# COMPLEX STRUCTURES ON TWISTED HILBERT SPACES

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Abstract. We investigate complex structures on twisted Hilbert spaces, with special attention paid to the Kalton-Peck $Z_2$ space and to the hyperplane problem. For any non-trivial twisted Hilbert space, we show there are always complex structures on the natural copy of the Hilbert space that cannot be extended to the whole space. Regarding the hyperplane problem we show that no complex structure on $\ell_2$ can be extended to a complex structure on an hyperplane of $Z_2$ containing it.

## 1. Introduction

A twisted Hilbert space is a Banach space $X$ admitting a subspace $H_1$ isomorphic to a Hilbert space and so that the quotient $X/H_1$ is also isomorphic to a Hilbert space $H_2$. Or else, using the homological language, a Banach space $X$ that admits an exact sequence

$$0 \longrightarrow H_1 \longrightarrow X \longrightarrow H_2 \longrightarrow 0$$

in which both $H_1$ and $H_2$ are Hilbert spaces. The space $X$ is usually denoted $H_1 \oplus_\Omega H_2$ and the so-called quasi-linear map $\Omega$ is there to specify the form in which the norm of the direct product must be "twisted". The most interesting example for us is the Kalton-Peck space $Z_2$ [28], which is the twisted Hilbert space associated to the quasi-linear map (defined on finitely supported sequences as)

$$\mathcal{K}\left(\sum_i x_i e_i\right) = \sum_i x_i \left(\log \frac{|x_i|}{\|x\|_2}\right) e_i$$

with the understanding that $\log(|x_i|/\|x\|_2) = 0$ if $x_i = 0$.

The following problem related to real twisted Hilbert spaces remains open:

**Existence problem:** Does every twisted Hilbert space admit a complex structure?

But of course one can be more specific.

**Extension problem:** Given a twisted Hilbert space $H_1 \oplus_\Omega H_2$, can a complex structure defined on $H_1$ (resp. $H_2$) be extended (resp. lifted) to a complex structure on $H_1 \oplus_\Omega H_2$?

In this paper we shall concentrate on the extension problem, with special attention to the case of $Z_2$. Our motivation for the study of this problem is the hyperplane problem, which has its origins in Banach's book [4], and asks whether every closed hyperplane (i.e., 1-codimensional closed subspace) of any infinite dimensional Banach space is linearly isomorphic to the whole space. The first space conjectured not to be isomorphic to its hyperplanes was precisely the space $Z_2$, and whether it is so is still an open problem. Banach's hyperplane problem was solved by W.T. Gowers in 1994, [24], with the construction of a Banach space

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with unconditional basis not isomorphic to any of its proper subspaces. His space is an unconditional version of the first hereditarily indecomposable (in short H.I.) space constructed by Gowers and Maurey [25]; after Gowers' result, Gowers and Maurey showed that their space –as, in fact, any H.I. space– cannot be isomorphic to any of its proper subspaces. Since then many other examples "with few operators" and consequently not isomorphic to their hyperplanes, have appeared, including Argyros-Haydon's space on which every operator is a compact perturbation of a multiple of the identity [3]. In comparison to those examples the Kalton-Peck space has a relatively simple and natural construction: it also appears as the "derivative" (as defined in [37]) induced in $\ell_2$ by the interpolation scale of $\ell_p$ spaces; see also [9] and more recently [10]. This motivates our study of the hyperplane problem in this setting. Noting that $Z_2$ itself admits complex structures, our quest for an isomorphic invariant to distinguish between $Z_2$ and its hyperplanes leads us to consider whether the hyperplanes of $Z_2$ admit complex structures; or else, in terms of Ferenczi-Galego [22], we investigate whether $Z_2$ is "even".

Note that if hyperplanes of $Z_2$ did not admit complex structures then this would in particular imply a negative answer to the Existence problem.

For non-trivial twisted Hilbert spaces we provide a negative answer to the Extension problem. Regarding $Z_2$, we address the question of whether its complex structure is unique. Then as a partial answer to the hyperplane problem for $Z_2$, we show that no complex structure on $\ell_2$ can be extended to a complex structure on an hyperplane of $Z_2$ containing it; more generally, that hyperplanes of $Z_2$ do not admit, in our terminology, compatible complex structures.

## 2. Basic facts on complex structures on twisted sums

2.1. **Complex structures.** A complex structure on a real Banach space $X$ is a complex Banach space $Z$ which is $\mathbb{R}$-linearly isomorphic to $X$. Let $Z$ be a complex structure on $X$ and $T : X \to Z$ be an $\mathbb{R}$-linear isomorphism; then $X$ can be seen as a complex space where the multiplication by the complex number $i$ on vectors of $X$ is given by the operator $\tau x = T^{-1}(iTx)$ which is clearly an automorphism on $X$ satisfying $\tau^2 = -id_X$. Conversely, if there exists a linear operator $\tau$ on $X$ such that $\tau^2 = -id_X$, we can define on $X$ a $\mathbb{C}$-linear structure by declaring a new law for the scalar multiplication: if $\lambda, \mu \in \mathbb{R}$ and $x \in X$, then $(\lambda + i\mu).x = \lambda x + \mu\tau(x)$. The resulting complex Banach space will be denoted by $X^\tau$.

**Definition 2.1.** *A real Banach space $X$ admits a complex structure if there is a bounded linear operator $u : X \to X$ such that $u^2 = -id_X$. Two complex structures $u, v$ on $X$ are equivalent if there is a linear automorphism $\phi$ of $X$ such that $u = \phi v \phi^{-1}$ or, equivalently, if $X^u$ and $X^v$ are $\mathbb{C}$-linearly isomorphic.*

The first example in the literature of a infinite dimensional real Banach space that does not admit a complex structure was the James space, proved by Dieudonné [20]. Other examples of spaces without complex structures are the uniformly convex space constructed by Szarek [39] or the hereditary indecomposable space of Gowers and Maurey [25]. At the other extreme, we find the classical spaces $\ell_p$ or $L_p$, which admit a unique complex structure up to $\mathbb{C}$-linear isomorphism. A simple proof of this fact was provided by N.J. Kalton and appears in [23, Thm. 22].

