SINGULAR TWISTED SUMS GENERATED BY COMPLEX INTERPOLATION

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Abstract. We present new methods to obtain singular twisted sums $X \oplus_\Omega X$ (i.e., exact sequences $0 \to X \to X \oplus_\Omega X \to X \to 0$ in which the quotient map is strictly singular), where Ω is the centralizer arising from a complex interpolation schema and X is precisely the interpolation space. We are mainly concerned with the choice of X as either a Hilbert space or Ferenczi's uniformly convex Hereditarily Indecomposable space. In the first case our methods produce new singular twisted Hilbert spaces, of which the only one known was the Kalton-Peck Z_2 space. In the second case we obtain the first example of an H.I. twisted sum of an H.I. space. We then use Rochberg's description of iterated twisted sums to show that there is a sequence $\mathcal{F}_n$ of H.I. spaces so that $\mathcal{F}_{2n}$ is a singular twisted sum of $\mathcal{F}_n$ with itself, while for $n > k$ the space $\mathcal{F}_k \oplus \mathcal{F}_{n+m}$ is a nontrivial twisted sum of $\mathcal{F}_n$ and $\mathcal{F}_{k+m}$.

1. Introduction

For all unexplained notation see the background Sections 2 (exact sequences and quasi-linear maps) and 3 (complex interpolation and centralizers).

This paper focuses on the study of the existence and properties of exact sequences

$$0 \longrightarrow X \xrightarrow{\ j\ } E \xrightarrow{\ q\ } X \longrightarrow 0, \tag{1}$$

in which the Banach space X has been obtained by complex interpolation. The exact sequence will be called *nontrivial* when $j(X)$ is not complemented in the middle space E, which will be called a (nontrivial) twisted sum of X (or a twisting of X, or even a twisted X). The exact sequence will be called *singular* when the operator q is strictly singular. The key example on which all the theory is modeled is the Kalton-Peck twisted Hilbert space Z_2 obtained in [32], which provides the first and only known singular sequence

$$0 \longrightarrow \ell_2 \longrightarrow Z_2 \longrightarrow \ell_2 \longrightarrow 0.$$

In [26] Kalton showed that exact sequences (1) are in correspondence with certain non-linear maps $F : X \to X$, called quasi-linear maps, so, they can be written in the form

$$0 \longrightarrow X \longrightarrow X \oplus_F X \longrightarrow X \longrightarrow 0. \tag{2}$$

As in [10, 14], we will say that a quasi-linear map F is *singular* if the associated exact sequence (2) is singular. In [32] Kalton and Peck refined the quasi-linear method to show an explicit construction of (a special type of quasi-linear maps called) *centralizers* on Banach spaces with unconditional basis. The main example are the so called *Kalton-Peck maps:*

$$\mathcal{K}_\phi(x) = x\phi\left(-\log\frac{|x|}{\|x\|}\right)$$

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where ϕ is a certain Lipschitz map. The choice of the function $\phi_r(t) = t^r$ (when $t \geq 1$), $\phi_r(t) = t$ (when $0 \leq t \leq 1$) and $\phi_r(t) = 0$ (when $t < 0$); with $0 < r \leq 1$ will have a especial interest for us. We will simply write $\mathcal{K}$ instead of $\mathcal{K}_{\phi_1}$. In [32] it is shown that $\mathcal{K}$ is singular on ℓ_p spaces for $1 < p < \infty$; in [14] for $p = 1$; and in [10] for the whole range $0 < p < \infty$. The Kalton-Peck map $\mathcal{K}$ on ℓ_2 can be also obtained as the centralizer generated by the interpolation scale of ℓ_p spaces. Taking this as starting point, Kalton unfolds in [28, 29] the existence of a correspondence between centralizers defined on Köthe function spaces and interpolation scales of Köthe function spaces. This opens the door to the possibility of obtaining nontrivial quasi-linear maps in Banach spaces generated by an interpolation scale, even when no unconditional structure is present. Such is the point of view we adopt in this paper to tackle the study of singular centralizers and singular quasi-linear maps on Banach spaces obtained by complex interpolation. In the case of centralizers this will lead us to obtain new singular twisted Hilbert spaces, and in the case of quasi-linear maps we will obtain the first H.I. twisted sum of an H.I. space.

A description of the contents of the paper is in order: After this introduction and a preliminary Section 2 on basic facts about exact sequences and quasi-linear maps, Section 3 takes root in Kalton's work and so it contains an analysis of centralizers arising from an interpolation schema; the analysis is centered in an interpolation couple (X_0, X_1) and the centralizer Ω_θ obtained at the interpolation space $X_\theta = (X_0, X_1)_\theta$; although the results extend (see subsection 5.3) to cover the case of a measurable family of spaces. We observe, and derive a few consequences, from the fact that such centralizers admit an overall "Kalton-Peck form" as $\Omega_\theta(x) = x \log \frac{a_0(x)}{a_1(x)}$, where $a_0(x)^{1-\theta} a_1(x)^\theta$ is a Lozanovskii factorization of $|x|$ with respect to the couple (X_0, X_1),

Section 4 contains the two fundamental estimates we will use through the paper: Lemma 4.2 (estimate for trivial maps) and Lemma 4.4 (general estimate for centralizers arising from an interpolation schema). Section 5 contains several criteria for singularity based on the previous two lemmata: the first two subsections treat the unconditional case and the third one the conditional case which will be needed to cover H.I. spaces. In Section 6 we obtain new singular twisted Hilbert spaces; we also complete previous results by showing that a certain family of centralizers $\mathcal{K}_\phi$ is singular under rather mild conditions, satisfied in particular by the complex versions [30] of $\mathcal{K}$. In Section 7 we connect the results about singular sequences with the twisting of H.I. spaces: a twisted sum of two H.I. spaces is H.I. if and only if it is singular; then we show that the difficulty of obtaining an H.I. twisting sum is that a nontrivial twisted sum of two H.I. spaces can be decomposable. Note that it was known [23, Theorem 1] that such twisted sums should be at most 2-decomposable. Section 8 applies the previous techniques to the quasi-linear map associated to the construction of Ferenczi's H.I. space $\mathcal{F}$ [21] by complex interpolation of a suitable family of Banach spaces. In Section 9 we complete and improve the results in Sections 7 and 8 by showing that there is a sequence $(\mathcal{F}_n)$ of H.I. spaces so that:

(i) For each $n, m \geq 1$ there is a singular exact sequence

$$0 \longrightarrow \mathcal{F}_n \longrightarrow \mathcal{F}_{n+m} \longrightarrow \mathcal{F}_m \longrightarrow 0.$$

(ii) For each $k, n, m \geq 1$ with $n > k$ there is a nontrivial exact sequence

$$0 \longrightarrow \mathcal{F}_n \longrightarrow \mathcal{F}_k \oplus \mathcal{F}_{n+m} \longrightarrow \mathcal{F}_{m+k} \longrightarrow 0.$$

2. Exact sequences, twisted sums and quasi-linear maps

A *twisted sum* of two Banach spaces Y and Z is a space X which has a subspace M isomorphic to Y with the quotient X/M isomorphic to Z. The space X is a quasi-Banach space in general [32]. Recall that a Banach space is *B-convex* when it does not contain ℓ_1^n uniformly. Theorem 2.6 of [26] implies that a twisted sum of two B-convex Banach spaces is isomorphic to a Banach space.

An exact sequence $0 \to Y \to X \to Z \to 0$ with Y, Z Banach spaces and (bounded) operators is a diagram in which the kernel of each arrow coincides with the image of the preceding one. By the open mapping theorem this means that the middle space X is a twisted sum of Y and Z.

Two exact sequences $0 \to Y \to X_1 \to Z \to 0$ and $0 \to Y \to X_2 \to Z \to 0$ are *equivalent* if there exists an operator $T : X_1 \to X_2$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "i"] \arrow[d, equal] & X_1 \arrow[r, "q"] \arrow[d, "T"] & Z \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r, "j"'] & X_2 \arrow[r, "p"'] & Z \arrow[r] & 0 \end{tikzcd}$$

The classical 3-lemma (see [13, p. 3]) shows that T must be an isomorphism. An exact sequence is said to split if it is equivalent to the trivial sequence $0 \to Y \to Y \oplus Z \to Z \to 0$.

A map $F : Z \to X$ is called quasi-linear if it is homogeneous and there is a constant M such that $\|F(u+v) - F(u) - F(v)\| \le M\|u+v\|$ for all $u, v \in Z$. There is a correspondence (see [13, Theorem 1.5.c, Section 1.6]) between exact sequences $0 \to Y \to X \to Z \to 0$ of Banach spaces and a special kind of quasi-linear maps $\omega : Z \to X$, called z-linear maps, which satisfy $\|\omega(\sum_{i=1}^n u_i) - \sum_{i=1}^n \omega(u_i)\| \le M \sum_{i=1}^n \|u_i\|$ for all finite sets $u_1, \ldots, u_n \in Z$. $0 \to Y \xrightarrow{j} Y \oplus_F Z \xrightarrow{p} Z \to 0$ in which $Y \oplus_F X$ means the vector space $Y \times X$ endowed with the quasi-norm $\|(y, x)\|_F = \|y - F(x)\| + \|x\|$. The embedding is $j(y) = (y, 0)$ while the quotient map is $p(y, z) = z$. When F is z-linear, this quasi-norm is equivalent to a norm [13, Chapter 1]. On the other hand, the process to obtain a z-linear map out from an exact sequence $0 \to Y \xrightarrow{i} X \xrightarrow{q} Z \to 0$ of Banach spaces is the following one: get a homogeneous bounded selection $b : Z \to X$ for the quotient map q, and then a linear $\ell : Z \to X$ selection for the quotient map. Then $\omega = b - \ell$ is a z-linear map. The commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "i"] \arrow[d, equal] & X \arrow[r, "q"] \arrow[d, "T"] & Z \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r, "j"'] & Y \oplus_\omega Z \arrow[r, "p"'] & Z \arrow[r] & 0 \end{tikzcd}$$

obtained by taking as $T : X \to Y \oplus_\omega Z$ the operator $T(x) = (x - \ell qx, qx)$ shows that the upper and lower exact sequences are equivalent. Two quasi-linear maps $F, F' : Z \to Y$ are said to be equivalent, denoted $F \equiv G$, if the difference $F - F'$ can be written as $B + L$, where $B : Z \to Y$ is a homogeneous bounded map and $L : Z \to Y$ is a linear map. Of course that two quasi-linear maps are equivalent if and only if the associated exact sequences are equivalent. Thus, two exact sequences

$$0 \to Y \to Y \oplus_\Omega Z \to Z \to 0 \quad \text{and} \quad 0 \to Y \to Y \oplus_\Psi Z \to Z \to 0$$

(or two quasi-linear maps Ω, Ψ) are equivalent ($\Omega \equiv \Psi$) if there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Y \oplus_\Omega Z & \longrightarrow & Z & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & Y & \longrightarrow & Y \oplus_\Psi Z & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

with $\alpha = id_Y$ and $\gamma = id_Z$. Imposing other conditions on the maps α, β, γ yields other notions of equivalence appeared in the literature:

(1) Projective equivalence [32]: asking α, γ to be scalar multiples of the identity. Equivalently, $\Omega \equiv \mu\Psi$ for some scalar μ.
(2) Isomorphic equivalence [7, 15]: asking α, β, γ to be isomorphisms. In quasi-linear terms, this means that $\alpha\Omega \equiv \Psi\gamma$.
(3) Bounded equivalence [28, 29] (see Section 3 below): asking that $\Omega - \Psi$ is bounded.
(4) We will need in this paper "permutative projective equivalence": asking $T_\sigma\Omega \equiv \mu\Psi T_\sigma$ for some scalar μ and some operator $T_\sigma(\sum_i x_i e_i) = \sum_i x_i e_{\sigma(i)}$ induced by a permutation σ of the integers. When $\mu = 1$ we will just say that Ω and Λ are permutatively equivalent.

A few facts about the connections between quasi-linear maps and the associated exact sequences will be needed in this paper, and can be explicitly found in [16]. Given an exact sequence $0 \to Y \to X \to Z \to 0$ with associated quasi-linear map F and an operator $\alpha : Y \to Y'$, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{i} & X & \xrightarrow{q} & Z & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle T}\big\downarrow & & \big\| & & \\ 0 & \longrightarrow & Y' & \xrightarrow{i'} & X' & \xrightarrow{q'} & Z & \longrightarrow & 0. \end{array} \tag{3}$$

The lower sequence is usually called the push-out sequence, its associated quasi-linear map is (equivalent to) $\alpha \circ F$, and the middle space X' is called the push-out space. When F is z-linear, so is $\alpha \circ F$. Given a commutative diagram like (3) the *diagonal push-out sequence* is the exact sequence generated by the quasi-linear map $F \circ q'$, and is equivalent to the exact sequence

$$0 \longrightarrow Y \xrightarrow{d} Y' \oplus X \xrightarrow{m} X' \longrightarrow 0$$

where $d(y) = (-\alpha y, iy)$ and $m(y', x) = i'y' + Tx$.

3. Complex interpolation and centralizers

Here we explain the connections between complex interpolation, twisted sums and quasi-linear maps that we use throughout the paper.

3.1. **Complex interpolation and twisted sums.** We describe the complex interpolation method for a pair of spaces following [5]. Other general references are [17, 29, 31, 36].

Let $\mathbb{S}$ denote the closed strip $\{z \in \mathbb{C} : 0 \leq \Re z \leq 1\}$ in the complex plane, and let $\mathbb{S}^\circ$ be its interior. Given an admissible pair (X_0, X_1) of complex Banach spaces, we denote by $\mathcal{H} = \mathcal{H}(X_0, X_1)$ the space of functions $g : \mathbb{S} \to \Sigma := X_0 + X_1$ satisfying the following conditions:

(1) g is $\|\cdot\|_\Sigma$-bounded and $\|\cdot\|_\Sigma$-continuous on $\mathbb{S}$, and $\|\cdot\|_\Sigma$-analytic on $\mathbb{S}^\circ$;
(2) $g(it) \in X_0$ for each $t \in \mathbb{R}$, and the map $t \in \mathbb{R} \mapsto g(it) \in X_0$ is bounded and continuous;
(3) $g(it+1) \in X_1$ for each $t \in \mathbb{R}$, and the map $t \in \mathbb{R} \mapsto g(it+1) \in X_1$ is bounded and continuous;

The space $\mathcal{H}$ is a Banach space under the norm $\|g\|_{\mathcal{H}} = \sup\{\|g(j+it)\|_j : j = 0,1; t \in \mathbb{R}\}$. For $\theta \in [0,1]$, define the interpolation space

$$X_\theta = (X_0, X_1)_\theta = \{x \in \Sigma : x = g(\theta) \text{ for some } g \in \mathcal{H}\}$$

with the norm $\|x\|_\theta = \inf\{\|g\|_H : x = g(\theta)\}$. So $(X_0, X_1)_\theta$ is the quotient of $\mathcal{H}$ by $\ker \delta_\theta$, and thus it is a Banach space.

