an S_θ-normalized constant coefficients vector,*
(c) *for each j, $1 \le \|v_j\|_F \le 1 + \epsilon$,*
(d) *for any $a_1, \dots, a_n$ of modulus 1, $\|a_1 v_1 + \dots + a_n v_n\|_F \le (1+\epsilon) n^{\theta/q}$.*

Proof. Let $u_1, \dots, u_n$ be given by Proposition 5.4. Write each u_j in the form

$$u_j = \frac{f(m_j)}{m_j} \sum_{i \in M_j} e_i,$$

where $m_j = |M_j|$. Consider for each j the analytic function F_j defined on $\mathcal{S}$ by

$$F_j(z) = \frac{f(m_j)^{1-z}}{m_j^{1-z+z/q}} \sum_{i \in M_j} e_i.$$

Let $v_j = F_j(\theta)$ for each j. Observe that

$$v_j = \frac{f(m_j)^{1-\theta}}{m_j^{1/p}} \sum_{i \in M_j} e_i,$$

and therefore v_j is an S_θ-normalized constant coefficients vector and has norm at least 1. Also the v_j's are disjointly supported. Let $a_1, \dots, a_n$ be an arbitrary sequence of complex numbers of modulus 1. Denote $y = a_1 v_1 + \dots + a_n v_n$ and $F = a_1 F_1 + \dots + a_n F_n$. Since $\mathcal{F}$ is an analytic bounded function on S satisfying $F(\theta) = y$, it belongs to the set $\mathcal{A}_\theta(y)$ of analytic functions defined at the beginning of [22] 1.2. By Lemma 1 of [22] the following formula holds for $x \in c_{00}$:

$$\|x\|_F = \inf_{G \in \mathcal{A}_\theta(x)} \Big(\int_R \|G(it)\|_t d\mu_0(t) \Big)^{1-\theta} \Big(\int_R \|G(1+it)\|_q d\mu_1(t) \Big)^{\theta} \qquad (2),$$

where μ_0 and μ_1 are some probability measures on $\mathbb{R}$ whose definitions may be found in [22]. Therefore

$$\|y\|_F \le \Big(\int_R \|F(it)\|_t d\mu_0(t) \Big)^{1-\theta} \Big(\int_R \|F(1+it)\|_q d\mu_1(t) \Big)^{\theta} \qquad (3),$$

Now for any t in S_0^∞,

$$F_j(it) = \frac{f(m_j)}{m_j} \Big(\frac{f(m_j)}{m_j^{1-1/q}} \Big)^{-it} (\sum_{i \in M_j} e_i) = a_{j,t} u_j,$$

where $a_{j,t}$ has modulus 1. Therefore

$$F(it) = \sum_{j=1}^n a_j a_{j,t} u_j,$$

and by Proposition [35],

$$\|F(it)\|_t \le 1 + \epsilon \qquad (4).$$

On the ℓ_q-side we compute that

$$F_j(1+it) = \frac{1}{m_j^{1/q}} \Big(\frac{f(m_j)}{m_j^{1-1/q}} \Big)^{-it} \sum_{i \in M_j} e_i,$$

therefore $\|F_j(1+it)\|_q = 1$, and since the vectors $F_j(1+it)$ are disjointly supported,

$$\|F(1+it)\|_q = n^{1/q}. \qquad (5)$$

Combining (3)(4) and (5),

$$\|y\|_F \leq (1+\epsilon)^{1-\theta} n^{\theta/q} \leq (1+\epsilon) n^{\theta/q}.$$

Applying (2) to each v_j and considering the estimates obtained for $F_j(it)$ and $F_j(1+it)$, we also obtain that

$$\|v_j\|_F \leq (1+\epsilon)^{1-\theta} 1^{\theta} \leq 1+\epsilon.$$

□

We pass to the proof of Proposition 4.4

Proof of Proposition 4.4 First we observe that since $p < q/\theta$ it follows from these estimates that $\mathcal{F}$ can never be near-Hilbert. To prove the estimates, first note that by [22] Proposition 1, for any successive sequence of normalized vectors $x_1, \ldots, x_n$ in $\mathcal{F}$ we have that