In addition to that, there are currently known examples of spaces admitting exactly $n$ complex structures (Ferenczi [21]), infinite countably many (Cuellar [19]) and uncountably many (Anisca [1]). See also [2] for a study of complex structures in the context of classification of analytic equivalence relations on Polish spaces.

2.2. **Exact sequences and quasi-linear maps.** For a rather complete background on the theory of twisted sums see [12]. We recall that a twisted sum of two Banach spaces $Y$, $Z$ is a quasi-Banach space $X$ which has a closed subspace isomorphic to $Y$ such that the quotient $X/Y$ is isomorphic to $Z$. Equivalently, $X$ is a twisted sum of $Y$, $Z$ if there exists a short exact sequence

$$0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0.$$

According to Kalton and Peck [28], twisted sums can be identified with homogeneous maps $\Omega : Z \to Y$ satisfying

$$\|\Omega(z_1 + z_2) - \Omega z_1 - \Omega z_2\| \leq C(\|z_1\| + \|z_2\|),$$

which are called quasi-linear maps, and induce an equivalent quasi-norm on $X$ (seen algebraically as $Y \times Z$) by

$$\|(y, z)\|_\Omega = \|y - \Omega z\| + \|z\|.$$

When $Y$ and $Z$ are, for example, Banach spaces of non-trivial type, the quasi-norm above is equivalent to a norm; therefore, the twisted sum obtained is a Banach space. And when $Y$ and $Z$ are both isomorphic to $\ell_2$, the twisted sum is called a *twisted Hilbert space.* Two exact sequences $0 \to Y \to X_1 \to Z \to 0$ and $0 \to Y \to X_2 \to Z \to 0$ are said to be *equivalent* if there exists an operator $T : X_1 \to X_2$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X_1 & \longrightarrow & Z & \longrightarrow & 0 \\ & & \Vert & & \downarrow{\scriptstyle T} & & \Vert & & \\ 0 & \longrightarrow & Y & \longrightarrow & X_2 & \longrightarrow & Z & \longrightarrow & 0. \end{array}$$

The classical 3-lemma (see [12, p. 3]) shows that $T$ must be an isomorphism. An exact sequence is trivial if and only if it is equivalent to $0 \to Y \to Y \oplus Z \to Z \to 0$, where $Y \oplus Z$ is endowed with the product norm. In this case we say that the exact sequence *splits.* Two quasi-linear maps $\Omega, \Omega' : Z \to Y$ are said to be equivalent, denoted $\Omega \equiv \Omega'$, if the difference $\Omega - \Omega'$ can be written as $B + L$, where $B : Z \to Y$ is a homogeneous bounded map (not necessarily linear) and $L : Z \to Y$ is a linear map (not necessarily bounded). Two quasi-linear maps are equivalent if and only if the associated exact sequences are equivalent. A quasi-linear map is trivial if it is equivalent to the 0 map, which also means that the associated exact sequence is trivial. Given two Banach spaces $Z, Y$ we will denote by $\ell(Z, Y)$ the vector space of linear (not necessarily continuous) maps $Z \to Y$. The distance between two homogeneous maps $T, S$ will be the usual operator norm (the supremum on the unit ball) of the difference; i.e., $\|T - S\|$, which can make sense even when $S$ and $T$ are unbounded. So a quasi-linear map $\Omega : Z \longrightarrow Y$ is trivial if and only if $d(\Omega, \ell(Z, Y)) \leq C < +\infty$, in which case we will say that $\Omega$ is $C$-trivial.

In many situations it is convenient to define $\Omega$ only on a dense linear subspace $Z'$ of $Z$. In this case $\Omega$ may be extended to a quasi-linear map on $Z$ in different manners which are all equivalent; so the actual choice of the extension does not matter, and this situation may still be denoted by $\Omega : Z \to Y$.

Given an exact sequence $0 \to Y \to X \to Z \to 0$ with associated quasi-linear map $\Omega$ and an operator $\alpha : Y \to Y'$, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Y \oplus_\Omega Z & \longrightarrow & Z & \longrightarrow & 0 \\ & & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha'} & & \Vert & & \\ 0 & \longrightarrow & Y' & \longrightarrow & \text{PO} & \longrightarrow & Z & \longrightarrow & 0 \end{array} \tag{1}$$

whose lower sequence is called the *push-out sequence* and the space PO is called the *push-out space*: just set $\mathrm{PO} = Y' \oplus_{\alpha\Omega} Z$ and $\alpha'(y,z) = (\alpha y, z)$.

2.3. **Compatible complex structures on twisted sums.** The approach to complex structures as operators allows us to consider their extension/lifting properties. The commutator notation will be helpful here. Recall that given two maps $A, \Omega$ for which what follows makes sense, their commutator is defined as $[A,\Omega] = A\Omega - \Omega A$. Given three maps $A, \Omega, B$ for which what follows makes sense, we define their "commutator" as $[A,\Omega,B] = A\Omega - \Omega B$.

**Definition 2.2.** *Let $\Omega : Z \to Y$ be a quasi-linear map. Let $\tau$ and $\sigma$ be two operators on $Y$ and $Z$, respectively. The couple $(\tau,\sigma)$ is compatible with $\Omega$ if $\tau\Omega \equiv \Omega\sigma$.*

The following are well-known equivalent formulations to compatibility.