For $0 < \theta < 1$, we will consider the maps $\delta_\theta^n : \mathcal{H} \to \Sigma$ –evaluation of the n^{th}-derivative at θ– that appear in Schechter's version of the complex method of interpolation [37]. Note that $\delta_\theta \equiv \delta_\theta^0$ is bounded by the definition of $\mathcal{H}$, and this fact and the Cauchy integral formula imply the boundedness of δ_θ^n for $n \geq 1$ (see also [9]). We will also need the following result (see [12, Theorem 4.1]):

Lemma 3.1. $\delta'_\theta : \ker \delta_\theta \to X_\theta$ *is bounded and onto for* $0 < \theta < 1$.

Lemma 3.1 provides the connection with exact sequences and twisted sums through the following push-out diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \delta_\theta & \xrightarrow{i_\theta} & \mathcal{H} & \xrightarrow{\delta_\theta} & X_\theta & \longrightarrow & 0 \\ & & {\scriptstyle \delta'_\theta}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & X_\theta & \longrightarrow & \text{PO} & \longrightarrow & X_\theta & \longrightarrow & 0 \end{array} \tag{4}$$

whose lower row is obviously a twisted sum of X_θ.

Apart from the obvious description as a push-out space, PO can be represented as:

(1) *A twisted sum space.* Let $B_\theta : X_\theta \to \mathcal{H}$ be a bounded homogeneous selection for δ_θ, and let $L_\theta : X_\theta \to \mathcal{H}$ be a linear selection. The map $\omega_\theta = B_\theta - L_\theta : X_\theta \to \ker \delta_\theta$ is an associated quasi-linear for the upper sequence in diagram (4). The lower push-out sequence will then comes defined by the quasi-linear map $\delta'_\theta \omega_\theta$. Hence, $\text{PO} \simeq X_\theta \oplus_{\delta'_\theta \omega_\theta} X_\theta$.

(2) *A derived space.* With the same notation as above, set

$$d_{\delta'_\theta B_\theta}(X_\theta) = \{(y,z) \in \Sigma \times \Sigma : \, z \in X_\theta, \, y - \delta'_\theta B_\theta z \in X_\theta\}$$

endowed with the quasi-norm $\|(y,z)\|_d = \|y - \delta'_\theta B_\theta z\|_{X_\theta} + \|z\|_{X_\theta}$. This is a twisted sum of X_θ since the embedding $y \to (y,0)$ and quotient map $(y,z) \to z$ yield an exact sequence

$$0 \longrightarrow X_\theta \longrightarrow d_{\delta'_\theta B_\theta}(X_\theta) \longrightarrow X_\theta \longrightarrow 0.$$

Moreover, the two exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_\theta & \longrightarrow & X_\theta \oplus_{\delta'_\theta \omega_\theta} X_\theta & \longrightarrow & X_\theta & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle T} & & \| & & \\ 0 & \longrightarrow & X_\theta & \longrightarrow & d_{\delta'_\theta B_\theta}(X_\theta) & \longrightarrow & X_\theta & \longrightarrow & 0. \end{array}$$

are isometrically equivalent via the isometry $T(y,z) = (y + \delta'_\theta L_\theta z, z)$.

Thus, we can pretend that the quasi-linear map naturally associated to the push-out sequence is $\delta'_\theta B_\theta$, sometimes much more intuitive than the true quasi-linear map $\delta'_\theta(B_\theta - L_\theta)$. Such map has been sometimes called "the Ω-operator". Needless to say, the Ω-operator depends on the choice of B_θ. The difference between two associated Ω-operators must be bounded:

$$\|\delta'_\theta(\tilde{B}_\theta - B_\theta)x\|_{X_\theta} \leq \|\delta'_{\theta|\ker \delta_\theta}\|(\|\tilde{B}_\theta\| + \|B_\theta\|)\|x\|_{X_\theta}.$$

The derived space admits the following useful representation (see [36, p.323] for an embryonic finite-dimensional version; also quoted in [18, p.218]; see [12, Prop.7.1] for a general version involving two compatible interpolators and [9] for a rather complete exposition, variations and applications of that representation.

Proposition 3.2.

$$d_{\delta'_\theta B_\theta}(X_\theta) = \{\big(f'(\theta), f(\theta)\big) \,:\, f \in \mathcal{H}\},$$

and the quotient norm of $\mathcal{H}/(\ker\delta_\theta \cap \ker\delta'_\theta)$ *is equivalent to the quasi-norm* $\|(\cdot,\cdot)\|_d$.

Proof. That $\big(f'(\theta), f(\theta)\big) \in d_{\delta'_\theta B_\theta}(X)$ is clear: since $f - B_\theta(f(\theta)) \in \ker\delta_\theta$, by Lemma 3.1 one has

$$f'(\theta) - \delta'_\theta B_\theta(f(\theta)) = \delta'_\theta(f - B_\theta(f(\theta))) \in X_\theta.$$

Conversely, let $(y, z) \in d_{\delta'_\theta B_\theta}(X)$. We have $z \in X_\theta$, so $B_\theta z \in \mathcal{H}$. Since $y - \delta'_\theta B_\theta z \in X_\theta$, there exists $g \in \ker\delta_\theta$ such that $y - \delta'_\theta B_\theta z = g'(\theta)$. Thus taking $f = B_\theta z + g$ we have $f(\theta) = z$ and $f'(\theta) = y$, and the equality is proved.

For the equivalence, given $(y, z) \in d_{\delta'_\theta B_\theta}(X)$, take $f \in \mathcal{H}$ with $\|f\| \leq 2\,\mathrm{dist}(f, \ker\delta_\theta \cap \ker\delta'_\theta)$ such that $y = f'(\theta)$ and $z = f(\theta)$. Then $\|z\|_{X_\theta} = \mathrm{dist}(f, \ker\delta_\theta)$ and

$$\|y - \delta'_\theta B_\theta z\|_{X_\theta} = \|\delta'_\theta(f - B_\theta z)\|_{X_\theta}.$$

Since $f - B_\theta z \in \ker\delta_\theta$, we get

$$\|(y,z)\|_d \leq \|\delta'_{\theta|\ker\delta_\theta}\|(1 + \|B_\theta\|)\|f\| + \|f\| \leq 2(\|\delta'_{\theta|\ker\delta_\theta}\|(1 + \|B_\theta\|) + 1)\,\mathrm{dist}(f, \ker\delta_\theta \cap \delta'_\theta),$$

and there exists a constant C so that $\mathrm{dist}(f, \ker\delta_\theta \cap \delta'_\theta) \leq C\|(y,z)\|_d$ by the open-mapping theorem. □

Remark 3.3. In Section 8 we will need to consider the complex interpolation method associated to a family $(X_{(0,t)}, X_{(1,t)})_{t\in\mathbb{R}}$ of complex Banach spaces given in [17]. For this method the results mentioned here remain valid because it is a special case of the general method of interpolation considered in [31, Section 10].

3.2. **Centralizers.** Here we consider Köthe function spaces X over a measure space (Σ, μ) with their L_∞-module structure. As a particular case, we have Banach spaces with a 1-unconditional basis with their associated ℓ_∞-structure. We denote by L_0 the space of all μ-measurable functions, and given $g \in L_0$, we understand that $\|g\|_X < \infty$ implies $g \in X$.

Definition 1. *A centralizer on a Köthe function space* X *is a homogeneous map* $\Omega : X \to L_0$ *such that* $\|\Omega(ax) - a\Omega(x)\|_X \leq C\|x\|_X\|a\|_\infty$ *for all* $a \in L_\infty$ *and* $x \in X$.

A centralizer on X will be denoted by $\Omega : X \curvearrowright X$. This notion coincides with that of Kalton's "strong centralizer" introduced in [28].

Centralizers arise naturally in a complex interpolation scheme in which the interpolation scale of spaces share a common L_∞-module structure: in such case, the space $\mathcal{H}$ also enjoys the same L_∞-module structure in the form $(u \cdot f)(z) = u \cdot f(z)$. In this way, the fundamental sequence of the interpolation scheme $0 \to \ker\delta_\theta \to \mathcal{H} \to X_\theta \to 0$ is an exact sequence in the category of L_∞-modules. In an interpolation scheme starting with a couple (X_0, X_1) of Köthe function spaces, the map ${\delta_\theta}' B_\theta$ is a centralizer on X_θ. We will denote it by Ω_θ.

For a centralizer $\Omega : X \curvearrowright X$ on a Köthe function space X, it was proved in [28, Lemma 4.2] that there exists $M > 0$ such that $\|\Omega(u+v) - \Omega(u) - \Omega(v)\|_X \leq M(\|u\|_X + \|v\|_X)$. So we can assume that Ω is a quasi-linear map. The smallest of the constants M above will be called

$\rho(\Omega)$. For example, $\Omega : X \curvearrowright X$ induces an exact sequence in the category of (quasi-)Banach L_∞-modules $0 \to X \to d_\Omega(X) \to X \to 0$, where

$$d_\Omega(X) = \{(w, z) \in L_0 \times X : w - \Omega z \in X\}$$

endowed with the quasi-norm $\|(w, z)\|_\Omega = \|w - \Omega z\|_X + \|z\|_X$; with embedding $y \to (y, 0)$ and quotient map $(w, z) \to z$. The derived space $d_\Omega(X)$ admits a L_∞-module structure defined by $a(w, z) = (aw, az)$. Kalton proved in [28, Section 4] that every self-extension of a Köthe function space X is (equivalent to) the extension induced by a centralizer on X. Sometimes we will take the restriction of Ω to a closed subspace Y of X, and consider $d_\Omega(X, Y)$ defined in the same way as a subspace of $L_0 \times Y$.

A centralizer $\Omega : X \curvearrowright X$ is said to be *bounded* when it takes values in X and $\|\Omega(x)\| \leq C\|x\|$ for all $x \in X$. Two centralizers $\Omega_1 : X \curvearrowright X$ and $\Omega_2 : X \curvearrowright X$ are equivalent if and only if the induced exact sequences are equivalent, which happens if and only if there exists a linear map $L : X \to L_0$ so that $\Omega_1 - \Omega_2 - L$ is bounded. Two centralizers $\Omega_1 : X \curvearrowright X$ and $\Omega_2 : X \curvearrowright X$ are said to be *boundedly* equivalent when $\Omega_1 - \Omega_2$ is bounded. The interest in this notion (which, to some extent, plays the role of triviality for quasi-linear maps) stems from the following outstanding result of Kalton [29, Theorem 7.6]:

Theorem 3.4. *Let X be a separable superreflexive Köthe function space. Then there exists a constant c (depending on the concavity of a q-concave renorming of X) such that if $\Omega : X \curvearrowright X$ is a real centralizer on X with $\rho(\Omega) \leq c$, then*

1. *There is a pair of Köthe function spaces X_0, X_1 such that $X = (X_0, X_1)_{1/2}$ and $\Omega - \Omega_{1/2}$ is bounded.*
2. *The spaces X_0, X_1 are uniquely determined up to equivalent renorming.*

An example is in order: taking the couple (ℓ_1, ℓ_∞), the map $B(x) = x^{2(1-z)}$ is a homogeneous bounded selection for the evaluation map $\delta_{1/2} : \mathcal{H} \to \ell_2$; hence the interpolation procedure yields the centralizer $-2\mathcal{K}$; while the couple (ℓ_p, ℓ_{p^*}) yields $-2(\frac{1}{p} - \frac{1}{p^*})\mathcal{K}$. As we see the two centralizers are the same up to the scalar factor. Theorem 3.4 shows however that the scalar factor cannot be overlooked: it actually determines the end points X_0, X_1 in the interpolation scale (see also Proposition 3.8). We note for future use that the condition on $\rho(\Omega)$, which is necessary for existence, is not necessary for uniqueness:

Proposition 3.5. *Let X be a separable superreflexive Köthe function space. Assume that $X = (X_0, X_1)_\theta = (Y_0, Y_1)_\theta$, where $0 < \theta < 1$ and X_i, Y_i are Köthe function spaces, and that the associated maps Ω_X and Ω_Y are boundedly equivalent. Then $X_0 = Y_0$ and $X_1 = Y_1$.*

Proof. Following Kalton's notation and proof, since Ω_X and Ω_Y are boundedly equivalent, $\Omega_X^{[1]}$ and $\Omega_Y^{[1]}$ are boundedly equivalent. Hence on a suitable strict semi-ideal, Φ^{Ω_X} is equivalent to $\Phi_{Y_1} - \Phi_{Y_0}$, while $(1-\theta)\Phi_{Y_0} + \theta\Phi_{Y_1}$ is equivalent to Φ_X . Thus, up to equivalence Φ_{Y_0} and Φ_{Y_1} are uniquely determined. [29, Proposition 4.5] shows then that the spaces Y_0 and Y_1 are unique up to equivalence of norm. □

3.3. **Centralizers and Lozanovskii's decomposition.** Here we obtain a formula for the centralizer Ω_θ attached to the interpolation of a couple of Köthe function spaces (X_0, X_1).

Let $0 < \theta < 1$, and suppose that one of the spaces X_0, X_1 has the Radon-Nikodym property. The Lozanovskii decomposition formula allows us to show (see [31, Theorem 4.6]) that the complex interpolation space X_θ is isometric to the space $X_0^{1-\theta} X_1^\theta$, with

$$\|x\|_\theta = \inf\{\|y\|_0^{1-\theta}\|z\|_1^\theta : y \in X_0, z \in X_1, |x| = |y|^{1-\theta}|z|^\theta\}.$$

By homogeneity we may always assume that $\|y\|_0 = \|z\|_1$ for y, z in this infimum. When $\|y\|_0, \|z\|_1 \leq K\|x\|_\theta$ we shall say that $|x| = |y|^{1-\theta}|z|^\theta$ is a *K-optimal decomposition* for x. When x is finitely supported or X is uniformly convex a 1-optimal (or simply, optimal) decomposition may be achieved. A simple choice of $B_\theta(x)$ can be made for positive x as follows: Let $a_0(x), a_1(x)$ be a $(1+\epsilon)$-optimal (or optimal when possible) Lozanovskii decomposition for x. Since $\|x\|_\theta = \|a_0(x)\|_0 = \|a_1(x)\|_1$, set $B_\theta(x) \in \mathcal{H}$ given by $B_\theta(x)(z) = |a_0(x)|^{1-z}|a_1(x)|^z$. One thus gets for positive x the formula:

$$\Omega_\theta(x) = \delta'_\theta B_\theta(x) = |a_0(x)|^{1-\theta}|a_1(x)|^\theta \log \frac{|a_1(x)|}{|a_0(x)|} x = x \log \frac{|a_1(x)|}{|a_0(x)|}. \tag{5}$$

Using $B_\theta(x) = (\operatorname{sgn} x)B_\theta(|x|)$ for general x one still gets

$$\Omega_\theta(x) = x \log \frac{|a_1(x)|}{|a_0(x)|}.$$

Recall that a unit $u \in L_\infty$ is an element which only takes the values ± 1. Thus one has:

Lemma 3.6. *The centralizer $\Omega_\theta = \delta'_\theta B_\theta$ on $X_\theta = (X_0, X_1)_\theta$ satisfy the following properties:*

1. *$\Omega_\theta(ux) = u\Omega_\theta(x)$ for every unit u and $x \in X_\theta$;*
2. *$\operatorname{supp} \Omega_\theta(x) \subset \operatorname{supp} x$ for every $x \in X_\theta$;*
3. *when X_1 and X_2 are spaces with an unconditional basis (e_n), $\Omega_\theta(e_n) = 0$ for all n.*

The Lozanovskii approach can be used to make explicit the Kalton correspondence between centralizers and interpolation scales in some special cases: Recall that the p-convexification of a Köthe function space X is defined by the norm $|||x||| = |||x|^p||^{1/p}$. Conversely, when X is p-convex, the p-concavification of X is given by $|||x||| = |||x|^{1/p}||^p$. Modulo the fact that every uniformly convex space may be renormed to be p-convex for some $p > 1$, the following proposition interprets Kalton-Peck maps defined on uniformly convex spaces as induced by specific interpolation schemes.