$$\|x_1 + \ldots + x_n\| \geq \frac{n^{1/p}}{f(n)^{1-\theta}}.$$

It follows that if $\mathcal{F}$ has type t then $n^{1/p} \leq M f(n)^{1-\theta} n^{1/t}$ for some constant M, and since $\mathcal{F}$ is logarithmic, that $t \leq p$. Therefore $p(F) \leq p$. On the other hand from Proposition 5.7, we see immediately that if $\mathcal{F}$ has cotype c then c must be at least q/θ, so $q(F) \geq q/\theta$. If $q \geq 2$ then it follows from the inequality appearing in [22] Proposition 3 that the modulus of convexity in $\mathcal{F}$ has power type q/θ; from which by results of Figiel and Pisier [39, Thm. 1.e.16] $\mathcal{F}$ has cotype q/θ. If $q \leq 2$ then the inequality in [22] provides modulus of power type $2/\theta$ and therefore cotype $2/\theta$.

It only remains to show that $\mathcal{F}$ has type $[1-(\theta/2)]^{-1}$ in case (1) and p in case (2). So pick n vectors $x_1, \ldots, x_n$ in $\mathcal{F}$ and without loss of generality assume that they are finitely supported and non-zero. By a result of [19], see Theorem 2 of [22], we may find for each x_j an interpolation function F_j such that $F_j(\theta) = x_j$, and such that almost everywhere in t,

$$\|F_j(it)\|_t = \|x_j\| \quad \text{and} \quad \|F_j(1+it)\|_q = \|x_j\|.$$

Fixing $\lambda_j > 0$ for each j, we define

$$G_j(z) = \lambda_j^{z-\theta} F_j(z),$$

and observe that $G_j(\theta) = x_j$ and that almost everywhere in t,

$$\|G_j(it)\|_t = \lambda_j^{-\theta}\|x_j\| \quad \text{and} \quad \|G_j(1+it)\|_q = \lambda_j^{1-\theta}\|x_j\|.$$

Let $\epsilon_j = \pm 1$ for each j. By the formula (2) and this observation we have

$$\|\sum_j \epsilon_j x_j\|_F =\leq \Big(\int_R \|\sum_j \epsilon_j G_j(it)\|_t d\mu_0(t)\Big)^{1-\theta}\Big(\int_R \|\sum_j \epsilon_j G_j(1+it)\|_q d\mu_1(t)\Big)^{\theta}$$

$$\leq \Big(\sum_j \lambda_j^{-\theta}\|x_j\|\Big)^{1-\theta}\Big(\int_R \|\sum_j \epsilon_j G_j(1+it)\|_q d\mu_1(t)\Big)^{\theta}.$$

Therefore

$$\|\sum_j \epsilon_j x_j\|_F^{1/\theta} \leq \Big(\sum_j \lambda_j^{-\theta}\|x_j\|\Big)^{\frac{1-\theta}{\theta}}\Big(\int_R \|\sum_j \epsilon_j G_j(1+it)\|_q d\mu_1(t)\Big),$$

and

$$2^{-n}\sum_{\epsilon_j=\pm1}\|\sum_j \epsilon_j x_j\|_F^{1/\theta} \leq \Big(\sum_j \lambda_j^{-\theta}\|x_j\|\Big)^{\frac{1-\theta}{\theta}}\Big(\int_R 2^{-n}\sum_{\epsilon_j=\pm1}\|\sum_j \epsilon_j G_j(1+it)\|_q d\mu_1(t)\Big).$$

Now it is known that ℓ_q has type $r=\min(2,q)$, therefore there is a constant C_q such that

$$2^{-n}\sum_{\epsilon_j=\pm1}\|\sum_j \epsilon_j x_j\|_F^{1/\theta} \leq C_q\Big(\sum_j \lambda_j^{-\theta}\|x_j\|\Big)^{\frac{1-\theta}{\theta}}\Big(\int_R (\sum_j \|G_j(1+it)\|_q^r)^{1/r} d\mu_1(t)\Big)$$

$$\leq C_q\Big(\sum_j \lambda_j^{-\theta}\|x_j\|\Big)^{\frac{1-\theta}{\theta}}\Big(\sum_j \lambda_j^{(1-\theta)r}\|x_j\|^r\Big)^{1/r}.$$

Picking each λ_j of the form $\|x_j\|^\alpha$, $\alpha\in\mathbb{R}$,

$$\Big(2^{-n}\sum_{\epsilon_j=\pm1}\|\sum_j \epsilon_j x_j\|_F^{1/\theta}\Big)^\theta \leq C_q^\theta\Big(\sum_j \|x_j\|^{1-\alpha\theta}\Big)^{1-\theta}\Big(\sum_j \|x_j\|^{r+\alpha(1-\theta)r}\Big)^{\theta/r}.$$