**Lemma 2.1.** *The following are equivalent:*

- *$(\tau,\sigma)$ is compatible with $\Omega$.*
- *$[\tau,\Omega,\sigma] \equiv 0$.*
- *$\tau$ can be extended to an operator $\beta : Y \oplus_\Omega Z \to Y \oplus_\Omega Z$ whose induced operator on the quotient space is $\sigma$.*
- *$\sigma$ can be lifted to an operator $\beta : Y \oplus_\Omega Z \to Y \oplus_\Omega Z$ whose restriction to the subspace is $\tau$.*
- *There is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Y\oplus_\Omega Z & \longrightarrow & Z & \longrightarrow & 0\\ & & \tau\downarrow & & \downarrow\beta & & \downarrow\sigma & & \\ 0 & \longrightarrow & Y & \longrightarrow & Y\oplus_\Omega Z & \longrightarrow & Z & \longrightarrow & 0; \end{array} \tag{2}$$

*Proof.* That $[\tau,\Omega,\sigma] \equiv 0$ means that $\tau\Omega - \Omega\sigma = B + L$ for some homogeneous bounded map $B$ and some linear map $L$. In which case, the operator $\beta(y,z) = (\tau y - Lz, \sigma z)$ is linear and continuous, and makes the diagram (2) commutative. □

The last commutativity condition can be formulated without an explicit reference to any associated quasi-linear map. So, $(\tau,\sigma)$ make a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0\\ & & \tau\downarrow & & \downarrow\beta & & \downarrow\sigma & & \\ 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

if and only $(\tau,\sigma)$ is compatible with any quasi-linear map $\Omega$ associated to the exact sequence.

A simple computation shows that if, moreover, $\tau,\sigma$ are complex structures then the operator $\beta$ defined above is a complex structure, $\beta^2 = -id$, if and only if $\tau L + L\sigma = 0$. Note also that when $(\tau,\sigma)$ is compatible with $\Omega$, the decomposition $\tau\Omega - \Omega\sigma = B + L$ above is not unique. This makes worthwhile the following corollary, which is stated to include the case when $\Omega$ is only defined on some dense linear subspace $Z'$ of $Z$.

**Corollary 2.2.** *Let $\tau$ and $\sigma$ be complex structures defined on Banach spaces $Y$ and $Z$, respectively. Assume $Z'$ is a dense linear subspace of $Z$ and that $\sigma(Z') = Z'$. Let $\Omega : Z' \to Y$ be a quasi-linear map. If $[\tau,\Omega,\sigma] : Z' \to Y$ is linear or bounded then $\tau$ can be extended to a complex structure $\beta : Y \oplus_\Omega Z \to Y \oplus_\Omega Z$ whose induced operator on the quotient space is $\sigma$.*

*Proof.* If $[\tau,\Omega,\sigma]$ is linear, set $L=[\tau,\Omega,\sigma]$ and $B=0$. This makes $\tau L+L\sigma=\tau(\tau\Omega-\Omega\sigma)+(\tau\Omega-\Omega\sigma)\sigma=-\Omega-\tau\Omega\sigma+\tau\Omega\sigma+\Omega=0$. If $[\tau,\Omega,\sigma]$ is bounded then set $B=[\tau,\Omega,\sigma]$ and $L=0$. In both cases, applying the observation after Lemma 2.1 above, we obtain an extension of $\tau$ to a complex structure on $Y\oplus_\Omega Z'$ inducing $\sigma_{|Z'}$; this complex structure extends in an unique way to a complex structure $\beta$ on $Y\oplus_\Omega Z$. □

Observe that in the case in which $[\tau,\Omega,\sigma]$ is bounded, the associated complex structure $\beta$ on $Y\oplus_\Omega Z$ appearing in the corollary is defined by $\beta(y,z)=(\tau y,\sigma z)$ on $Y\oplus_\Omega Z'$. We shall therefore denote it as $\beta=(\tau,\sigma)$. We deduce:

**Corollary 2.3.** *Let $\sigma:\mathbb{N}\to\mathbb{N}$ denote the permutation $\sigma=(2,1)(4,3)\ldots(2n,2n-1)\ldots$ and consider the complex structure $\omega$ defined on $\ell_2$ by*

$$\omega((x_n))=((-1)^n x_{\sigma(n)}).$$

*Then the map $(\omega,\omega)$ defines a complex structure on $Z_2$.*

*Proof.* If $c_{00}$ is the subspace of finitely supported vectors of $\ell_2$ we note that $\omega(c_{00})=c_{00}$. Since $[\omega,\mathcal{K}]=0$ on $c_{00}$, we deduce that $(\omega,\omega)$ is a complex structure. □

## 3. The extension problem for twisted Hilbert spaces

In this section we work with arbitrary twisted Hilbert spaces. In this context, we provide a negative answer to the extension problem:

**Theorem 3.1.** *Let $H$ be a Hilbert space and $\Omega:H\to H$ be a non-trivial quasi-linear map. Then there is a complex structure on $H$ that can not be extended to an operator on $H\oplus_\Omega H$.*

The proof follows from a criteria that will be established in Lemma 3.3. It will be helpful to recall that an operator between Banach spaces is said to be *strictly singular* if no restriction to an infinite dimensional closed subspace is an isomorphism. In this line, a quasi-linear map is said to be *singular* if its restriction to every infinite dimensional closed subspace is never trivial. It is well known [14] that a quasi-linear map is singular if and only if the associated exact sequence has strictly singular quotient map. Singular quasi-linear maps have been studied in e.g. [8], [10]. The Kalton-Peck map on $\ell_2$ is singular [28], while the function space version of the Kalton-Peck map on $L_2$ is not [5] (although it is "disjointly" singular, according to the terminology of [10]).

**Definition 3.1.** *Let $Z,Y$ be Banach spaces and $\Omega:Z\to Y$ be a quasi-linear map. Given a finite sequence $b=(b_k)_{k=1}^n$ of vectors in $Z$ we will call $\nabla_{[b]}\Omega$ the number*

$$\nabla_{[b]}\Omega=\mathrm{Ave}_\pm\left\|\Omega\left(\sum_{k=1}^n \pm b_k\right)-\sum_{k=1}^n \pm\Omega(b_k)\right\|_Y,$$

*where the average is taken over all the signs $\pm1$.*

Note that this expression is invariant by linear perturbations (even unbounded) of $\Omega$. Also the triangle inequality holds for $\nabla_{[b]}\Omega$: if $\Omega$ and $\Psi$ are quasi-linear, then $\nabla_{[b]}(\Omega+\Psi)\leq\nabla_{[b]}\Omega+\nabla_{[b]}\Psi$. If $\lambda=(\lambda_k)_k$ is a finite sequence of scalars and $x=(x_k)_k$ a sequence of vectors of $X$, we write $\lambda x$ to denote the finite sequence obtained by the non-zero vectors of $(\lambda_1x_1,\lambda_2x_2,\ldots)$.