Proposition 3.7. *Let $0 < \theta < 1 < p < \infty$, and let X be a Banach space with unconditional basis (respectively a Köthe function space). Then $X_\theta = (\ell_\infty, X)_\theta$ (respectively $(L_\infty(\mu), X)_\theta$) is the θ^{-1}-convexification of X, and the induced centralizer on X_θ is*

$$\Omega(x) = \theta^{-1}\, x \log(|x|/||x||_\theta).$$

Conversely if X is p-convex and X^p is the p-concavification of X then $X = (\ell_\infty, X^p)_{1/p}$ (respectively $X = (L_\infty(\mu), X^p)_{1/p}$), and the induced centralizer is defined on X by

$$\Omega(x) = p\, x \log(|x|/||x||).$$

Proof. We write down the proof for unconditional basis, the other being analogous. For normalized positive x in X_θ, write $x = a_0(x)^{1-\theta}a_1(x)^\theta$ and look for such a (normalized) decomposition which is optimal. Since $a_0(x) \in \ell_\infty$, $a_0(x)$ will have constant coefficients equals to 1 on the support of x: otherwise, we may increase the non 1 coordinates of $a_0(x)$ to 1, therefore diminishing the corresponding coordinates of $a_1(x)$ and non-increasing the norm of $a_1(x)$ by 1-unconditionality, and still get something optimal. So $a_0(x) = 1_{\text{supp}}(x)$ and $x = a_1(x)^\theta$. Therefore $\|x\|_\theta = \|a_1(x)\|^\theta = \|x^{1/\theta}\|^\theta$. So X_θ is the θ^{-1}-convexification of X and

$$\Omega_\theta(x) = x \log(a_1(x)/a_0(x)) = \frac{1}{\theta}\, x \log(x).$$

As for the converse, note that when we interpolate ℓ_∞ and some Y we have $|a_1(x)| = |x|^p$ for x normalized in Y_θ, so if we interpolate ℓ_∞ and $Y = X^{(p)}$ we obtain for such x

$$\|x\|_{Y_\theta} = 1 = \|a_1(x)\|_Y = \||x|^p\|_Y = \|(|x|^p)^\theta)\|_X^p = \|x\|_X^p,$$

therefore $X = Y_\theta = (\ell_\infty, X^{(p)})_\theta$.

The second part of the proposition is an immediate consequence of the first one. $\square$

The following result shows the reason behind the constant factor which appears multiplying a centralizer.

Proposition 3.8. *Let (X_0, X_1) be an admissible pair of Köthe function spaces and for $0 < \alpha < \beta < 1$, consider also the admissible pair (X_α, X_β). Let Ω (resp. Ω') denote the centralizers generated by the couple (X_0, X_1) (resp. (X_α, X_β)). Assume that for some $\alpha < \theta < \beta$ and $0 < \rho < 1$ one has $(X_0, X_1)_\theta = (X_\alpha, X_\beta)_\rho$. Then $\Omega'_\rho = (\beta - \alpha)\Omega_\theta$.*

Proof. It is easy to check (see [31, Theorem 4.5]) that ρ is given by $\alpha(1-\rho) + \beta\rho = \theta$. Let us consider the centralizers

$$\Omega_\theta(x) = x \log \frac{|a_1(x)|}{|a_0(x)|} \quad \text{and} \quad \Omega'_\rho(x) = x \log \frac{|a_\beta(x)|}{|a_\alpha(x)|}.$$

Since $x = a_0(x)^{1-\theta} a_1(x)^\theta$, $1 - \theta = (1-\alpha)(1-\rho) + (1-\beta)\rho$ and $\theta = \alpha(1-\rho) + \beta\rho$ we get

$$x = \big(a_0(x)^{1-\alpha} a_1(x)^\alpha\big)^{1-\rho} \big(a_0(x)^{1-\beta} a_1(x)^\beta\big)^\rho.$$

Thus taking $a_\alpha(x) = a_0(x)^{1-\alpha} a_1(x)^\alpha$ and $a_\beta(x) = a_0(x)^{1-\beta} a_1(x)^\beta$ it is not difficult to check that the minimality of $x = a_0(x)^{1-\theta} a_1(x)^\theta$ implies the minimality of $x = a_\alpha(x)^{1-\rho} a_\beta(x)^\rho$, and the equality $\Omega'_\rho(x) = (\beta - \alpha)\Omega_\theta(x)$ follows from the properties of the logarithm function. $\square$

Next we describe the centralizers associated to Orlicz function spaces over a measure space (Σ, μ). Recall that an N-function is a map $\varphi : [0, \infty) \to [0, \infty)$ which is strictly increasing, continuous, $\varphi(0) = 0$, $\varphi(t)/t \to 0$ as $t \to 0$, and $\varphi(t)/t \to \infty$ as $t \to \infty$. An N-function φ satisfies the Δ_2-property if there exists a number $C > 0$ such that $\varphi(2t) \leq C\varphi(t)$ for all $t \geq 0$. For $1 < p < \infty$, $\varphi(t) = t^p$ is N-function satisfying the Δ_2-property.

When an N-function φ satisfies the Δ_2-property, the Orlicz space $L_\varphi(\mu)$ is given by

$$L_\varphi(\mu) = \{f \in L_0(\mu) : \varphi(|f|) \in L_1(\mu)\}.$$

The following result was proved in [25], and a clear exposition can be found in [11].

Proposition 3.9. *Let φ_0 and φ_1 be two N-functions satisfying the Δ_2-property, and let $0 < \theta < 1$. Then the formula $\varphi^{-1} = \big(\varphi_0^{-1}\big)^{1-\theta} \big(\varphi_1^{-1}\big)^\theta$ defines an N-function φ satisfying the Δ_2-property, and $\big(L_{\varphi_0}(\mu), L_{\varphi_1}(\mu)\big)_\theta = L_\varphi(\mu)$.*

Next we give an expression for the centralizer associated to a Hilbert space obtained by complex interpolation of Orlicz spaces. Note that once we have defined a centralizer Ω for non-zero $0 \leq f \in X$, we can define $\Omega(0) = 0$ and $\Omega(g) = g \cdot \Omega(|g|/\|g\|)$ for $0 \neq g \in X$.

Proposition 3.10. *Let φ_0 and φ_1 be two N-functions satisfying the Δ_2-property and such that $t = \varphi_0^{-1}(t) \cdot \varphi_1^{-1}(t)$. Then $\big(L_{\varphi_0}(\mu), L_{\varphi_1}(\mu)\big)_{1/2} = L_2(\mu)$ and the induced centralizer is*

$$\Omega_{1/2}(f) = f \log \frac{\varphi_1^{-1}(f^2)}{\varphi_0^{-1}(f^2)} = 2f \log \frac{\varphi_1^{-1}(f^2)}{f} \quad (0 \leq f \in L_2(\mu), f \neq 0).$$

Proof. First we consider the general case $\varphi^{-1} := \big(\varphi_0^{-1}\big)^{1-\theta} \big(\varphi_1^{-1}\big)^\theta$, as in Proposition 3.9. For $0 \leq f \in L_\varphi(\mu)$ we can write $f = \big(\varphi_0^{-1}\varphi(f)\big)^{1-\theta} \big(\varphi_1^{-1}\varphi(f)\big)^\theta$. Thus a selection of the quotient map $\mathcal{H} \to L_\varphi(\mu)$ is given by $B_\theta(f)(z) = \big(\varphi_0^{-1}\varphi(f)\big)^{1-z} \big(\varphi_1^{-1}\varphi(f)\big)^z$. Differentiating $B_\theta(f)'(z) = B_\theta(f)(z) \log \frac{|\varphi_1^{-1}(\varphi(f))|}{|\varphi_0^{-1}\varphi(f)|}$, hence $\Omega_{1/2}(f) = B_{1/2}(f)'(1/2) = f \log \frac{|\varphi_1^{-1}(\varphi(f))|}{|\varphi_0^{-1}\varphi(f)|}$, which gives the desired result when $\varphi(t) = t^2$. $\square$

3.4. **Additional properties.** The properties of Ω_θ obtained in Lemma 3.6 will turn out essential for our estimates, so they deserve a definition.

Definition 2. *Let X be a Köthe function space. A centralizer $\Omega : X \curvearrowright X$ will be called* exact *if for each $x \in X$ and every unit u one has $\Omega(ux) = u\Omega x$. It will be called* contractive *if* $\operatorname{supp} \Omega(x) \subset \operatorname{supp} x$ *for every $x \in X$.*

One has:

Lemma 3.11. *Let X be a Köthe function space.*
1. *Every exact quasi-linear map on X is contractive.*
2. *If X is complemented in its bidual, then every exact trivial centralizer Ω on X admits an exact linear map Λ such that $\Omega - \Lambda$ is bounded.*
3. *If X has unconditional basis (e_n) and is complemented in its bidual, and if Ω is exact and trivial on X, and such that $\Omega(e_n) = 0$, then Ω is bounded.*

Proof. (1) Let $u \in L_\infty$ be the function with value 1 on the support of x and -1 elsewhere, then $ux = x$, therefore $u\Omega(x) = \Omega(ux) = \Omega(x)$ which means that the support of $\Omega(x)$ is included in supp (x).

(2) Let Ω be a centralizer with constant C and assume it is trivial, some linear map $\ell : X \to L_0$ is such that $B := \Omega - \ell$ is bounded. Let U denotes the abelian group of units. It is therefore amenable, so there exists a left invariant finitely additive mean m on U allowing to define for any bounded $f : U \to \mathbb{R}$ an integral $\int_U f(u)dm$. Since X is complemented in its bidual we may then define for any bounded $f : U \to X$ an element $\int_U f(u)dm \in X$. One can therefore define a map $\Lambda : X \to L_0$ as follows:

$$\Lambda(x) = \Omega(x) - \int_U uB(ux)dm.$$

By exactness of Ω and invariance of m, we have that Λ is exact. It is also easy to check that Λ is linear. Indeed, denoting by $\Delta(x, y)$ the element $\Omega(x+y)-\Omega x-\Omega y = B(x+y)-Bx-By \in X$, and observing that $\Delta(ux, uy) = u\Delta(x, y)$, we obtain

$$\begin{aligned} \Omega(x+y) - \Lambda(x+y) &= \textstyle\int_U uB(ux+uy)dm \\ &= \textstyle\int_U u\Delta(ux, uy)dm + \int_U uBuxdm + \int_U uBuydm \\ &= \Delta(x, y) + \Omega(x) - \Lambda(x) + \Omega(y) - \Lambda(y) \\ &= \Omega(x+y) - \Lambda(x) - \Lambda(y). \end{aligned}$$

(3) We claim that $\Lambda(x) = ax$ for all $x \in X$, where $\Lambda(e_n) = a_n e_n$. Indeed

$$\Lambda(x) = \Lambda(x - x_n e_n) + \Lambda(x_n e_n) = \Lambda(x - x_n e_n) + a_n x_n e_n$$

which, since $\Lambda(x - x_n e_n)$ has support disjoint from n, implies that the n-th entry of $\Lambda(x)$ is $a_n x_n$. Since $\Omega(e_n) = 0$, $a_n e_n = -B(e_n)$, and therefore $(a_n)_n$ is a bounded sequence. So unconditionality applies to make Λ bounded. Since $\Omega - \Lambda$ is also bounded, Ω is bounded. □

A reformulation of (3) will provide us in due time a criterion to distinguish between permutatively projectively equivalent centralizers:

Corollary 3.12. *Let Ω and Ψ be exact centralizers on a reflexive space X with* 1*-unconditional basis (e_n), and such that $\Omega(e_n) = \Psi(e_n) = 0$ for all $n \in \mathbb{N}$. If Ω and Λ are equivalent then they are boundedly equivalent.*

Proof. $\Omega - \Lambda$ is still an exact centralizer vanishing on the e_n. Thus, if it is trivial then it is bounded. □

Lemma 3.11 can be generalized for maps between two different modules. We are interested in the particular case in which one has to combine two related actions: let X be an L_∞-Banach module and let $W \subset X$ be a subspace generated by disjointly supported elements $W = [u_n]$. Consider in this case the subspace $L_\infty^W \subset L_\infty$ formed by the elements which are constant on the supports of all the u_n. Let U_W be its group of units. We say that a map $\Omega : W \to X$ is *relatively exact* if $\Omega(ux) = u\Omega(x)$ for all $u \in U_W$ and $x \in W$, and we say that Ω is *relatively contractive* if $\mathrm{supp}_X \Omega(x) \subset \mathrm{supp}_X x$, for all $x \in W$. One has:

Lemma 3.13. *Let X be a Köthe function space, and let W be a subspace of X generated by disjointly supported elements. Then:*

(1) *If $\Omega : X \curvearrowright X$ is a exact centralizer then the restriction $\Omega_{|W}$ is relatively exact.*
(2) *Every relatively exact map $W \curvearrowright X$ is relatively contractive.*
(3) *Assume X is complemented in its bidual. If some relatively exact $\Omega : W \curvearrowright X$ is trivial then there exists a relatively exact linear map $\Lambda : W \to X$ such that $\Omega - \Lambda$ is bounded.*

Proof. Assertion (1) is obvious, (2) has the same proof as before. For (3), assuming $\Omega = B + \ell$, where $B : W \to X$ is bounded and $\ell : W \to L_0$ is linear, define for $x \in W$,

$$\Lambda(x) = \Omega(x) - \int_{U_W} uB(ux)dm,$$

where m is a left invariant finitely additive mean on U_W. □

Lemma 3.14.

(1) *Every centralizer Ω on a Köthe function space admits a exact centralizer ω such that $\Omega - \omega$ is bounded.*
(2) *Every exact centralizer (resp. quasi-linear map) Ω between Banach spaces with unconditional basis admits a exact centralizer (resp. quasi-linear map) ω such that $\omega(e_n) = 0$ and $\Omega - \omega$ is linear and exact.*
(3) *Every contractive centralizer (resp. quasi-linear map) Ω between Köthe function spaces admits, for every sequence (f_n) of disjointly supported vectors, a contractive centralizer (resp. quasi-linear map) ω such that $\omega(f_n) = 0$ and $\Omega - \omega$ is linear and contractive.*

Proof. Assertion (1) is in [28, Prop. 4.1]. In fact, $\omega(x) = \|x\| \,\mathrm{sgn}(x)\Omega(|x|/\|x\|)$ for $x \neq 0$. To prove (2), note that since Ω is contractive, $\Omega(e_n) = \mu_n e_n$, and we may define the multiplication linear map $\ell(x) = \mu x$, where $\mu = (\mu_n)_n$. Thus $\omega = \Omega - \ell$ is the desired map. To prove (3), define as above a linear map by $\ell(f_n) = \Omega(f_n)$. If Ω is contractive, so is ℓ and thus $\omega = \Omega - \ell$ is the desired map. □

4. Singularity and estimates for exact centralizers

Recall that an operator between Banach spaces is said to be *strictly singular* if no restriction to an infinite dimensional closed subspace is an isomorphism.