Choosing α such that

$$1-\alpha\theta = r+\alpha(1-\theta)r,$$

or equivalently

$$\alpha = \frac{1-r}{\theta+(1-\theta)r},$$

we obtain

$$\Big(2^{-n}\sum_{\epsilon_j=\pm1}\|\sum_j \epsilon_j x_j\|_F^{1/\theta}\Big)^\theta \leq C_q^\theta\Big(\sum_j \|x_j\|^{1-\alpha\theta}\Big)^{1-\theta+\theta/r}.$$

Letting $1/m = 1-\theta+\theta/r$, it is immediate by the choice of α that $1-\alpha\theta=m$, and therefore

$$\Big(2^{-n}\sum_{\epsilon_j=\pm1}\|\sum_j \epsilon_j x_j\|_F^{1/\theta}\Big)^\theta \leq C_q^\theta\Big(\sum_j \|x_j\|^m\Big)^{1/m}.$$

Since $1/\theta>1$, by [39, Thm. 1.e.13] this is enough to deduce that $\mathcal{F}$ has type m. Now if $q\leq 2$, then $m=p$ and $\mathcal{F}$ has type p; if $q\geq 2$ then $1/m = 1-\theta/2$ and $\mathcal{F}$ has type $[1-(\theta/2)]^{-1}$. This concludes the proof of the proposition. □

We conclude with the proof of Lemma 5.1.

Proof of Lemma 5.1 It is quite similar to the proof of Lemma 3.3 of [2], up to some change and simplification of notation. By the definition of $\|\cdot\|_t$ in [22], a norming subset B_t of the unit ball of X_t^* is obtained by the following inductive procedure. Let

$$D_1 = \{\lambda_n e_n, n\in\mathbb{N}, |\lambda|\leq 1\}.$$

Given D_{n-1} a subset of c_{00}, let D_n^1 be the set of functionals of the form

$$z^* = E\sum_{i=1}^{l}\alpha_i z_i^*,$$

where $\sum_{i=1}^l |\alpha_i| \le 1$, $z_i^* \in D_{n-1}$ and E is an interval. Let D_n^2 be the set of functionals of the form

$$z^* = E(\frac{1}{f(l)}\sum_{i=1}^l z_i^*),$$

where $z_i^* \in D_{n-1}$, $z_1^* < \cdots < z_n^*$, and E is an interval. Let D_n^3 be the set of functionals of the form

$$z^* = EG(it),$$

where E is an interval of integers and G a special analytic function, therefore

$$G = \frac{1}{\sqrt{f(k)}^{1-z} k^{z-z/q}}(\sum_{i=1}^k G_i), \text{ with } G_i = \frac{1}{f(m_i)^{1-z} m_i^{z-z/q}}(\sum_{j=1}^{m_i} G_{i,j}),$$

where $m_1 = j_{2k}$ and $m_{j+1} = \sigma(G_1, \ldots, G_j)$.

Then let

$$D_n = D_n^1 \cup D_n^2 \cup D_n^3$$

and let

$$B_t = \cup_{n=0}^\infty D_n.$$

The result stated in the lemma will be proved for $z^* \in D_n$ by induction on n. For $n = 0$, that is, $z^* = \lambda e_i^*$ where $|\lambda| = 1$, we have that $J(z^*) = \emptyset$, and we just define $T_0(z^*) = z^*$. Now assuming the conclusion is proved for any functional in D_n, we need to prove it for any z^* in D_{n+1}^1, D_{n+1}^2 or D_{n+1}^3.

When $z^* = 0$ we have $T_0(z^*) = 0$ and define $T_j(z^*) = 0$ for all $j \in J$. Although $J(0) = \emptyset$, and therefore $T_j(0)$ does not appear in the formula of statement of the lemma, notation will be simplified by giving a value to any $T_j(0)$. We now turn our attention to $z^* \neq 0$.

If $z^* \in D_{n+1}^1$, then z^* has the form $E(\sum_{i=1}^l \alpha_i z_i^*)$, where $z_i^* \in D_n$, $\sum_{i=1}^l |\alpha_i| \le 1$ and $E = \operatorname{ran} z^*$. Then we may apply the formula of [2], Lemma 3.3, Case 1, using as they do the fact that $\cup_{i=1}^l J(Ez_i^*) \subset J(z^*)$. That is,

$$T_0(z^*) = \sum_{i=1}^l \alpha_i T_0(Ez_i^*),$$

and

$$T_j(z^*) = \sum_{1\le i\le l, j\in J(Ez_i^*)} \alpha_i T_j(Ez_i^*),$$

for $j \in J(z^*)$ (this sum being possibly 0 if j belongs to no $J(Ez_i^*)$).