Recall that two sequences $(x_n)_n$ and $(y_n)_n$ of a Banach space $X$ are said to be equivalent if the linear map $T$ defines on the closed span of $(x_n)_n$ by $Tx_n=y_n$, $\forall n\in\mathbb{N}$ is an isomorphism onto the closed linear span of $(y_n)_n$. That is, $(x_n)_n$ and $(y_n)_n$ are equivalent if there exists a constant $C > 0$ such that

$$C^{-1}\left\|\sum_{k=1}^{n}\lambda_k x_k\right\| \leq \left\|\sum_{k=1}^{n}\lambda_k y_k\right\| \leq C\left\|\sum_{k=1}^{n}\lambda_k x_k\right\|,$$

for every sequence of scalars $(\lambda_k)_{k=1}^n$ and every $n \in \mathbb{N}$.

Felix Cabello has pointed out to us that the behaviour of the average $\nabla\Omega$ characterizes triviality for a general quasi-linear map $\Omega$ on Hilbert spaces.

**Lemma 3.2.** *Let $H$ be a Hilbert space and $\Omega : H \to H$ be a quasi-linear map. Then the restriction of $\Omega$ to a closed subspace $W \subset H$ is trivial if and only if there is a constant $C > 0$ such that for every finite sequence $x = (x_k)_{k=1}^n$ of normalized vectors in $W$ and every finite sequence of scalars $\lambda = (\lambda_k)_{k=1}^n$,*

$$\nabla_{[\lambda x]}\Omega \leq C\|\lambda\|_2.$$

*Proof.* Suppose that $\Omega|_W$ is trivial for a closed subspace $W$ of $H$. Then we can write $\Omega|_W = B + L$ with $B$ bounded homogeneous and $L$ linear. Let $x = (x_k)_{k=1}^n$ be a normalized sequence of vectors in $W$ and $\lambda = (\lambda_k)_{k=1}^n$ be a sequence of scalars. Since $H$ has type 2 one gets:

$$\begin{aligned}
\nabla_{[\lambda x]}\Omega &= \mathrm{Ave}_{\pm}\left\|B\left(\sum_{k=1}^{n}\epsilon_k\lambda_k x_k\right) - \sum_{k=1}^{n}\epsilon_k B(\lambda_k x_k)\right\| \\
&\leq \|B\|\mathrm{Ave}_{\pm}\left\|\sum_{k=1}^{n}\epsilon_k\lambda_k x_k\right\| + Ave_{\pm}\left\|\sum_{k=1}^{n}\epsilon_k B\lambda_k x_k\right\| \\
&\leq \|B\|\left(\sum_{k=1}^{n}\|\lambda_k x_k\|^2\right)^{1/2} + \left(\sum_{k=1}^{n}\|B\lambda_k x_k\|^2\right)^{1/2} \\
&\leq 2\|B\|\|\lambda\|_2.
\end{aligned}$$

Conversely, let $(y_i, \lambda_i w_i)_{i=1}^n$ be a finite sequence of vectors in $H \oplus_\Omega W$, with $w = (w_i)_{i=1}^n$ normalized, and $\epsilon = (\epsilon_i)_{i=1}^n$ be a sequence of signs, then

$$\begin{aligned}
\left\|\sum_{i=1}^{n}\epsilon_i(y_i, \lambda_i w_i)\right\|_\Omega &= \left\|\sum_{i=1}^{n}\epsilon_i y_i - \Omega\left(\sum_{i=1}^{n}\epsilon_i\lambda_i w_i\right)\right\| + \left\|\sum_{i=1}^{n}\epsilon_i\lambda_i w_i\right\| \\
&\leq \left\|\sum_{i=1}^{n}\epsilon_i\Omega(\lambda_i w_i) - \Omega\left(\sum_{i=1}^{n}\epsilon_i\lambda_i w_i\right)\right\| + \left\|\sum_{i=1}^{n}\epsilon_i\left(y_i - \Omega(\lambda_i w_i)\right)\right\| + \left\|\sum_{i=1}^{n}\epsilon_i\lambda_i w_i\right\|.
\end{aligned}$$

Now, by taking the average in both sides and using that $H$ has type 2, we have

$$\begin{aligned}
\mathrm{Ave}_{\pm}\left\|\sum_{i=1}^{n}\epsilon_i(y_i, \lambda_i w_i)\right\|_\Omega &\leq \nabla_{[\lambda w]}\Omega + c\left(\sum_{i=1}^{n}\|y_i - \Omega(\lambda_i w_i)\|^2\right)^{1/2} + c\left(\sum_{i=1}^{n}\|\lambda_i w_i\|^2\right)^{1/2} \\
&\leq (C + 2c)\left(\sum_{i=1}^{n}\|(y_i, \lambda_i w_i)\|_\Omega^2\right)^{1/2}.
\end{aligned}$$

Therefore $H \oplus_\Omega W$ has type 2. It follows from the Maurey's extension theorem [31] that if a Banach space $X$ has type 2, then every subspace $Y$ of $X$ isomorphic to a Hilbert space is complemented in $X$. This means that H is complemented in $H \oplus_\Omega W$, and then the restriction of $\Omega$ to $W$ is trivial. □

From the Lemma, we deduce that if the restriction of a quasi-linear map $\Omega : H \curvearrowright H$ to an infinite dimensional closed subspace $W \subseteq H$ is non-trivial, then there exist for every $n \in \mathbb{N}$, a finite sequence $u_n = (u_n^i)_{i \in F_n}$ of normalized vectors in $W$ and $\lambda_n \in c_{00}$ such that

$$\nabla_{[\lambda_n u_n]} \Omega > n \|\lambda_n\|_2,$$

with $u_n^i$ orthogonal to $u_m^j$ $(n \neq m)$. We use the fact that, for every finite dimensional subspace $F$ of $W$, we can write $W = F \oplus F^\perp$ and $\Omega_{|F^\perp}$ is non-trivial.