Definition 3. *A quasi-linear map (in particular, a centralizer) is said to be* singular *if its restriction to every infinite dimensional closed subspace is never trivial. An exact sequence induced by a singular quasi-linear map will be called a* singular sequence.

A quasi-linear map on a Köthe function space will be called disjointly singular *if its restriction to every subspace generated by a disjoint sequence is never trivial.*

It can be shown [14] that a quasi-linear map is singular if and only if the associated exact sequence has strictly singular quotient map. It is clear that singularity implies disjoint singularity. We shall see that the reverse implication does not hold in general, although both notions are equivalent on Banach spaces with unconditional basis. The following "transfer principle" ([14], [10]) will be essential for us.

Lemma 4.1. *If a quasi-linear map defined on a Banach space with basis is trivial on some infinite dimensional subspace then it is also trivial on some subspace $W = [w_n]$ spanned by normalized blocks of the basis.*

Observe that if F is a quasi-linear map on a Köthe space X, and for some sequence (u_n) of disjointly supported vectors and some constant K one has

$$\left\| F(\sum \lambda_l u_j) - \sum \lambda_l F(u_j) \right\| \leq K \| \sum \lambda_l u_j \|$$

for all choices of scalars (λ_j) then F is not singular: indeed, the estimate above means that the linear map $[u_j] \to X \oplus_F [u_j]$ given by $u_j \to (0, u_j)$ is continuous. Under exactness conditions we can get a partial converse.

Lemma 4.2. *Let $\Omega : X \curvearrowright X$ be an exact centralizer on a Köthe function space. If Ω is not disjointly singular, there exists a subspace W of X generated by a disjoint sequence and a constant K such that given vectors $u_1, \ldots, u_n$ in W there are vectors $z_1, ..., z_n$ in X with* $\mathrm{supp} z_i \subset \mathrm{supp} u_i$ *and $\|z_i\| \leq K\|u_i\|$ such that for all scalars $\lambda_1, \ldots, \lambda_n$ one has*

$$\|\Omega(\sum_{i=1}^{n} \lambda_i u_i) - \sum_{i=1}^{n} \lambda_i \Omega(u_i)\| \leq K \left(\| \sum_{i=1}^{n} \lambda_i u_i \| + \| \sum_{i=1}^{n} \lambda_i z_i \| \right). \tag{6}$$

Proof. Since Ω is not disjointly singular, it is trivial on some subspace $W = [u_n]$ spanned by disjointly supported vectors. Then by Lemma 3.13 there exists a linear relatively exact map $\Lambda : W \to X$ so that $\Omega_{|W} - \Lambda$ is bounded. Since both Ω and Λ (by Lemma 3.13 (2)) are relatively contractive, so is $\Omega - \Lambda$. Set $z_i = (\Omega - \Lambda)(u_i)$ and $K = \|\Omega_{|W} - \Lambda\|$. □

Remarks. The preceding estimate can be considered as a subtler version of the "upper p-estimates" argument for non-splitting, which can be quickly described as: if the space X verifies some type of upper p-estimate and the twisted sum $X \oplus_\Omega X$ splits then the space $X \oplus_\Omega X$ must also verify the upper p-estimate (the key here is the p since, in general, if X has type p then $X \oplus_\Omega X$ only needs to have type $p + \varepsilon$ for every ε (see [27]). Therefore, given suitable vectors (u_n) in X the elements $(0, u_n)$ in $X \oplus_\Omega X$ should verify an upper p-estimate; which amounts

$$\|\Omega(\sum_{i=1}^{n} u_i) - \sum_{i=1}^{n} \Omega(u_i)\| \leq C \sqrt[n]{p}.$$

We now define the notion of *standard class* of finite families of elements of Köthe spaces.

Definition 4. *A* standard class *$\mathcal{S}$ is a class of finite families (n-tuples) of elements of Köthe function spaces (respect. spaces with* 1*-unconditional bases) X satisfying*

(i) *whenever $(x_i) \in \mathcal{S}$ and* $\mathrm{supp} z_i \subset \mathrm{supp} x_i$ *for all i then $(z_i) \in \mathcal{S}$;*
(ii) *assume that W is a subspace generated by disjoint vectors (resp. generated by successive vectors) of X, and (x_i) is n-tuple of elements of W; if (x_i) belongs to $\mathcal{S}$ as a family in W, then it also belongs to $\mathcal{S}$ as a family in X.*

The three standard classes we shall use are: disjointly supported vectors in Köthe spaces, successive vectors on 1-unconditional bases, and "Schreier" successive vectors on 1-unconditional bases (i.e. families $(x_1, \ldots, x_n)$ such that $n < \text{supp } x_1 < \cdots < \text{supp } x_n$).

Given a standard class $\mathcal{S}$ and a space X, we consider the following indicator function:

$$M_{X,\mathcal{S}}(n) := \sup\{\|x_1 + \ldots + x_n\| : \ (x_j) \in \mathcal{S}, \ \|x_j\| \leq 1\}.$$

Lemma 4.2 can be rewritten as:

Lemma 4.3. *Let $\mathcal{S}$ be a standard class, and let $\Omega : X \curvearrowright X$ be an exact centralizer on a Köthe function space. If Ω is not disjointly singular, then there exists a subspace W of X generated by a disjoint sequence and a constant K such that given a n-tuple $(u_i) \in \mathcal{S}$ belonging to the unit ball of W, one has*

$$\Big\|\Omega(\sum_{i=1}^n u_i) - \sum_{i=1}^n \Omega(u_i)\Big\| \leq K M_{X,\mathcal{S}}(n).$$

The following estimate holds for many real centralizers (after Kalton's Theorem 3.4).

Lemma 4.4. *Let (X_0, X_1) be an admissible couple of Köthe function spaces, fix $0 < \theta < 1$, and let Ω_θ be the induced centralizer on X_θ. If $(y_i) \in \mathcal{S}$ is a n-tuple in the unit ball of X_θ, then*

$$\Big\|\Omega_\theta\Big(\sum_{i=1}^n y_i\Big) - \sum_{i=1}^n \Omega_\theta(y_i) - \log\frac{M_{X_0,\mathcal{S}}(n)}{M_{X_1,\mathcal{S}}(n)}\Big(\sum_{i=1}^n y_i\Big)\Big\| \leq 3 M_{X_0,\mathcal{S}}(n)^{1-\theta} M_{X_1,\mathcal{S}}(n)^{\theta}. \tag{7}$$

Proof. To simplify notation, let us write $M(n,z) = M_{X_0,\mathcal{S}}(n)^{1-z} M_{X_1,\mathcal{S}}(n)^z$. Given $\epsilon > 0$, let $(x_i) \in \mathcal{S}$ be a n-tuple in the unit ball of X_θ. Let B_θ be a $(1+\epsilon)$-bounded selection $X_\theta \to \mathcal{H}$ such that supp $B_\theta(x) \subset$ supp x for all x. Let $F_i = B_\theta(x_i)$ for each i. Note that $\big(F_i(z)\big)$ is a n-tuple in $\mathcal{S}$ for any z in the strip. Let F be the function

$$F(z) = \frac{F_1(z) + \cdots + F_n(z)}{M(n,z)}$$

for $z \in \mathbb{S}$. We know that $\|F\| \leq 1 + \epsilon$ and

$$F(\theta) = \frac{1}{M(n,\theta)}(x_1 + \ldots + x_n).$$

Set $k = \|F(\theta)\|^{-1}$. The map $\Phi : F(\theta) \to F$ defines a linear bounded selection on the one-dimensional subspace $[F(\theta)]$ having norm at most k. Therefore $\|B_\theta - \Phi\| \leq 1 + \epsilon + k$. Thus, if $x \in [F(\theta)]$,

$$\|(\delta' B_\theta - \delta'\Phi)(x)\| \leq 2k\|x\|_\theta,$$

in particular

$$\Big\|(\delta' B_\theta - \delta'\Phi)(\sum_{i=1}^n x_i)\Big\| \leq 2k \Big\|\sum_{i=1}^n x_i\Big\|_\theta.$$

On the other hand,

$$F'(\theta) = F(\theta)\log\frac{M_{X_0,\mathcal{S}}(n)}{M_{X_1,\mathcal{S}}(n)} + \frac{1}{M(n,\theta)}\sum_i B_\theta(x_i)'(\theta),$$

which means

$$\delta'\Phi(\sum_i x_i) = \log\frac{M_{X_0,\mathcal{S}}(n)}{M_{X_1,\mathcal{S}}(n)}\big(\sum_i x_i\big) + \sum_i \delta' B_\theta(x_i).$$

Therefore

$$\delta'\Phi(\sum_i x_i) - \delta' B_\theta(\sum_i x_i) = \sum_i \delta' B_\theta(x_i) - \delta' B_\theta(\sum_i x_i) + \log \frac{M_{X_0,\mathcal{S}}(n)}{M_{X_1,\mathcal{S}}(n)}(\sum_i x_i)$$

which yields

$$\Big\| \sum_i \delta' B_\theta(x_i) - \delta' B_\theta(\sum_i x_i) + \log \frac{M_{X_0,\mathcal{S}}(n)}{M_{X_1,\mathcal{S}}(n)}(\sum_i y_i)\Big\| \leq 2k \Big\|\sum_{i=1}^n x_i\Big\|_\theta,$$

hence

$$\Big\|\Omega_\theta(\sum_{i=1}^n x_i) - \sum_{i=1}^n \Omega_\theta(x_i) - \log \frac{M_{X_0,\mathcal{S}}(n)}{M_{X_1,\mathcal{S}}(n)}(\sum_i x_i)\Big\| \leq 2k \Big\|\sum_i x_i\Big\|_\theta \leq 3M(n,\theta) \tag{8}$$

as desired. $\square$

Note here the dependence of the indicator functions on the parameter in the interpolation scale:

Lemma 4.5. *Given an interpolation scale (X_θ) of Köthe function spaces associated to a pair (X_0, X_1), the function $\theta \mapsto M_{X_\theta,\mathcal{S}}(n)$ is log-convex.*

Proof. Let $F(z) = (F_1(z) + \cdots + F_n(z))/M(n,z)$ be the function in the proof Lemma 4.4. The inequalities $\|F(\theta)\|_\theta \leq \|F\| \leq 1+\epsilon$ imply $\|x_1 + \cdots + x_n\|_\theta \leq (1+\epsilon)M(n,\theta)$. Thus $M_{X_\theta,\mathcal{S}}(n) \leq M_{X_0,\mathcal{S}}(n)^{1-\theta} M_{X_1,\mathcal{S}}(n)^\theta$. $\square$

5. Criteria for singularity

We set now the core of our criterion to obtain disjointly singular sequences: to combine Lemma 4.3, Lemma 4.4 and Lemma 4.5 to get the following result.

Proposition 5.1. *Let $\mathcal{S}$ be a standard class. Let (X_0, X_1) be an interpolation couple of Köthe function spaces generating the interpolation scale (X_θ); and let Ω_θ be the induced centralizer on X_θ. If Ω_θ is not disjointly singular then there exists a subspace $W \subset X_\theta$ spanned by disjointly supported vectors and a constant K such that*

$$\left|\log \frac{M_{X_0,\mathcal{S}}(n)}{M_{X_1,\mathcal{S}}(n)}\right| M_{W,\mathcal{S}}(n) \leq K M_{X_0,\mathcal{S}}(n)^{1-\theta} M_{X_1,\mathcal{S}}(n)^\theta. \tag{9}$$

Remark. An even more general criterion can be obtained by using in the definition of M_X sequences of vectors whose norms are at most some prescribed varying values, instead of vectors of norm at most 1. We shall not write it since it will not be needed to deal with the applications we are interested in.

We consider firstly the standard class $\mathcal{D}$ of all disjointly supported sequences in a Köthe function space X, and simplify notation to:

$$M_X(n) = M_{X,\mathcal{D}}(n) = \sup\{\|x_1 + \ldots + x_n\| : x_1, \ldots, x_k \text{ disjoint in the unit ball of } X\}.$$

Recall that two functions $f, g : \mathbb{N} \to \mathbb{R}$ will be called equivalent, and denoted $f \sim g$, if $0 < \liminf f(n)/g(n) \leq \limsup f(n)/g(n) < +\infty$. As a direct application of the criterion in Proposition 5.1 we have:

Proposition 5.2. *Let (X_0, X_1) be an interpolation couple of two Köthe function spaces so that M_{X_0} and M_{X_1} are not equivalent. Assume that X_θ is "self-similar" in the sense that $M_W \sim M_{X_\theta}$ for every infinite-dimensional subspace generated by a disjoint sequence $W \subset X_\theta$, and $M_{X_\theta} \sim M_{X_0}^{1-\theta} M_{X_1}^{\theta}$. Then Ω_θ is disjointly singular.*

Proof. Otherwise, the estimate (9) yields that, on some subspace W, one gets

$$\left|\log \frac{M_{X_0}(n)}{M_{X_1}(n)}\right| M_W(n) = O(M(n,\theta)) = O(M_{X_\theta}(n)) = O(M_W(n)),$$

which is impossible unless M_{X_0} and M_{X_1} are equivalent. □

Let us see these criteria at work. The simplest case of course concerns the scale of ℓ_p spaces, $1 \le p < +\infty$. These spaces are self similar with $M_{\ell_p}(n) = n^{1/p}$, while reiteration theorems allow one to fix X_0 and X_1 at any two different values p, q so that $\lim |\log \frac{M_{X_0}(n)}{M_{X_1}(n)}| = \lim |\log n^{1/p-1/q}| = +\infty$. Thus, the induced centralizer, which actually is (projectively equivalent to) the Kalton-Peck ℓ_∞-centralizer $\mathcal{K}$, in ℓ_p is lattice singular, hence singular. The case of L_p spaces, $1 \le p \le +\infty$ is also simple: Proposition 5.1 yields that if the twisted sum fails to be disjointly singular then

$$\left|\log \frac{M_{L_\infty}(n)}{M_{L_1}(n)}\right| M_{\ell_p}(n) \le K M_{L_\infty}^{1-\frac{1}{p}}(n) M_{L_1}^{\frac{1}{p}}(n).$$

Therefore $(\log n) n^{1/p} \le K n^{1/p}$, which is impossible. So the induced centralizer, actually (projectively equivalento to) the Kalton-Peck L_∞-centralizer $\mathcal{K}$, in L_p is disjointly singular.

In [6] it was shown that no L_∞-centralizer on L_p can be singular for $0 < p < \infty$; previously, it had been shown in [38] that the Kalton-Peck L_∞-centralizer $\Omega(f) = f \log f/\|f\|$ on L_p is not singular (it becomes trivial on the Rademacher copy of ℓ_2). In [10, Theorem 2(b)] it was shown that the Kalton-Peck centralizer on ℓ_p is singular for $0 < p < \infty$.

A tricky question is what occurs with the scale of L_p-spaces in their ℓ_∞-module structure generated by the Haar basis. Is singular the associated Kalton-Peck ℓ_∞-centralizer? Khintchine's inequality makes possible to define $B_\theta(r) = f_r$ (the constant function $f_r(z) = r$ on the subspace ℓ_2^R generated by the Rademacher functions, so $\Omega_\theta(r) = \delta'_\theta B_\theta(r) = 0$ on ℓ_2^R and thus Ω_θ is not singular. It was shown in [10] that the Kalton-Peck centralizer (relative to the Haar basis) is singular for $2 \le p < \infty$. Which shows, in particular, that the Kalton-Peck ℓ_∞-centralizer relative to the Haar basis is not the L_∞-centralizer induced by the interpolation scale of L_p spaces in their ℓ_∞-module structure. Cabello [6] remarks that it would be interesting to know where there exist singular quasi-linear maps $L_p \to L_p$ for $p < 2$.