If $z^* \in D_{n+1}^2$, that is $z^* = E(\frac{1}{f(l)}\sum_{i=1}^l z_i^*)$, where the z_i^* are successive in D_n, then we observe that once again $\cup_{i=1}^l J(Ez_i^*) \subset J(z^*)$, and also, by the injectivity of σ, that $J(Ez_i^*) \cap J(Ez_s^*) = \emptyset$ whenever $i \neq s$. We therefore may apply the formula of [2], Lemma 3.3, Case 2:

$$T_0(z^*) = \frac{1}{f(l)}\sum_{i=1}^l T_0(Ez_i^*),$$

and

$$T_j(z^*) = \frac{1}{f(l)} T_j(Ez_i^*)$$

when j belongs to some $J(Ez_i^*)$, or

$$T_j(z^*) = 0$$

otherwise.

Finally, if $z^* \in D^3_{n+1}$, then

$$z^* = E(\frac{1}{\sqrt{f(k)}} \sum_{i=1}^{k} z_i^*), \text{ with } z_i^* = \frac{1}{f(m_i)} \sum_{j=1}^{m_i} z_{i,j}^*, ,$$

where $m_1 = j_{2k}$ and $m_{j+1} = \sigma(G_1, \ldots, G_j)$ for $G_1, \ldots, G_l$ associated to $z_1^*, \ldots, z_l^*$ by $z_j^* = G_j(it)$. Let

$$i_1 = \min\{i \in \{1, \ldots, l\} : E \cap \text{supp } (z_i^*) \neq \emptyset\}.$$

By the induction hypothesis, we have

$$\begin{aligned} Ez^* &= \frac{1}{\sqrt{f(l)}} \Big(\frac{1}{f(m_{i_1})} \sum_{j=1}^{m_{i_1}} Ez_{i_1,j}^* + \sum_{i=i_1+1}^{l} \frac{1}{f(m_i)} \sum_{j=1}^{m_i} Ez_{i,j}^* \Big) \\ &= \frac{1}{\sqrt{f(l)}} \frac{1}{f(m_{i_1})} \sum_{j=1}^{m_{i_1}} T_0(Ez_{i_1,j}^*) \\ &\quad + \sum_{i=i_1+1}^{l} \frac{1}{\sqrt{f(l)}} \frac{1}{f(m_i)} \sum_{j=1}^{m_i} T_0(Ez_{i,j}^*) \\ &\quad + \sum_{i=i_1}^{l} \sum_{j=1}^{m_i} \Big(\sum_{k \in J(Ez_{i,j}^*)} \frac{1}{\sqrt{f(l)}} \frac{1}{f(m_i)} T_k(Ez_{i,j}^*) \Big). \end{aligned}$$

We then set

$$T_0(Ez^*) = \frac{1}{\sqrt{f(l)}} \frac{1}{f(m_{i_1})} \sum_{j=1}^{m_{i_1}} T_0(Ez_{i_1,j}^*)$$

and after noting that, by injectivity of σ, $\{m_{i_1+1}, \ldots, m_l\}$ and $J(Ez_{i,j}^*)$, $i = i_1, \ldots, l$, $j = 1, \ldots, m_i$ are mutually disjoint subsets of $J(Ez^*)$ (possibly empty when $Ez_{i,j}^* = 0$), we set

$$T_k(Ez^*) = \frac{1}{\sqrt{f(l)}} \frac{1}{f(m_i)} \sum_{j=1}^{m_i} T_0(Ez_{i,j}^*),$$

if $k = m_i$ for some i in $\{i_1 + 1, \ldots, l\}$,

$$T_k(Ez^*) = \frac{1}{\sqrt{f(l)}} \frac{1}{f(m_i)} T_k(Ez_{i,j}^*),$$

if $k \in J(z_{i,j}^*)$ for some $i \in \{i_1, \ldots, l\}$ and $j \in \{1, \ldots, m_i\}$, and

$$T_k(Ez^*) = 0$$

if $k \in J(Ez^*)$ otherwise. It is then easy to see that the conclusion of the lemma is satisfied. $\square$

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