Let us recall from Lemma 2.1 that we say that an operator $u$ on a Hilbert space $H_2$ *can be lifted* to an operator on $H_1 \oplus_\Omega H_2$ if there exists an operator $\beta$ on $H_1 \oplus_\Omega H_2$ so that $\beta_{|H_1} : H_1 \to H_1$. The next lemma yields a method to construct a complex structure on $H$ which cannot be lifted to operators on $H \oplus_\Omega H$, provided there exist two separated sequences in $H$ with sufficiently different $\nabla\Omega$. Precisely,

**Lemma 3.3.** *Let $H$ be Hilbert space and let $\Omega : H \to H$ be a quasi-linear map. Suppose that $H$ contains two normalized sequences of vectors $a = (a_n)_n$, $b = (b_n)_n$, such that:*

(i) *$(a_n)_n$ and $(b_n)_n$ are equivalent sequences,*
(ii) *$[a] \oplus [b]$ forms a direct sum in $H$,*
(iii)

$$\sup_{\lambda \in c_{00}} \frac{\nabla_{[\lambda b]} \Omega}{\|\lambda\|_2 + \nabla_{[\lambda a]} \Omega} = +\infty$$

*Then $H$ admits a complex structure $u$ that cannot be lifted to any operator on $H \oplus_\Omega H$.*

*Proof.* There is no loss of generality in assuming that $\Omega$ takes values in $H$ and $[a] \oplus [b]$ spans an infinite codimensional subspace of $H$. Thus write $H = [a] \oplus [b] \oplus Y$ and let $v : Y \to Y$ be any choice of complex structure on $Y$. We define a complex structure $u$ on $H$ by setting $u(a_n) = b_n$, $u(b_n) = -a_n$, and $u_{|Y} = v$.

Note that if $\tau : H \to H$ is any bounded linear operator then for every $\lambda \in c_{00}$ one has

$$\nabla_{[\lambda b]} \left(\tau \Omega u^{-1}\right) \leq \|\tau\| \nabla_{[\lambda b]} \left(\Omega u^{-1}\right) = \|\tau\| \nabla_{[\lambda a]} \Omega. \tag{3}$$

Now, assume that $u$ lifts to a bounded operator $\beta$ on $H \oplus_\Omega H$, which means that there exists a bounded operator $\tau$ on $H$ such that $(\tau, u)$ is compatible with $\Omega$. Thus, by Lemma 2.1, $\tau\Omega - \Omega u = B + L$, with $B$ homogeneous bounded and $L$ linear. Therefore, $\tau \Omega u^{-1} = \Omega + B u^{-1} + L u^{-1}$. By the triangular inequality it follows

$$|\nabla_{[\lambda b]} \left(\tau \Omega u^{-1}\right) - \nabla_{[\lambda b]} \Omega| \leq \nabla_{[\lambda b]} (\tau \Omega u^{-1} - \Omega) = \nabla_{[\lambda a]} B \leq 2\|B\| \|\lambda\|_2, \tag{4}$$

for all $\lambda \in c_{00}$. Thus, the combination of (4) and (3) yields,

$$\nabla_{[\lambda b]} \Omega \leq \|\tau\| \nabla_{[\lambda a]} \Omega + 2\|B\| \|\lambda\|_2,$$

which contradicts (iii). □

Recall that an operator $T$ between two Banach spaces $X$ and $Y$ is said to be *super-strictly singular* if every ultrapower of $T$ is strictly singular; which, by localization, means that there does not exist a number $c > 0$ and a sequence of subspaces $E_n \subseteq X$, $\dim E_n = n$, such that $\|Tx\| \geq c\|x\|$ for all $x \in \cup_n E_n$. Super-strictly singular operators have also been called finitely strictly singular; they were first introduced in [32, 33], and form a closed ideal containing the ideal of compact and contained in the ideal of strictly singular operators. See also [17] for the study of such a notion in the context of twisted sums. An exact sequence

$0 \to Y \to X \xrightarrow{q} Z \to 0$ (resp. a quasi-linear map) is called *supersingular* if $q$ is super-strictly singular. It follows from the general theory of twisted sums that an exact sequence induced by a map $\Omega : Z \to Y$ is supersingular if and only if for all $C > 0$ there exists $n \in \mathbb{N}$ such that for all $F \subset Z$ of dimension $n$, and all linear maps $L$ defined on $F$ such that $\Omega_{|F} - L : F \to Y$ one has $\|\Omega_{|F} - L\| \geq C$.

From the study in [36] we know [36, Thm 3] that a super-strictly singular operator on a $B$-convex space has super-strictly singular adjoint. Therefore, no supersingular quasi-linear maps $\Omega : Z \to Y$ exist between $B$-convex spaces $Z$ and $Y$, since $B$-convexity is a 3-space property (see [12]) and the adjoint of a quotient map is an into isomorphism.

We are thus ready to obtain:

**Proof of Theorem 3.1** Let $\Omega : H \to H$ be a non-trivial quasi-linear map on a Hilbert space $H$, and pick the dual quasi-linear map $\Omega^* : H^* \to H^*$: if $\Omega$ produces the exact sequence

$$0 \longrightarrow H \longrightarrow H \oplus_\Omega H \longrightarrow H \longrightarrow 0,$$

then $\Omega^*$ is is the responsible to produce the dual exact sequence

$$0 \longrightarrow H^* \longrightarrow (H \oplus_\Omega H)^* \longrightarrow H^* \longrightarrow 0.$$

The existence and construction of $\Omega^*$ can be found in [6, 15].

We work from now on with $\Omega^*$ and the identification $H^* = H$. We claim that there exists a decomposition $H^* = H_1 \oplus H_2$, such that $\Omega^*_{|H_1}$ is non-trivial and $\Omega^*_{|H_2}$ is not supersingular. If, for example, $\Omega^*_{|H_1}$ is trivial then $\Omega^*_{|H_2}$ must be non-trivial and we are done. So we may assume that $\Omega^*_{|H_1}$ and $\Omega^*_{|H_2}$ are non-trivial. By the ideal properties of super-strictly singular operators and the characterization of supersingularity, we note that either $\Omega^*_{|H_1}$ or $\Omega^*_{|H_2}$ must be non supersingular. So up to appropriate choice of $H_1$ and $H_2$, we are done.