5.1. **The unconditional case.** We will consider now the following asymptotic variation of M_X:

$$A_X(n) = \sup\{\|x_1 + \ldots + x_n\|_\theta : \|x_i\| \le 1,\ n < x_1 < \ldots < x_k\},$$

with its associated standard class. A proof entirely similar to that of Lemma 4.4, using instead the function

$$F(z) = \frac{1}{A_{X_0}(n)^{1-z} A_{X_1}(n)^z} (B_\theta(y_1) + \cdots + B_\theta(y_n))(z),$$

immediately yields the estimate

$$\left\|\Omega_\theta(\sum_{i=1}^n y_i) - \sum_{i=1}^n \Omega_\theta(y_i) - \log \frac{\mathrm{A}_{X_0}(n)}{\mathrm{A}_{X_1}(n)} \sum_i y_i\right\| \le 3\mathrm{A}_{X_0}^{1-\theta} \mathrm{A}_{X_1}^{\theta}(n), \tag{10}$$

for all $n < y_1 < \cdots < y_n$ in the unit ball of X_θ. On the other hand the estimate in Lemma 4.2 can be rewritten as

$$\Big\|\Omega(\sum_{i=1}^{n}\lambda_i u_i) - \sum_{i=1}^{n}\lambda_i\Omega(u_i)\Big\| \leq KA_X(n). \tag{11}$$

for blocks $n < u_1 < u_2 < \cdots < u_n$. Now, given an admissible pair (X_0, X_1) of spaces with common 1-unconditional basis and $0 < \theta < 1$, one can prove that the function $\theta \mapsto A_{X_\theta}(n)$ is log-convex working as in Lemma 4.5. Thus, estimates (10) and (11) yield:

Proposition 5.3. *Let (X_0, X_1) be an admissible pair of Banach spaces with a common 1-unconditional basis, and $0 < \theta < 1$.*

a) *If the associated centralizer Ω_θ is not singular then there exists a block subspace $W \subset X_\theta$ and a constant K such that:*
$$\left|\log\frac{A_{X_0}(n)}{A_{X_1}(n)}\right| A_W(n) \leq K\mathrm{A}_{X_0}^{1-\theta}(n)\mathrm{A}_{X_1}^{\theta}(n).$$
b) *If $A_{X_0} \not\sim A_{X_1}$ and $A_{X_0}^{1-\theta}A_{X_1}^{\theta} \sim A_{X_\theta} \sim A_Y$ for all subspaces $Y \subset X_\theta$ then Ω_θ is singular.*

Recall that a Banach space with a basis is said to be *asymptotically ℓ_p* if there exists $C \geq 1$ such that for all n and normalized $n < x_1 < \ldots < x_n$ in X, the sequence $(x_i)_{i=1}^n$ is C-equivalent to the basis of ℓ_p^n. Apart from the ℓ_p spaces, Tsirelson's space is asymptotically ℓ_1 as well as a class of H.I. spaces defined by Argyros and Delyanii [2]. One has:

Corollary 5.4. *Let (X_0, X_1) be an interpolation pair of Banach spaces with a common 1-unconditional basis. Let $p_0 \neq p_1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. The induced centralizer $\Omega_\theta : X_\theta \curvearrowright X_\theta$ is singular in any of the following cases:*

(1) *The spaces X_j, $j = 0, 1$ are reflexive asymptotically ℓ_{p_j}.*
(2) *Successive vectors in X_j, $j = 0, 1$ satisfy an asymptotic upper ℓ_{p_j}-estimate; and for every block-subspace W of X_θ, there exist a constant C and, for each n, a finite block-sequence $n < y_1 < \ldots < y_n$ in B_W such that $\|y_1 + \cdots + y_n\| \geq C^{-1}n^{1/p}$.*

Corollary 5.5. *Let X be a p-convex Köthe function space. The Kalton-Peck map*
$$\mathcal{K}(x) = x\log\frac{|x|}{\|x\|}$$
is disjointly singular on X in any of the following two cases:

(a) *$M_X(n) \sim M_Y(n)$ for every sublattice Y of X,*
(b) *X is a sequence space and $A_X(n) \sim A_Y(n)$ for every block-subspace Y of X.*

Proof. (a) Since X is p-convex we may write $X = (L_\infty, X^p)_{1/p}$. Furthermore the centralizer induced by this interpolation scheme is a multiple of the Kalton-Peck map. In particular, the two twisted sums are projectively equivalent in the sense of Remark **??**. Thus one is singular if and only if the other is. Since the norm on X^p is defined as $\|x\| = \||x|^{1/p}\|_X^p$, we have immediately that $M_{X^p}(n) = M_X(n)^p$. Since X is p-convex, $M_X(n)$ is not bounded and so $M_X(n)^p$ is not equivalent to $M_{L_\infty}(n) = 1$. Furthermore
$$M_{L_\infty}(n)^{1-\frac{1}{p}}M_{X^p}(n)^{\frac{1}{p}} = (M_X(n)^p)^{1/p} = M_X(n),$$
and by Proposition 5.2 the centralizer (hence the Kalton-Peck map) is disjointly singular.

(b) The proof is entirely similar applying Proposition 5.3. $\square$

5.2. **The conditional case.** Let $\Omega : X \to X$ be a quasi-linear map acting on a space with 1-monotone basis. This case does not fit under the umbrella of Kalton theorem, so it could well occur that Ω could not be recovered from an interpolation scheme. Without the lattice structure, supports cannot be used as before. One can instead use the range of vectors (ran$x$ is the minimal interval of integers containing its support) to then define "successive" vectors and "asymptotic limits" in X, which means that the function A_X still makes sense. In the general case of 1-monotone bases the maps Ω_θ appearing in the interpolation process are not ℓ_∞-centralizers or contractive. However, the maps can be chosen to be "range" contractive, in the sense of verifying ran $\Omega_\theta(x) \subset$ ran x. Indeed if for $x \in c_{00}$, $b_\theta(x)$ is an almost optimal selection, then $B_\theta(x) = 1_{\mathrm{ran}x} b_\theta(x)$ will also be almost optimal and range contractive, so $\delta'_\theta B_\theta$ will be the required map. The transfer principle still works and thus a non-singular $\Omega : X \curvearrowright X$ must be trivial on some subspace W generated by blocks of the basis.

Proposition 5.6. *Assume we have a complex interpolation scheme of two spaces X_0, X_1 with a common 1-monotone basis. Assume that for every block-subspace W of X_θ, there exists for every n a finite successive sequence $n < y_1 < \cdots < y_n$ with $\|y_i\| \leq 1$ $\forall i = 1, \ldots, n$, and constants $\varepsilon_n, \lambda_n, M_n$ satisfying*

(i) *The block sequence is ε_n-optimal, in the sense that $\|\sum_{i=1}^n y_i\| \geq \varepsilon_n A_{X_0}(n)^{1-\theta} A_{X_1}(n)^\theta$;*
(ii) *The block sequence $\{y_1, \ldots, y_n\}$ is λ_n-unconditional;*
(iii) *the space $[y_1, \ldots, y_n]$ is M_n-complemented in X_θ;*

and so that

$$\liminf_{n\to+\infty} \frac{\lambda_n^2 M_n}{\varepsilon_n \left|\log \frac{A_{X_0}(n)}{A_{X_1}(n)}\right|} = 0.$$

Then Ω_θ is singular.

Proof. Suppose that the restriction of Ω_θ to some subspace of X is trivial. By the hypothesis Ω_θ is trivial on some block subspace Y_θ subspace of X_θ, and we can pick a λ_n-unconditional finite sequence $[y_i]_{i=1}^n$ of blocks in B_{Y_θ} that is M_n-complemented in X_θ by a projection P_n.

Then a reasoning similar to the proof Lemma 3.13 (3) can be made. Namely, change the module structure to work with the subalgebra $\ell_\infty^{Y_\theta}$ of ℓ_∞ formed by those elements constant on the support of each of the y_n. The module action on Y_θ is clear. The group of units of $\ell_\infty^{Y_\theta}$ is now a compact part U_{Y_θ} of 2^ω, thus it admits an invariant mean $m_{Y_\theta} : \ell_\infty(U_{Y_\theta}) \to \mathbb{R}$. Let $\ell_{Y_\theta} : Y_\theta \to L_0$ be a linear map so that $\|\Omega_{|Y_\theta} - \ell_{Y_\theta}\| \leq K$. So $(uP_{Y_\theta}(\Omega_{|Y_\theta} - \ell_{Y_\theta})(uy))_{u\in\ell_\infty^{Y_\theta}}$ is bounded since $\|uP_{Y_\theta}(\Omega_{|Y_\theta} - \ell_{Y_\theta})(uy)\| \leq KM\|uy\| \leq KM\lambda\|y\|$. We can thus define $\psi_{Y_\theta}(y) \in Y_\theta$ as the only element so that for each $f \in {Y_\theta}^*$

$$< \psi_{Y_\theta}(y), f >= m\left(< uP_{Y_\theta}(\Omega_{|Y_\theta} - \ell_{Y_\theta})(uy), f >\right).$$

This map ψ_{Y_θ} is bounded by $KM\lambda$ and an exact $\ell_\infty^{Y_\theta}$-centralizer, so supp $\psi_{Y_\theta}(y) \subset$ supp y for $y \in Y_\theta$. This implies that $\psi_{Y_\theta}(y_n) = \mu_n y_n$ for some scalars μ_n with $|\mu_n| \leq KM\lambda$. Thus

$$\begin{aligned} \|\psi_{Y_\theta}(\textstyle\sum_n \lambda_n y_n) - \sum_n \lambda_n \psi_{Y_\theta}(y_n)\| &\leq KM\lambda\|\textstyle\sum_n \lambda_n y_n\| + \|\sum_n \lambda_n \mu_n y_n\| \\ &\leq KM\lambda(1+\lambda)\|\textstyle\sum_n \lambda_n y_n\|. \end{aligned} \tag{12}$$

Consider the estimate (10), and observe that replacing Ω_θ by $\Omega_\theta + \ell_\theta$ with ℓ_θ linear changes nothing, and projecting and averaging on $\pm$ signs only changes the estimate by $\|P_n\| \leq M_n$; so one gets

$$\left\|\psi_{Y_\theta}(\sum_{i=1}^n y_i) - \sum_{i=1}^n \psi_{Y_\theta}(y_i) - \log \frac{\mathrm{A}_{X_0}(n)}{\mathrm{A}_{X_1}(n)} \sum_{i=1}^n y_i\right\| \leq 3M_n A_{X_0}(n)^{1-\theta} A_{X_1}(n)^\theta.$$

On the other hand we can rewrite (12) as

$$\Big\|\psi_{Y_\theta}(\sum_i y_i) - \sum_i \psi_{Y_\theta}(y_i)\Big\| \leq K M_n \lambda_n (1+\lambda_n) \mathrm{A}_{X_0}^{1-\theta}(n) \mathrm{A}_{X_1}^{\theta}(n). \tag{13}$$

Putting both estimates together we get

$$\Big|\log \frac{A_{X_0}(n)}{A_{X_1}(n)}\Big| \cdot \Big\|\sum_{i=1}^n y_i\Big\| \leq \big(K\lambda_n(1+\lambda_n)+3\big) M_n \mathrm{A}_{X_0}^{1-\theta}(n)\mathrm{A}_{X_1}^{\theta}(n).$$

Condition (i) yields that

$$\varepsilon_n \left|\log \frac{A_{X_0}(n)}{A_{X_1}(n)}\right| \leq \big(K\lambda_n(1+\lambda_n)+3\big) M_n$$

in contradiction with the hypothesis. □

Corollary 5.7. *Assume we have an interpolation scheme of two spaces X_0 and X_1 with a common 1-monotone basis. Let $p_0 \neq p_1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and assume that the spaces X_j, $j = 0,1$ satisfy an asymptotic upper ℓ_{p_j}-estimate; and that for every block-subspace W of X_θ, there exist a constant C and for each n, a C-unconditional finite block-sequence $n < y_1 < \ldots < y_n$ in B_W such that $\|y_1 + \cdots + y_n\| \geq C^{-1}n^{1/p}$ and $[y_1, \cdots, y_n]$ is C-complemented in X_θ. Then Ω_θ is singular.*

Remark. It was proved by Pisier [34] that a B-convex Banach space contains ℓ_2^n uniformly complemented. Condition (ii) in Proposition 5.6 suggests to apply this result to B-convex Banach spaces. Proposition 7.2 below states that when X is B-convex, nontrivial twisted sums $X \oplus_F X$ always exist.

5.3. **Interpolation of families of spaces.** Here we apply the preceding criteria to spaces induced by complex interpolation of a family of spaces (see [17]), as we require in Section 8.

We take a family of compatible Banach spaces $\{X_{(j,t)} : j = 0,1; t \in \mathbb{R}\}$ with index in the boundary of $\mathbb{S}$, and denote by $\Sigma(X_{j,t})$ the algebraic sum of these spaces with the norm

$$\|x\|_\Sigma = \inf\{\|x_1\|_{(j_1,t_1)} + \cdots + \|x_n\|_{(j_n,t_n)} : x = x_1 + \cdots + x_n\}.$$

Let $\mathcal{H}(X_{j,t})$ denote the space of functions $g : \mathbb{S} \to \Sigma := \Sigma(X_{j,t})$ which are $\|\cdot\|_\Sigma$-bounded, $\|\cdot\|_\Sigma$-continuous on $\mathbb{S}$ and $\|\cdot\|_\Sigma$-analytic on $\mathbb{S}^\circ$; and satisfy $g(it) \in X_{(0,t)}$ and $g(it+1) \in X_{(1,t)}$ for each $t \in \mathbb{R}$. Note that $\mathcal{H}(X_{j,t})$ is a Banach space under the norm

$$\|g\|_{\mathcal{H}} = \sup\{\|g(j+it)\|_{(j,t)} : j = 0,1; t \in \mathbb{R}\}.$$

For each $\theta \in (0,1)$, or even $\theta \in \mathbb{S}$, we define

$$X_\theta := \{x \in \Sigma(X_{j,t}) : x = g(\theta) \text{ for some } g \in \mathcal{H}(X_{j,t})\}$$

with the norm $\|x\|_\theta = \inf\{\|g\|_{\mathcal{H}} : x = g(\theta)\}$. Clearly X_θ is the quotient of $\mathcal{H}(X_{j,t})$ by the kernel of the evaluation map $\ker \delta_\theta$, and thus it is a Banach space.

All the ingredients of our constructions straightforwardly adapt to this context, and the only relevant modification is to set $A_j(n) = \operatorname{ess\,sup}_{t\in\mathbb{R}} A_{X_{j+it}}(n)$ instead of $A_{X_j}(n)$, $j = 0,1$.