Since $\Omega^*_{|H_1}$ is non-trivial, the remark below Lemma 3.2 provides, for every $n \in \mathbb{N}$, a finite sequence $f^n = (f_i^n)_{i \in F_n}$ of normalized vectors in $H_1$ and $\lambda^n \in c_{00}$ such that $f_i^n$ is orthogonal to $f_j^m$ $(n \neq m)$ and

$$\nabla_{[\lambda^n f]}\Omega > n\|\lambda^n\|_2,$$

where $f = (f^1, f^2, \ldots)$ is the concatenation of all the $f^n$'s. On the other hand, since $\Omega^*_{|H_2}$ is not supersingular, there exists $C$ and a sequence $G_n$ of $n$-dimensional subspaces of $H_2$, such that $\Omega_{|G_n} = L_n + B_n$, where $L_n$ is linear and $\|B_n\| \leq C$. Now we choose, for every $n \in \mathbb{N}$, a finite sequence $g^n = (g_i^n)_{i \in \mathcal{G}_n}$ of normalized vectors in $G_n$ 1-equivalent to $f^n$. By Lemma 3.2, for all $\lambda \in c_{00}$ one has $\nabla_{[\lambda g^n]}\Omega \leq 2C\|\lambda\|_2$ and therefore

$$\nabla_{[\lambda^n g^n]}\Omega \leq 2C\|\lambda^n\|_2.$$

There is no loss of generality assuming that for different $n, m$ the elements of $g^n$ and $g^m$ are orthogonal. Thus, pasting all those pieces we construct a sequence $g = (g^1, g^2, \ldots, )$ of normalized vectors in $H_2$, and we note that $f$ and $g$ are equivalent sequences such that $[f]$ and $[g]$ form a direct sum. Then we apply Lemma 3.3 to get a complex structure $u$ on $H^*$ so that $(\tau, u)$ is not compatible with $\Omega^*$. Therefore $u^*$ is a complex structure on $H$ so that for every operator $\tau$ the couple $(u^*, \tau)$ is not compatible with $\Omega$. $\square$

We conclude this section with several remarks about the previous results and some of their consequences.

3.1. $\mathcal{K}$ **is not supersingular.** The papers [17, 38] contain proofs that the Kalton-Peck map $\mathcal{K}$ is not supersingular. A proof assuming some familiarity with complex interpolation and inducing an estimate of $\nabla\mathcal{K}$ can be presented: we consider the interpolation scale $\ell_2 = (\ell_p, \ell_{p'})_{1/2}$ on the strip $\mathbb{S}$, for a fixed $p$, where $1/p + 1/p' = 1$ and recall that $\mathcal{K}$ is the "centralizer" induced by this scale. This means that we choose $B_x : \mathbb{S} \to \Sigma = \ell_p + \ell_{p'}$ any holomorphic selection for the evaluation map $\delta_{1/2}$ (i.e. satisfying $B_x(1/2) = x$ for $x \in \ell_2$) which is homogeneous and bounded (with respect to the usual norm $\|f\|_{\mathcal{H}} = \sup_{z\in\delta\mathbb{S}} \|f(z)\|_z$). Then $B'_x(1/2)$ may be used to define the twisted sum, in the sense that there exist constants $\lambda, D$ such that for all $x \in \ell_2$,

$$\|\mathcal{K}(x) - \lambda B'_x(1/2)\| \leq D\|x\|_2. \tag{5}$$

For a complete exposition of this situation we refer to [10, Section 3] (more specifically, [10, Proposition 3.7] and the remark before [10, Proposition 3.2]).

For every $n$ even let us now consider the finite sequence $f^n = (f^n_k)_{k=1}^n$ of orthonormalized vectors in $\ell_2$ defined by

$$\begin{aligned}
f^n_1 &= \frac{1}{\sqrt{2^n}} \sum_{j=1}^{2^n} (-1)^{j+1} e_{2^n+j-1} \\
f^n_2 &= \frac{1}{\sqrt{2^n}} \sum_{j=1}^{2^{n-1}} (-1)^{j+1} (e_{2^n+2(j-1)} + e_{2^n+2j-1}) \\
&\vdots \\
f^n_n &= \frac{(e_{2^n} + e_{2^n+1} + \cdots + e_{2^n+2^{n-1}+1}) - (e_{2^n+2^{n-1}} + e_{2^n+2^{n-1}+1} + \cdots + e_{2^{n+1}-1})}{\sqrt{2^n}}.
\end{aligned}$$

For $x = \sum_{k=1}^n a_k f^n_k$ take the holomorphic function $g_x : \mathbb{S} \to \Sigma = \ell_p + \ell_{p'}$ given by $g_x(z) = 2^{n(1/2-1/p)(1-2z)} \sum_{k=1}^n a_k f^n_k$, which obviously satisfies $g_x(1/2) = x$. To compute the norm of $g_x$ recall Khintchine's inequality [30, Theorem 2.b.3], which yields constants $A_p$, $B_p$ such that

$$A_p 2^{(1/p-1/2)n} \sqrt{\sum_{k=1}^n \lambda_k^2} \leq \left\| \sum_{k=1}^n \lambda_k f^n_k \right\|_p \leq B_p 2^{(1/p-1/2)n} \sqrt{\sum_{k=1}^n \lambda_k^2}$$

for all $n$ and scalars $\lambda_1, \ldots, \lambda_n$. Thus, $\|g_x\|_{\mathcal{H}} \leq c_p\|x\|_2$ for some constant $c_p$ independent of $n$ and the scalars $a_k$, and therefore $x \mapsto g_x$ is a homogeneous bounded selection for the evaluation map $\delta_{1/2}$. Thus, by the estimate (5), we have a constant $D_p$ such that

$$\|\mathcal{K}(x) - g'_x(1/2)\| \leq D_p\|x\|_2. \tag{6}$$

for $x$ in the span of $\{f^n_k\}$. Since $g'_x(1/2) = -2n \log n(1/2 - 1/p) \sum_{k=1}^n a_k f^n_k$, it is a linear map, which in particular shows that $\mathcal{K}$ is not supersingular.