Proposition 5.8. *Consider an interpolation scheme of a family $\{X_{(j,t)} : j = 0,1; t \in \mathbb{R}\}$ of spaces with a common 1-monotone basis. Let $p_0 \neq p_1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.*

Assume that all the spaces $X_{j,t}$ satisfy an asymptotic upper ℓ_{p_j}-estimate with uniform constant; and for every block-subspace W of X_θ, there exist a constant C and for each n, a C-unconditional finite block-sequence $n < y_1 < \ldots < y_n$ in B_W such that $\|y_1 + \cdots + y_n\| \geq C^{-1}n^{1/p}$ and $[y_1, \cdots, y_n]$ is C-complemented in X_θ.

Then Ω_θ is singular.

Proof. The arguments are similar to those in the proof of Proposition 5.6. □

6. Singular twisted Hilbert spaces

In many cases, complex interpolation between a Banach space and its dual gives $(X, X^*)_{1/2} = \ell_2$. See e.g., the comments at [35, around Theorem 3.1]. Also Watbled [39] claims that her results cover the case of spaces with a 1-unconditional basis X. We do not know whether there could be counterexamples with monotone basis. So, for the sake of clarity, let us briefly explain the situation.

Given a Banach space X with a normalized basis (e_n), we denote (e_n^*) the corresponding sequence of biorthogonal functionals. We identify X with $\{\big(e_n^*(x)\big) : x \in X)\}$, and its antidual space $\hat{X}^*$ with $\{\overline{\big(x^*(e_n)\big)} : x^* \in X)\}$, both linear subspaces of ℓ_∞, in such a way that $X \cap \hat{X}^*$ is continuously embedded in ℓ_2. Indeed, $x = (a_n) \in X \cap \hat{X}^*$ implies $x(x) = \sum |a_n|^2 \leq \|x\|_X \cdot \|x\|_{\hat{X}^*}$.

Proposition 6.1. *Let X be a Banach space with a monotone shrinking basis. Then $(X, \hat{X}^*)_{1/2} = \ell_2$ with equality of norms.*

Proof. It is enough to show that ℓ_2 is continuously embedded in $X + \hat{X}^*$ and apply [39, Corollary 4]. Let $T : X \cap \hat{X}^* \to \ell_2$ be the embedding. Since the basis is shrinking, $X \cap \hat{X}^*$ is dense in both X and $\hat{X}^*$. Thus the dual of $X \cap \hat{X}^*$ is $X^* + (\hat{X}^*)^* = X^{**} + \hat{X}^*$ [5, 2.7.1 Theorem], and the conjugate operator T^* embeds ℓ_2 into $X + \hat{X}^*$, which is a closed subspace of $X^{**} + \hat{X}^*$ by the arguments in [39, p. 204]. □

We have a similar result for Köthe function spaces X. Observe that in this case X^* and $\hat{X}^*$ coincide as sets.

Proposition 6.2. [39, Corollary 5] *Let X be a Köthe function space on a complete σ-finite measurable space S. Suppose that $X \cap X^*$ is dense in X and*

$$L_1(S) \cap L_\infty(S) \subset X \cap X^* \subset L_2(S) \subset X + X^* \subset L_1(S) + L_\infty(S).$$

Then $(X, X^*)_{1/2} = L_2(S)$.

Remark 6.3. Arguing like in Proposition 6.1, we can show that the conditions X and X^* intermediate spaces between $L_1(S)$ and $L_\infty(S)$, and $X \cap X^*$ dense in both X and X^* imply the hypothesis of Proposition 6.2.

In all the previous situations the twisted sum space induced by the interpolation of a space and its antidual is a twisted Hilbert space. Proposition 5.2 fits appropriately in this situation since ℓ_2 is "asymptotically self-similar" in the sense that $A_W(n) = n^{1/2}$ for all infinite dimensional subspaces. Thus, we are ready to construct singular exact sequences

$$0 \longrightarrow \ell_2 \longrightarrow E \longrightarrow \ell_2 \longrightarrow 0.$$

The first consequence of Corollary 5.7 is:

Proposition 6.4. *The interpolation of a reflexive asymptotically ℓ_p space, $p \neq 2$, with its antidual induces a singular twisted Hilbert space.*

Thus interpolation of Tsirelson's space $\mathcal{T}$ with its dual $\mathcal{T}^*$; or interpolation of Argyros-Deliyanni's H.I. asymptotically ℓ_1-space [2] with its antidual produce new singular exact sequences

$$0 \longrightarrow \ell_2 \longrightarrow X \longrightarrow \ell_2 \longrightarrow 0$$

which are not boundedly equivalent to

$$0 \longrightarrow \ell_2 \longrightarrow Z_2 \longrightarrow \ell_2 \longrightarrow 0.$$

Thus, by Corollary 3.12, they cannot be even equivalent. In favorable situations this can be improved to be non-permutatively projectively equivalent. Indeed, given a reflexive Banach space X with normalized subsymmetric basis (e_n), we denote as usual [33]

$$\lambda_X(n) := \Big\| \sum_{i=1}^{n} e_i \Big\|_X.$$

Then $\lambda_{X^*}(n) \simeq n/\lambda_X(n)$ (see [33, Proposition 3.a.6]). One has

Proposition 6.5. *Let ℓ_M be the symmetric Orlicz space with function $M_\alpha(t) = e^{-t^{-\alpha}}, \alpha > 0$. The induced centralizers at $\ell_2 = (\ell_M, \ell_M^*)_{1/2}$ for different values of α are not permutatively projectively equivalent.*

Proof. Let X and Y be reflexive spaces with normalized 1-unconditional and 1-subsymmetric bases, and let Ω (resp. Ψ) be the induced centralizers at ℓ_2 defined on terms of the Lozanovskii decompositions associated to $(X, X^*)_{1/2}$ (resp. $(Y, Y^*)_{1/2}$). Then

$$\big(\Omega - \mu\Psi\big)(x) = \Big(\log \frac{|a_0(x)|}{|a_1(x)|} - \mu \log \frac{|a_0'(x)|}{|a_1'(x)|} \Big) x.$$

Pick $x = \sum_{i=1}^{n} x_i e_i$ with $x_i = 1/\sqrt{n}$ and apply the above formula with

$$|a_0(x)| = \lambda_X(n)^{-1} 1_{[1,n]}, \qquad |a_1(x)| = \frac{\lambda_X(n)}{n} 1_{[1,n]},$$

and

$$|a_0'(x)| = \lambda_Y(n)^{-1} 1_{[1,n]}, \qquad |a_1'(x)| = \frac{\lambda_Y(n)}{n} 1_{[1,n]}.$$

If $\Omega - \mu\Psi$ is bounded then the function $\log(n\lambda_X(n)^{-2}) - \mu \log(n\lambda_Y(n)^{-2})$ on $\mathbb{N}$ is bounded, which means that the functions $n\lambda_X(n)^{-2}$ and $(n\lambda_Y(n)^{-2})^\mu$ are equivalent. It is not difficult to check that that is impossible for different $\alpha, \beta \geq 0$ since the choice of M_α in the statement yields $\lambda_{\ell_{M_\alpha}}(n) \simeq (\log n)^{1/\alpha}$. Since the symmetric Orlicz spaces have symmetric bases, the corresponding induced centralizers are not even permutatively projectively equivalent. $\square$

We have found no specific criterion to show when twisted Hilbert sums induced by interpolation of spaces with subsymmetric bases are singular. Let us move our attention back to asymptotically ℓ_p spaces.

Proposition 6.6. *Let X, Y be spaces with asymptotically ℓ_p 1-unconditional bases. Then the singular twisted Hilbert sums induced by the interpolation couples $(X, \hat{X}^*)$ and $(Y, \hat{Y}^*)$ at $1/2$ are (permutatively) projectively equivalent if and only if the bases of X and Y are (permutatively) equivalent.*

Proof. The key is to show that projective equivalence actually implies equivalence, hence bounded equivalence; which implies, by Kalton's result (Proposition 3.5), that the bases of X and Y are equivalent.

Assume thus that the induced centralizers are λ-projectively equivalent. By Corollary 3.12

$$\sum_i a_i^2 \Big(\log \frac{\mu_i}{\nu_i} - \lambda \log \frac{\mu_i'}{\nu_i'} \Big)^2 \leq K,$$

whenever $x = \sum_i a_i e_i$ in ℓ_2 is normalized, and $a_i^2 = \nu_i \mu_i = \nu_i' \mu_i'$ with

$$1 \leq \| \sum_i \nu_i e_i \|_X, \| \sum_i \mu_i e_i \|_{X^*}, \| \sum_i \nu_i' e_i \|_Y, \| \sum_i \mu_i' e_i \|_{Y^*} \leq c.$$

Taking x with support far enough on the basis, we may choose $a_i = n^{-1/2}$ and $\nu_i = \nu_i' \simeq n^{-1/p}$, $\mu_i = \mu_i' \simeq n^{-1/p'}$. Then $|(1-\nu)\log n|^2 \leq K'$, which means that $\lambda = 1$. Therefore we have equivalence.

To deduce the permutative projective equivalence case from the projective equivalence case just note that if a basis (e_n) is asymptotically ℓ_p then any permutation of (e_n) is again asymptotically ℓ_p "in the long distance", which means that there exists $C \geq 1$ and a function $f : \mathbb{N} \to \mathbb{N}$ such that for all n and normalized $f(n) < x_1 < \ldots < x_n$ in X, the sequence $(x_i)_{i=1}^n$ is C-equivalent to the basis of ℓ_p^n. □

From the purely Banach space theory it is interesting to decide whether twisted Hilbert spaces are isomorphic. We can obtain non-isomorphic singular twisted Hilbert spaces as follows.

Definition 5. *A Lispchitz function $\phi : [0 + \infty) \to \mathbb{C}$ with $\phi(0) = 0$ will be called expansive if for every M there exists N such that $|s - t| \geq N \Rightarrow |\phi(s) - \phi(t)| \geq M$.*

Remark. Lipschitz functions for which $\lim_{t\to\infty} \phi'(t) = 0$ are not expansive. In particular the functions ϕ_r for $0 < r < 1$ are not expansive, while ϕ_1 is expansive.

Proposition 6.7. *Let X be a Köthe function space that is self-similar, in the sense that $M_X \sim M_Y$ for all subspaces $Y \subset X$ generated by a disjoint sequence and $\lim_{n\to\infty} M_X(n) = \infty$. Then*

(1) *The Kalton-Peck map*

$$\mathcal{K}_\phi(x) = x\phi\left(-\log \frac{|x|}{\|x\|}\right)$$

is disjointly singular.

(2) *If X has an unconditional basis $\mathcal{K}_\phi$ is singular.*

Proof. To simplify notation we will write $\Omega = \mathcal{K}_\phi$. Observe that Ω is a contractive centralizer. Assume that Y is a sublattice of X such that $\Omega_{|Y}$ is trivial. Let M be arbitrary positive, N be such that $|s-t| \geq N \Rightarrow |\phi(s)-\phi(t)| \geq M$, and n be such that $M_Y(n) \geq 2e^N$. We may consider disjoint vectors $y_1, \ldots, y_n$ in Y of norm at most 1 such that $\|y_1 + \cdots + y_n\| \geq M_Y(n)/2$. An easy calculation shows that

$$\Omega(\sum_i y_i) - \sum_i \Omega(y_i) = \sum_i y_i \big(\phi(-\log(\sum_i y_i/K)) - \phi(-\log(\sum_i y_i))\big),$$

where $K = \|\sum_{i=1}^n y_i\|$. Each coordinate of the vector $\log(\sum_i y_i)) - \log(\sum_i y_i/K)$ is $\log K$ which is larger than $\log(M_Y(n)/2) \geq N$. Therefore each coordinate of the vector $\phi(-\log(\sum_i y_i)) - \phi(-\log(\sum_i y_i/K))$ is larger than M in modulus. We deduce that

$$\|\Omega(\sum_i y_i) - \sum_i \Omega(y_i)\| \geq M\|\sum_i y_i\| \geq MM_Y(n)/2.$$

By Lemma 4.3, this implies for some fixed constant k that $kM_X(n) \geq MM_Y(n)/2$, therefore $M_X \not\sim M_Y$, a contradiction which proves that Ω is singular (resp. disjointly singular). □

Remark. Observe that $\lim_{n\to\infty} M_X(n)=\infty$ can be obtained assuming that X is self-similar and does not contain c_0.

In [30] Kalton obtained a family $Z_2(\alpha)$ of complex twisted Hilbert spaces induced by the centralizers
$$\mathcal{K}_{i\alpha}(x)=x\left(-\log\frac{|x|}{\|x\|}\right)^{1+i\alpha}$$
for $-\infty<\alpha<\infty$ (see also [28]). Since these are not real centralizers they probably do not appear as induced by interpolation of two spaces (although, according to [29] they are induced by the interpolation of three spaces) . Let us see that they are singular.

Lemma 6.8. *The Lispchitz function $\phi(t)=t^{1+i\alpha}$ is expansive.*

Proof. $|\phi(s)-\phi(t)|=|se^{i\alpha\log(s)}-te^{i\alpha\log(t)}|\geq ||s|-|t||=|s-t|$. □

Thus, applying Proposition 6.7 [30] we get:

Proposition 6.9. *Given $\alpha\in\mathbb{R}$, the exact sequences*
$$0\longrightarrow \ell_2\longrightarrow Z_2(\alpha)\longrightarrow \ell_2\longrightarrow 0$$
are singular and for $\alpha\neq\beta$ the spaces $Z_2(\alpha)$ and $Z_2(\beta)$ are not isomorphic.

Let us turn our attention to the Lipschitz functions $\phi_r(t)=t$ for $0\leq t\leq 1$, and $\phi_r(t)=t^r$ for $1<t<\infty$, and the centralizers
$$\mathcal{K}_{\phi_r}(x)=x\phi_r\big(-\log(|x|/\|x\|_2)\big),$$
and twisted Hilbert spaces $\ell_2(\phi_r)$ they induce, introduced and considered by Kalton and Peck in [32]. Note that $\ell_2(\phi_1)=Z_2$, the twisted Hilbert space generated by the interpolation scale (ℓ_1,c_0). It follows from Kalton's theorem 3.4 ([29, Theorem 7.6]) that also $\ell_2(\phi_r)$ appears generated by some interpolation scale. Let us show that is a scale of Orlicz spaces.

Proposition 6.10. *Let $0<r<1$ and $\varphi_0,\ \varphi_1$ be the maps $[0,\infty)\to[0,\infty)$ defined by*
$$\varphi_0^{-1}(t)=t^{\frac12+\frac14(-\log t)^{r-1}},\quad \varphi_1^{-1}(t)=t^{\frac12-\frac14(-\log t)^{r-1}},$$
on a neighborhood of 0, and extended to $[0,\infty)$ to be N-functions with the Δ_2-property.

Then the twisted Hilbert space induced by the interpolation scale $(\ell_{\varphi_0},\ell_{\varphi_1})$ at $1/2$ is isomorphic to $\ell_2(\phi_r)$.

Proof. We note that everything here is well defined since by choice of r and after an easy calculation, $t^{3/4}\leq\varphi_0^{-1}(t)\leq t^{1/4}$, $t^{3/4}\leq\varphi_1^{-1}(t)\leq t^{1/4}$ and $\varphi_1^{-1}(t)$ and $\varphi_0^{-1}(t)$ are increasing, for t in some neighborhood of 0. This is enough to make sure that φ_1 and φ_0 define N-function Orlicz spaces. The Δ_2-property is also satisfied on a neighborhood of 0. Indeed
$$\begin{aligned}\varphi_0^{-1}(9t)&=3t^{\frac12+\frac14(-\log 9t)^{r-1}}=3\varphi_0^{-1}(t)t^{\frac14[(-\log 9-\log t)^{r-1}-(-\log t)^{r-1}]}\\&=3\varphi_0^{-1}(t)\exp\big(-\tfrac14(-\log t)^r[(1+\tfrac{\log 9}{\log t})^{r-1}-1]\big).\end{aligned}$$
The exponential in this expression is easily seen to tend to 1 when t tends to 0, so close enough to 0, $\varphi_0^{-1}(9t)\geq 2\varphi_0^{-1}(t)$, and φ_0 satisfies the Δ_2 condition $\varphi_0(2s)\leq 9\varphi(s)$ for s in a neighborhood of 0. The same holds for φ_1. Since $\varphi_0^{-1}(t)\varphi_1^{-1}(t)=t$ on a neighborhood of 0, the equality $(\ell_{\varphi_0},\ell_{\varphi_1})_{1/2}=\ell_2$ holds up to equivalence of bases.

Let ψ be the map so that
$$\varphi_1^{-1}(t)=t^{\frac12-\frac14\psi(-\log(t))}.$$

Note that ψ is continuous, $\psi(s) = s^{r-1}$ for s on a neighborhood V of $+\infty$, and only the value of $\psi(s)$ for $s \geq 0$ is relevant here. Suppose that $\|x\|_2 = 1$. Then the centralizer Ω associated to $(\ell_{\varphi_0}, \ell_{\varphi_1})_{1/2} = \ell_2$ (see Proposition 3.10), is given by

$$\Omega(x) = 2x \log \frac{\varphi_1^{-1}(|x|^2)}{|x|} = 2x \log |x|^{-\frac{1}{2}\psi(-\log|x|)} = x\psi(-\log|x|)(-\log|x|),$$

while $\mathcal{K}_{\phi_r}(x)_n = x_n \cdot (-\log|x_n|)^r$ whenever $|x_n|$ is less than some constant c depending on V. So we deduce that

$$\begin{aligned} \|\Omega(x) - \mathcal{K}_{\phi_r}(x)\|^2 &\leq \textstyle\sum_{|x_n| \geq c} 2(\Omega(x))_n^2 + (\mathcal{K}_{\phi_r}(x))_n^2 \\ &\leq 2\big((-\log c)^2 \sup_{[0,-\log c]} |\psi| + (-\log c)^{2r}\big). \end{aligned}$$

Since Ω and $\mathcal{K}_{\phi_r}$ are homogeneous, they are boundedly equivalent. Hence $\ell_2 \oplus_\Omega \ell_2$ and $\ell_2(\phi_r)$ are isomorphic. □

Recall from [32, Corollary 5.5] that the spaces $\ell_2(\phi_r)$ are mutually non-isomorphic for different values of $0 < r \leq 1$. We know [32, Corollary 5.5] that $\mathcal{K} = \mathcal{K}_{\phi_1}$ is singular but, being the function ϕ_r non-expansive for $r < 1$, we do not know if also $\mathcal{K}_{\phi_r}$ is singular for $0 < r < 1$.

7. The twisting of H.I. spaces

A Banach space X is said to be *indecomposable* if it cannot be decomposed as $A \oplus B$ for two infinite dimensional subspaces A, B. An infinite dimensional space X is said to be hereditarily indecomposable (H.I., in short) if all subspaces are indecomposable [24]. It is said to be Quotient Hereditarily Indecomposable (Q.H.I., in short) if all its infinite dimensional quotients are H.I. In particular, Q.H.I. spaces are H.I. The existence of Q.H.I. Banach spaces was proved in [22]. The simplest connection between H.I. spaces and the theory of singular exact sequences is described in the following folklore proposition; we present its proof for the sake of completeness.

Lemma 7.1. *Given an exact sequence of Banach spaces*

$$0 \longrightarrow Y \longrightarrow X \xrightarrow{\;q\;} Z \longrightarrow 0,$$

the space X is H.I. if and only if Y is H.I. and q is strictly singular.

Proof. Suppose X is H.I. Then clearly Y is H.I., and if q is not strictly singular, $q_{|V}$ is an isomorphism for some (infinite dimensional) subspace V of X, hence $Y \oplus V$ is a subspace of X and thus X cannot be H.I. Conversely, suppose that q is strictly singular. If X is not H.I. we can find a decomposable subspace $X_1 \oplus X_2$ of X, and q has compact (even nuclear) restrictions on some subspaces $Y_1 \subset X_1$ and $Y_2 \subset X_2$. Thus we can assume that there exists a bijective isomorphism $U : X \to X$ such that $U(Y_1)$ and $U(Y_2)$ are contained in Y. Since $U(Y_1) \oplus U(Y_2)$ is closed, we conclude that Y is not H.I. □

The basic question we tackle in this section is whether it is possible to obtain nontrivial twisted sums of H.I. spaces. The existence of a nontrivial twisted sum of A and B will be denoted $\mathrm{Ext}(B, A) \neq 0$. On one hand, if X is Q.H.I. and Y is a subspace of X with $\dim Y = \dim X/Y = \infty$, then X is a nontrivial twisted sum of the two H.I. spaces Y and X/Y. However, what one is looking for is to obtain methods to twist two specified H.I. spaces. Recall that the Kalton-Peck method [32] to twist spaces works, in principle, under unconditionality assumptions. A second method is to use the local theory of exact sequences as developed in [8]. The following result is a good example; we could not find it explicitly in the literature, but it is certainly known:

Proposition 7.2. *If X is a B-convex Banach space then* $\mathrm{Ext}(X,X)\neq 0$.

Proof. If X contains ℓ_2^n uniformly complemented, as it is the case of B-convex Banach spaces, then $\mathrm{Ext}(X,\ell_2)\neq 0$ [8]. And if $\mathrm{Ext}(X,X)=0$ then $\mathrm{Ext}(X,\ell_2)=0$ [8]. □

The only currently known B-convex H.I. space is the one constructed by Ferenczi in [21]. So, calling this space $\mathcal{F}$ one gets $\mathrm{Ext}(\mathcal{F},\mathcal{F})\neq 0$. However this is not entirely satisfactory since this twisting does not provide any information about the twisted sum space, apart from its existence. So we formulate the following question:

Problem 1. *Given an H.I. space X, does there exist an H.I. twisted sum of X?*

Focusing again on Ferenczi's space $\mathcal{F}$, since it is a space obtained via an interpolation scheme, i.e., $\mathcal{F}=X_\theta$ for a certain configuration of spaces, the induced centralizer Ω_θ provides a natural twisted sum of $\mathcal{F}$ with itself that we will call $\mathcal{F}_2$:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F} \longrightarrow 0.$$

We will show in Section 8 that this sequence is singular, which implies that $\mathcal{F}_2$ is H.I.

By the characterization in Lemma 7.1 it is tempting to believe that a twisted sum of two H.I. spaces is H.I. whenever is not trivial. However, this is not the case:

Proposition 7.3. *There exists a nontrivial twisted sum of two H.I. spaces which is indecomposable but not H.I.*

Proof. Recall that two Banach spaces A,B are said to be totally incomparable if no infinite dimensional subspace of A is isomorphic to a subspace of B. It was proved in [22, Prop. 25] that there exist two reflexive Q.H.I. spaces X_1,X_2 admitting infinite dimensional subspaces $Y_1\subset X_1$ and $Y_2\subset X_2$ such that Y_1 is isometric to Y_2 and X_1/Y_1 and X_2/Y_2 are infinite dimensional and totally incomparable. Note that X_1^* and X_2^* are Q.H.I.

Given a bijective isometry $U:Y_1\to Y_2$, we consider the subspace $\hat{Y}:=\{(y,Uy):y\in Y_1\}$ of $X_1\times X_2$, the quotient $\hat{X}:=(X_1\times X_2)/\hat{Y}$, and the quotient map $Q:X_1\times X_2\to\hat{X}$. Note that $\hat{X}_1:=Q(X_1\times\{0\})$ and $\hat{X}_1:=Q(\{0\}\times X_2)$ are subspaces of $\hat{X}$ isometric to X_1 and X_2 respectively, and $\hat{Z}:=\hat{X}_1\cap\hat{X}_2=Q(Y_1\times\{0\})=Q(\{0\}\times Y_2)$. Thus $\hat{X}/\hat{Z}$ is isomorphic to $\hat{X}_1/\hat{Z}\oplus\hat{X}_2/\hat{Z}$, hence $\hat{Z}^\perp$ is decomposable and $\hat{X}^*$ is not H.I. Let us see that $\hat{X}^*$ is a nontrivial twisted sum of two H.I. spaces: Since $\hat{X}$ is reflexive and H.I. [22, Proposition 23], the dual space $\hat{X}^*$ is indecomposable, hence the exact sequence

$$0 \longrightarrow \hat{X}_1^\perp \longrightarrow \hat{X}^* \longrightarrow \hat{X}^*/\hat{X}_1^\perp \longrightarrow 0$$

is nontrivial. Moreover, $\hat{X}_1^\perp$ and $\hat{X}^*/\hat{X}_1^\perp$ are H.I. because $\hat{X}_1\simeq X_1$ and $\hat{X}/\hat{X}_1\simeq X_2/Y_2$ are Q.H.I. and reflexive. □

We can present an alternative construction of nontrivial and non H.I. twisted sums of H.I. spaces. Let us say that a Banach space X *admits a singular extension* if there exists a singular exact sequence

$$0 \longrightarrow X \longrightarrow Y \xrightarrow{\ q\ } Z \longrightarrow 0;$$

i.e., an exact sequence with q strictly singular and Z infinite dimensional.

Proposition 7.4. *Every separable H.I. space X which admits a singular extension is a complemented subspace of a nontrivial twisted sum of two H.I. spaces.*

Proof. Let $0 \to X \xrightarrow{i} Y \xrightarrow{q} Z \to 0$ be a singular extension of Y with Y separable. It follows from Proposition 7.1 that Y is H.I. By [3, Theorems 14.5 and 14.8] there exists a separable H.I. space W and a surjective operator $p : W \to Y$ with infinite dimensional kernel. Note that p is strictly singular by Proposition 7.1. We consider the closed subspace $PB := \{(w, x) \in W \oplus X : p(w) = i(x)\}$ of $W \oplus X$ and the projection operators $\alpha : PB \to W$ and $\beta : PB \to X$. Note that β is strictly singular because $i\beta = q\alpha$, and that β is surjective with $\ker(\beta) = \ker(p)$ an H.I. space. Hence PB is H.I.

Since the operator $U : (w, x) \in Z \oplus X \longrightarrow i(x) - p(w) \in Y$ is surjective, we have a twisted sum of two H.I. spaces

$$0 \longrightarrow PB \longrightarrow W \oplus X \xrightarrow{U} Y \longrightarrow 0. \tag{14}$$

To finish the proof it is enough to show that this twisted sum is nontrivial. Indeed, otherwise U would be in the class Φ_r of operators with complemented kernel and finite codimensional closed range. By the stability of Φ_r under strictly singular perturbations [1, Theorem 7.23], the operator $T(w, x) \in Z \oplus W \longrightarrow i(x) \in Y$ would define an isomorphism of X onto a finite codimensional subspace of Y, which is not possible. □

Problem 2. *Does every separable H.I. space admit a singular extension?*

The exact sequence (14) also shows that there exists a nontrivial twisted sum of two H.I. spaces which is decomposable ("two" is the maximum number of summands, see [23, Theorem 1]). In Section 9 we will give other examples of this kind. To conclude this section, we could formulate the general problem about the twisting as:

Problem 3. *Does there exists an H.I. space X so that* $\mathrm{Ext}(X, X) = 0$*?*

Let us recall [4] that there are currently known only four solutions to the equation $\mathrm{Ext}(X, X) = 0$: $c_0, \ell_\infty, L_1(\mu)$ and ℓ_∞/c_0.

8. An H.I. twisted sum of $\mathcal{F}$

Ferenczi's H.I. uniformly convex space $\mathcal{F}$ [21] can be obtained from a complex interpolation scheme associated to a family of Banach spaces (briefly described in Subsection 5.3) setting $X_{(1,t)} = \ell_q$ and as $X_{(0,t)}$ certain Gowers-Maurey-like spaces ($t \in \mathbb{R}$).

We fix $\theta \in [0, 1]$, and define

$$\mathcal{F} = \{x \in \Sigma(X_{j,t}) : x = g(\theta) \text{ for some } g \in \mathcal{H}(X_{j,t})\}$$

with the quotient norm of $\mathcal{H}(X_{j,t})/\ker \delta_\theta$, given by $\|x\|_\theta = \inf\{\|g\|_{\mathcal{H}} : x = g(\theta)\}$.

In this section we will show:

Theorem 8.1. *The space $\mathcal{F}$ satisfies the hypotheses of Proposition 5.8 with $C = 1 + \epsilon$ for any $\epsilon > 0$. So the induced twisted sum*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F} \longrightarrow 0.$$

is singular. Therefore $\mathcal{F}_2$ is H.I.

We set $f(x) := \log_2(1 + x)$. We first state estimates relative to successive vectors in the space $\mathcal{F}$ [21, Proposition 1], as well as estimates for successive functionals in $\mathcal{F}^*$ obtained by standard duality arguments:

Lemma 8.2. *For all successive vectors $x_1 < \cdots < x_n$ in $\mathcal{F}$,*

$$\frac{1}{f(n)^{1-\theta}} \Big(\sum_{i=1}^{n} \|x_i\|^p \Big)^{1/p} \leq \Big\| \sum_{i=1}^{n} x_i \Big\| \leq \Big(\sum_{i=1}^{n} \|x_i\|^p \Big)^{1/p},$$

and for all successive functionals $\phi_1 < \cdots < \phi_n$ in $\mathcal{F}^*$,

$$\Big(\sum_{i=1}^n \|\phi_i\|^{p'}\Big)^{1/p'} \leq \Big\|\sum_{i=1}^n \phi_i\Big\| \leq f(n)^{1-\theta}\Big(\sum_{i=1}^n \|\phi_i\|^{p'}\Big)^{1/p'}.$$

In [21], ℓ_{p+}^n-averages are defined as normalized vectors of the form $\sum_{i=1}^n x_i$, where the x_i's are successive of norm at most $(1+\epsilon)n^{-1/p}$, and may be found in any block-subspace of $\mathcal{F}$ (see [21, Lemma 2]). However here we need to control not only the norm of $\sum_{i=1}^n x_i$ but also of $\sum_{i=1}^n \pm x_i$ for any choice of signs $\pm$, so [21] Lemma 2 is not quite enough. To this end we shall use RIS sequences as defined in [21, Definition 3].