3.2. **A computation of $\nabla\mathcal{K}$.** The use of supersingularity is necessary when $\Omega$ is singular, since then no linear map approximates $\Omega$ on $[g] = [g^1, g^2, \ldots]$. In the case of Kalton-Peck space, an explicit example of two sequences in which $\nabla\mathcal{K}$ behaves differently can be explicitly given: one is the canonical basis $a_n = (e_{n_k})_{k=1}^n$ where $\nabla_{[a_n]}\mathcal{K} = \frac{1}{2}\sqrt{n}\log n$ and the other the sequence of finite sequences $f^n = (f^n_k)_{k=1}^n$ on which $\nabla_{[f^n]}\mathcal{K} \leq D_p\sqrt{n}$ as it immediately follows from the inequality (6) above.

3.3. **Initial and final twisted Hilbert spaces.** Theorem 3.1 answers a question in [34] raised by homological consideration about the existence of initial or final objects in the category of twisted Hilbert spaces. In its bare bones, the question is whether there is an *initial* twisted Hilbert space

$$0 \longrightarrow \ell_2 \longrightarrow Z \longrightarrow \ell_2 \longrightarrow 0;$$

i.e., a twisted Hilbert space such that for any other twisted Hilbert space

$$0 \longrightarrow \ell_2 \longrightarrow TH \longrightarrow \ell_2 \longrightarrow 0$$

there is an operator $\tau : \ell_2 \to \ell_2$ and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_2 & \longrightarrow & Z & \longrightarrow & \ell_2 & \longrightarrow & 0 \\ & & {\scriptstyle\tau}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \ell_2 & \longrightarrow & TH & \longrightarrow & \ell_2 & \longrightarrow & 0. \end{array}$$

In particular it was asked whether the Kalton-Peck space could be an initial twisted Hilbert space. The dual question about the existence of a final twisted Hilbert space was also posed, meaning the existence of a twisted Hilbert space $Z$ such that for any twisted Hilbert space $TH$, there exists $\tau : \ell_2 \to \ell_2$ and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_2 & \longrightarrow & Z & \longrightarrow & \ell_2 & \longrightarrow & 0 \\ & & \| & & \downarrow & & {\scriptstyle\tau}\downarrow & & \\ 0 & \longrightarrow & \ell_2 & \longrightarrow & TH & \longrightarrow & \ell_2 & \longrightarrow & 0. \end{array}$$

A consequence of Theorem 3.1 is:

**Proposition 3.4.** *No twisted Hilbert space can be initial or final.*

*Proof.* The existence of non-trivial twisted Hilbert spaces implies that the Hilbert space itself cannot be initial or final. So let $H \oplus_\Omega H$ be a non-trivial twisted Hilbert space. Theorem 3.1 yields an isomorphism $u$ on $H$ that cannot be lifted to an operator on $H \oplus_\Omega H$. This means that $\Omega u$ is never equivalent to $\tau\Omega$ for any operator $\tau$. I.e., $\Omega$ is not initial. A similar argument, using an isomorphism on $H$ that cannot be extended to an operator on $H \oplus_\Omega H$ shows that $\Omega$ cannot be final. □

3.4. **About $\ell_2$-automorphy and $\ell_2$-extensibility.** Theorem 3.1 shows that no non-trivial twisted Hilbert sum can be $\ell_2$-*extensible.* A space $X$ is said to be $Y$-extensible if for every subspace $A$ of $X$ isomorphic to $Y$ every operator from $A$ to $X$ can be extended to $X$. This notion was introduced in [35] in connection with the automorphic space problem of Lindenstrauss and Rosenthal [29]. A space $X$ is *automorphic (resp. $Y$-automorphic)* when isomorphisms between subspaces of $X$ of same codimension (resp. and isomorphic to $Y$) can be extended to automorphisms of $X$. It was known [11, p. 675] that no non-trivial twisted Hilbert space can be automorphic and of course $Z_2$ cannot be $\ell_2$-automorphic since $Z_2$ is isomorphic to its square. As a rule, $Y$-automorphic implies $Y$-extensible although the converse fails (see [13, 35, 16, 11] for details).

## 4. Complex structures on $Z_2$

In this section we will study in detail natural complex structures on $Z_2$. Our motivation is investigating whether $Z_2$ admits unique complex up to isomorphism. Recall that $\sigma : \mathbb{N} \to \mathbb{N}$ is the permutation $\sigma = (2,1)(4,3)\dots(2n,2n-1)\dots$ and that $\omega$ is the linear map defined as

$$\omega((x_n)) = ((-1)^n x_{\sigma(n)}).$$

In Corollary 2.3 we noted that $Z_2$ admits the complex structure $(\omega,\omega)$, which extends $\omega$. And from Theorem 3.1 that there is a complex structure on $\ell_2$ that does not extend to an operator on $Z_2$. There are actually at least three natural ways to consider complex structures on $Z_2$. The first, as we already said, is to simply consider those extended from complex structures on $\ell_2$ commuting with $\mathcal{K}$, like $(\omega,\omega)$. We will call $Z_2^\omega$ the resulting space. Secondly, the space $Z_2\oplus Z_2$ can be endowed with the complex structure $U(x,y)=(-y,x)$. Since $Z_2\simeq Z_2\oplus Z_2$ this yields a complex structure on $Z_2$. We will call this complex space $Z_2\oplus_{\mathbb{C}} Z_2$. There is a third way to define a natural complex space associated to $Z_2$: by directly forming the the complex Kalton-Peck space $Z_2(\mathbb{C})$ [28] starting with the complex space $\ell_2(\mathbb{C})$ and using the complex Kalton-Peck map $\mathcal{K}^{\mathbb{C}}(x)=x\log\frac{|x|}{\|x\|}$. It turns out that all of them are isomorphic.