RIS sequences with constant $C > 1$ are successive sequences of $\ell_{p+}^{n_k}$-averages with a technical "rapidly" increasing condition on the n_k's and therefore are also present in every block subspace of $\mathcal{F}$. Every subsequence of a RIS sequence is again a RIS sequence. In what follows L is some lacunary infinite subset of $\mathbb{N}$ whose exact definition may be found in [21]. As a consequence of Lemma 8.2, [21, Lemma 10] and standard duality arguments we have:

Lemma 8.3. *Let $y_1 < \cdots < y_n$ be a RIS sequence in $\mathcal{F}$, with constant $1 + \epsilon^2/100$, where $n \in [\log N, \exp N]$ for some N in L, and $0 < \epsilon < 1/16$. Then*

$$\frac{n^{1/p}}{f(n)^{1-\theta}} \leq \|\sum_{i=1}^n y_i\| \leq (1+\epsilon)\frac{n^{1/p}}{f(n)^{1-\theta}}.$$

Furthermore if for all i, $\phi_i \in \mathcal{F}^*$ satisfies $\|\phi_i\| = \phi_i(y_i) = 1$ and $\operatorname{ran}\phi_i \subset \operatorname{ran} y_i$, then

$$(1+\epsilon)^{-1} f(n)^{1-\theta} n^{1/p'} \leq \|\sum_{i=1}^n \phi_i\| \leq f(n)^{1-\theta} n^{1/p'}.$$

Proposition 8.4. *Let Y be a block sequence of $\mathcal{F}$, $n \in \mathbb{N}$, and $\epsilon > 0$. Then there exists a block-sequence $y_1 < \cdots < y_n$ in Y and a block-sequence $\psi_1 < \cdots < \psi_n$ in X^* such that:*

1. *$(1+\epsilon)^{-1} \leq \|\psi_i\| \leq 1 \leq \|y_j\| \leq 1+\epsilon$ and $\psi_i(y_j) = \delta_{ij}$ for $i, j = 1, \dots, n$,*
2. *for any complex $\alpha_1, \dots, \alpha_n$, $\|\sum_{i=1}^n \alpha_i y_i\| \geq (1+\epsilon)^{-1}(\sum_{i=1}^n |\alpha_i|^p)^{1/p}$*
3. *for any complex $\alpha_1, \dots, \alpha_n$, $\|\sum_{i=1}^n \alpha_i \psi_i\| \leq (1+\epsilon)(\sum_{i=1}^n |\alpha_i|^{p'})^{1/p'}$*

Moreover the block sequence $y_1 < \cdots < y_n$ of Y is $(1+\epsilon)$-equivalent to the unit vector basis of ℓ_p^n and $[y_1, \dots, y_n]$ is $(1+\epsilon)$-complemented in Y.

Proof. Assuming $\epsilon \leq 1/16$, pick m such that $d(mn, N) < n$ for some $N \in L$ and big enough to ensure that m and mn belong to $[\log N, \exp N]$, and that $f(mn)/f(m) < 1 + \epsilon$. Denote $M = mn$. Let $x_1, \dots, x_M$ be a RIS in Y with constant $1 + \epsilon^2/100$ and $\phi_1, \dots, \phi_M$ be a sequence of successive norming functionals in X^* for $x_1, \dots, x_M$.

Now for $j = 1, \dots, n$, let

$$y_j = \frac{f(m)^{1-\theta}}{m^{1/p}} \sum_{i=(j-1)m+1}^{jm} x_i, \text{ and } \psi_j = \frac{1}{f(m)^{1-\theta} m^{1/p'}} \sum_{i=(j-1)m+1}^{jm} \phi_j.$$

Since $x_{(j-1)m+1}, \dots, x_{jm}$ is a RIS with constant $1 + \epsilon^2/100$, we have by Lemma 8.3 that for $j = 1, \dots, n$,

$$1 \leq \|y_j\| \leq (1+\epsilon), \quad (1+\epsilon)^{-1} \leq \|\psi_j\| \leq 1,$$

and clearly $\psi_j(y_k) = \delta_{j,k}$. For any complex $\alpha_1, \dots, \alpha_n$, Lemma 8.2 implies

$$\frac{m^{1/p}}{f(m)^{1-\theta}} \|\sum_{j=1}^n \alpha_j y_j\| \geq \frac{(\sum_{j=1}^n m|\alpha_j|^p)^{1/p}}{f(M)^{1-\theta}},$$

so

$$\|\sum_{j=1}^{n}\alpha_j y_j\| \geq (\sum_{j=1}^{n}|\alpha_j|^p)^{1/p}(\frac{f(m)}{f(M)})^{1-\theta} \geq (\sum_{j=1}^{n}|\alpha_j|^p)^{1/p}(1+\epsilon)^{-1}.$$

Lemma 8.2 also implies $f(m)^{1-\theta}m^{1/p'}\|\sum_{j=1}^n \alpha_j\psi_j\| \leq f(M)^{1-\theta}(\sum_{j=1}^n m|\alpha_j|^{p'})^{1/p'}$, so $\|\sum_{j=1}^n \alpha_j\psi_j\| \leq (1+\epsilon)(\sum_{j=1}^n|\alpha_j|^{p'})^{1/p'}$.

Clearly $(y_i)_{i=1}^n$ is $(1+\epsilon)$-equivalent to the unit basis of ℓ_p^n. We claim that $Px = \sum_{i=1}^n \psi_i(x)y_i$ defines a projection from $\mathcal{F}$ onto $[y_1,\ldots,y_n]$ of norm at most $(1+\epsilon)^{2p}$. Indeed for $x\in\mathcal{F}$,

$$\|Px\|^p \leq (1+\epsilon)^p(\sum_{i=1}^n|\psi_i(x)|^p) = (1+\epsilon)^p(\sum_{i=1}^n\alpha_i|\psi_i(x)|^{p-1}\psi_i(x))$$

for some $\alpha_1,\ldots,\alpha_n$ of modulus 1. So

$$\|Px\|^p \leq (1+\epsilon)^p\|x\|\|\sum_{i=1}^n\alpha_i|\psi_i(x)|^{p-1}\psi_i\| \leq (1+\epsilon)^{p+1}\|x\|(\sum_{i=1}^n|\psi_i(x)^{p-1}|^{p'})^{1/p'}.$$

Since

$$\sum_{i=1}^n|\psi_i(x)^{p-1}|^{p'} = \sum_{i=1}^n|\psi_i(x)|^p \leq (1+\epsilon)^p\|Px\|^p,$$

we deduce $\|Px\|^p \leq (1+\epsilon)^{p+1+p/p'}\|x\|\|Px\|^{p/p'}$, therefore $\|Px\| \leq (1+\epsilon)^{2p}\|x\|$. This concludes the proof of the claim, and up to appropriate choice of ϵ, that of the proposition. □

9. Iterated twisting of $\mathcal{F}$

To simplify the notation, let us set $\mathcal{F}_1 = \mathcal{F}$. As above, $\mathcal{F}_2$ will denote the self-extension of $\mathcal{F}_1$ obtained in Section 8. As it is showed in Proposition 3.2,

$$\mathcal{F}_2 = \{\big(g'(\theta), g(\theta)\big) \,:\, g\in\mathcal{H}(X_{j,t})\},$$

endowed with the quotient norm of $\mathcal{H}(X_{j,t})/(\ker\delta_\theta\cap\ker\delta'_\theta)$. Let us show that the twisting process can be iterated obtaining a sequence $(\mathcal{F}_n)$ of H.I. spaces such that $\mathcal{F}_{n+m}$ is a twisted sum of $\mathcal{F}_n$ and $\mathcal{F}_m$.

Given a function $g\in\mathcal{H}(X_{j,t})$ and an integer $k\in\mathbb{N}$, we denote $\hat{g}[k] := g^{(k-1)}(\theta)/(k-1)!$, the (k)-th coefficient of the Taylor series of g at θ. Following the constructions in [9], we define for $n\geq 3$:

$$\mathcal{F}_n := \{\big(\hat{g}[n],\ldots,\hat{g}[2],\hat{g}[1]\big) \,:\, g\in\mathcal{H}(X_{j,t})\}$$

endowed with the quotient norm of $\mathcal{H}(X_{j,t})/\bigcap_{k=0}^{n-1}\ker\delta_\theta^{(k)}$. Our next result is modelled upon similar ones for twisted Hilbert spaces in [9], although the proofs may differ.

Proposition 9.1. *Let $m,n\in\mathbb{N}$ with $m>n$.*

1. *The expression $\pi_{m,n}(x_m,\ldots,x_n,\ldots,x_1) := (x_n,\ldots,x_1)$ defines a surjective operator $\pi_{m,n}:\mathcal{F}_m\to\mathcal{F}_n$.*
2. *The expression $i_{n,m}(x_n,\ldots,x_1) := (x_n,\ldots,x_1,0,\ldots,0)$ defines a isomorphic embedding $i_{n,m}:\mathcal{F}_n\to\mathcal{F}_m$ with $\mathrm{ran}(i_{n,m}) = \ker(\pi_{m,m-n})$.*
3. *The operator $\pi_{m,n}$ is strictly singular.*

Proof. (1) Since $\operatorname{dist}(g, \bigcap_{k=0}^{n-1} \ker \delta_\theta^{(k)}) \leq \operatorname{dist}(g, \bigcap_{k=0}^{m-1} \ker \delta_\theta^{(k)})$, we have $\|\pi_{m,n}\| \leq 1$. And it is obvious that $\pi_{m,n}$ is surjective.

(2) Let $\phi \in H^\infty(\mathbb{S})$ be a scalar function such that $\hat{\phi}[k] = \delta_{k,m-n}$ for $1 \leq k \leq m$. For the existence of ϕ, we consider a conformal equivalence $\varphi : \mathbb{S} \to \mathbb{D}$ satisfying $\varphi(\theta) = 0$, and the polynomial $p(z) := (z - \theta)^{m-n}$. The function $p \circ \varphi^{-1} \in H(\mathbb{D})$ admits a representation $p \circ \varphi^{-1}(\omega) = \sum_{l=0}^{\infty} a_l \omega^l$, and it is not difficult to check that $\phi(z) := \sum_{l=0}^{m} a_l \varphi(z)^l$ defines a function that satisfies the required conditions.

Given $(x_n, \dots, x_1) \in \mathcal{F}_n$, we take $g \in \mathcal{H}(X_{j,t})$ such that $\hat{g}[k] = x_k$ for $k = 1, \dots, n$. Then $f := \phi \cdot g \in \mathcal{H}(X_{j,t})$ with $\|f\| \leq \|\phi\|_\infty \cdot \|g\|$ and, by the Leibnitz rule,

$$\hat{f}[k] = \sum_{l=1}^{k} \hat{\phi}[l]\hat{g}[k-l].$$

Thus $\hat{f}[k] = 0$ for $1 \leq k \leq m-n$ and $\hat{f}[k] = \hat{g}[k-m+n]$ for $m-n < k \leq m$; i.e., $(\hat{f}[m], \dots, \hat{f}[1]) = (x_n, \dots, x_1, 0, \dots, 0)$. Hence $i_{n,m}$ is well-defined and $\|i_{n,m}\| \leq \|\phi\|_\infty$.

Clearly $i_{n,m}$ is injective and $\operatorname{ran}(i_{n,m}) \subset \ker(\pi_{m,m-n})$. Let $(y_n, \dots, y_1, 0, \dots, 0)$ in $\ker(\pi_{m,m-n})$. Then there exists $g \in \mathcal{H}(X_{j,t})$ such that $\hat{g}[k] = 0$ for $1 \leq k \leq m-n$ and $\hat{g}[k] = y_{k-m+n}$ for $m-n < k \leq m$. Since g has a zero of order $m-n$ at θ, there exists $f \in \mathcal{H}(X_{j,t})$ such that $g(z) = f(z)(z-\theta)^{m-n}$, and it is not difficult to check that $i_{n,m}(\hat{f}[n], \dots, \hat{f}[1]) = (y_n, \dots, y_1, 0, \dots, 0)$.

(3) Since $\pi_{m,n} = \pi_{m-1,n}\pi_{m,m-1}$ for $m > n+1$, it is enough to prove that $\pi_{m,m-1}$ is strictly singular. We will do it by induction:

We proved in Theorem 8.1 that $\pi_{2,1}$ is strictly singular. Let $m > 2$ and assume that $\pi_{m-1,m-2}$ is strictly singular. Note that $\pi_{m,1} = \pi_{m,2}\pi_{2,1}$; hence $\pi_{m,1}$ is also strictly singular.

We consider the following commuting diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}_{m-1} & \xrightarrow{i_{m-1,m}} & \mathcal{F}_m & \xrightarrow{\pi_{m,1}} & \mathcal{F}_1 & \longrightarrow & 0 \\ & & {\scriptstyle \pi_{m-1,m-2}}\downarrow & & \downarrow{\scriptstyle \pi_{m,m-1}} & & \| & & \\ 0 & \longrightarrow & \mathcal{F}_{m-2} & \xrightarrow[i_{m-2,m-1}]{} & \mathcal{F}_{m-1} & \xrightarrow[\pi_{m-1,1}]{} & \mathcal{F}_1 & \longrightarrow & 0. \end{array} \tag{15}$$

By (1) and (2), the two rows are exact. Suppose that M is an infinite dimensional closed subspace of $\mathcal{F}_m$ such that $\pi_{m,m-1}|_M$ is an isomorphism. Since $\pi_{m,m-1}i_{m-1,m}$ is strictly singular and $\operatorname{ran}(i_{m-1,m}) = \ker(\pi_{m,1})$, $M \cap \ker(\pi_{m,1})$ is finite dimensional and $M + \ker(\pi_{m,1})$ is closed. But this is impossible, because $\pi_{m,1}$ is strictly singular. $\square$

As an immediate consequence we get:

Corollary 9.2. *Let $m, n \in \mathbb{N}$. Then the sequence*

$$0 \longrightarrow \mathcal{F}_m \xrightarrow{i_{m,m+n}} \mathcal{F}_{m+n} \xrightarrow{\pi_{m+n,n}} \mathcal{F}_n \longrightarrow 0$$

is exact and singular. Therefore all the spaces $\mathcal{F}_n$ are H.I.

Next we show that there are natural nontrivial twisted sums of spaces $\mathcal{F}_n$ which are not H.I. Let $l, m, n \in \mathbb{N}$ with $l > n$. We consider the following push-out diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}_l \arrow[r, "i_{l,l+m}"] \arrow[d, "\pi_{l,n}"'] & \mathcal{F}_{l+m} \arrow[r, "\pi_{l+m,m}"] \arrow[d, "\pi_{l+m,k+m}"] & \mathcal{F}_m \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{F}_n \arrow[r, "i_{n,n+m}"'] & \mathcal{F}_{n+m} \arrow[r, "\pi_{n+m,m}"'] & \mathcal{F}_m \arrow[r] & 0. \end{tikzcd} \tag{16}$$

Proposition 9.3. *Let $l, m, n \in \mathbb{N}$ with $l > n$. Then the diagonal push-out sequence*

$$0 \longrightarrow \mathcal{F}_l \xrightarrow{\ i\ } \mathcal{F}_n \oplus \mathcal{F}_{l+m} \xrightarrow{\ \pi\ } \mathcal{F}_{m+n} \longrightarrow 0 \tag{17}$$

obtained from diagram (16) *is a nontrivial exact sequence.*

Proof. As we saw in Section 2, the maps i and π are given by

$$i(x) = (-\pi_{l,n}\, x,\, i_{l,l+m}\, x) \quad \text{ and } \quad \pi(y, z) = i_{n,n+m}\, y + \pi_{l+m,k+m}\, z,$$

and it is easy to check that the sequence (17) is exact. Since $l > n$, every operator from $\mathcal{F}_l$ or $\mathcal{F}_{m+n}$ into $\mathcal{F}_n$ is strictly singular. Thus $\mathcal{F}_l \oplus \mathcal{F}_{m+n}$ is not isomorphic to $\mathcal{F}_n \oplus \mathcal{F}_{l+m}$, and the exact sequence (17) is nontrivial. □

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