**Proposition 4.1.** *The spaces $Z_2\oplus_{\mathbb{C}} Z_2$, $Z_2^\omega$ and $Z_2(\mathbb{C})$ are isomorphic.*

*Proof.* As usual, $\{e_n\}$ will be the canonical basis of $\ell_2$. Thus, if $x=\sum x_ne_n\in\ell_2$, then $u(x)=\sum -x_{2n}e_{2n-1}+\sum x_{2n-1}e_{2n}$. The isomorphism $T: Z_2^\omega\to Z_2\oplus_{\mathbb{C}} Z_2$ can be defined by the $\mathbb{C}$-linear map

$$T(x,y)=\left(\left(\sum x_{2n-1}e_n,\sum y_{2n-1}e_n\right),\left(\sum x_{2n}e_n,\sum y_{2n}e_n\right)\right),$$

for $(x,y)=(\sum x_ne_n,\sum y_ne_n)\in Z_2$. Let us write, $x_i=\sum x_{2n-1}e_n$ and $x_p=\sum x_{2n}e_n$. One has, $\|T(x,y)\|=\|((x_i,y_i),(x_p,y_p))\|=\|(x_i,y_i)\|+\|(x_p,y_p)\|$.

Without loss of generality, suppose $\|y_i\|_2=1$ e $\|y_p\|_2\le 1$. In this case $\|y\|_2^2=\|y_i\|_2^2+\|y_p\|_2^2=1+\|y_p\|_2^2$. Now, since $\|(x_i,y_i)\|=\|x_i-\mathcal{K}y_i\|_2+\|y_i\|_2$, for every $n$ we have,

$$\begin{aligned}
|(x_i)_n-(\mathcal{K}y_i)_n| &= \left|x_{2n-1}-y_{2n-1}\log\frac{\|y_i\|_2}{|y_{2n-1}|}\right|\\
&\le \left|x_{2n-1}-y_{2n-1}\log\frac{\|y\|_2}{|y_{2n-1}|}\right|+|y_{2n-1}|\left|\log\frac{\|y\|_2}{|y_{2n-1}|}-\log\frac{\|y_i\|_2}{|y_{2n-1}|}\right|\\
&= \left|x_{2n-1}-y_{2n-1}\log\frac{\|y\|_2}{|y_{2n-1}|}\right|+|y_{2n-1}|\,|\log\|y\|_2|
\end{aligned}$$

Therefore

$$\begin{aligned}
\|(x_i,y_i)\| &\le \|(x,y)\|+\|y_i\|_2\,|\log\|y\|_2|+\|y_i\|_2\\
&= \|(x,y)\|+\|y_i\|_2\left(1+\frac{1}{2}\left|\log(1+\|y_p\|_2^2)\right|\right)\\
&\le \|(x,y)\|+(1+\log\sqrt{2})\|y\|_2\\
&\le (2+\log\sqrt{2})\|(x,y)\|
\end{aligned}$$

Analogously, since $\|(x_p,y_p)\|=\|x_p-\mathcal{K}y_p\|_2+\|y_p\|_2$, we have for every $n$,

$$\begin{aligned}
|(x_p)_n-(\mathcal{K}y_p)_n| &= \left|x_{2n}-y_{2n}\log\frac{\|y_p\|_2}{|y_{2n}|}\right|\\
&\le \left|x_{2n}-y_{2n}\log\frac{\|y\|_2}{|y_{2n}|}\right|+|y_{2n}|\left|\log\frac{\|y\|_2}{|y_{2n}|}-\log\frac{\|y_p\|_2}{|y_{2n}|}\right|\\
&= \left|x_{2n}-y_{2n}\log\frac{\|y\|_2}{|y_{2n}|}\right|+|y_{2n}|\left|\log\frac{\|y\|_2}{\|y_p\|_2}\right|\\
&= \left|x_{2n}-y_{2n}\log\frac{\|y\|_2}{|y_{2n}|}\right|+\frac{1}{2}|y_{2n}|\left|\log\left(1+\frac{1}{\|y_p\|_2^2}\right)\right|
\end{aligned}$$

Therefore

$$\begin{aligned}\|(x_p,y_p)\| &\le \|(x,y)\|+\frac{1}{2}\|y_p\|_2\left|\log\left(1+\frac{1}{\|y_p\|_2^2}\right)\right|+\|y_p\|_2\\ &= \|(x,y)\|+\frac{1}{2}+\|y_p\|_2\\ &\le \|(x,y)\|+2\|y\|_2\\ &\le 3\|(x,y)\|\end{aligned}$$

because $|t\log(1+1/t^2)|\le 1$ for $0<t<1$. So, we conclude that $\|T(x,y)\|\le(5+\log\sqrt{2})\|(x,y)\|$. The general case is immediate.

We show the isomorphism between $Z_2^\omega$ and $Z_2(\mathbb{C})$. Denote by $\{f_n\}$ the canonical basis for $\ell_2(\mathbb{C})$ and define the complex isomorphism $A:\ell_2^\omega\to\ell_2(\mathbb{C})$ given by

$$A(\sum x_nf_n)=\sum(x_{2n-1}+ix_{2n})e_n.$$

The (real) Kalton-Peck map $\mathcal{K}$ makes sense as a complex quasi-linear map $\ell_2^\omega\longrightarrow\ell_2^\omega$ since it is $\mathbb{C}$-homogeneous: $\mathcal{K}(i\cdot x)=\mathcal{K}(\omega x)=\omega\mathcal{K}(x)=i\cdot\mathcal{K}(x)$. Let us show that there is a commutative diagram:

$$\begin{array}{ccccccccc}0&\longrightarrow&\ell_2^\omega&\longrightarrow&\ell_2^\omega\oplus_{\mathcal{K}}\ell_2^\omega&\longrightarrow&\ell_2^\omega&\longrightarrow&0\\ &&{\scriptstyle A}\downarrow&&{\scriptstyle (A,A)}\downarrow&&\downarrow{\scriptstyle A}&&\\ 0&\longrightarrow&\ell_2(\mathbb{C})&\longrightarrow&\ell_2(\mathbb{C})\oplus_{\mathcal{K}^c}\ell_2(\mathbb{C})&\longrightarrow&\ell_2(\mathbb{C})&\longrightarrow&0;\end{array}$$

for which we will check that $A\mathcal{K}-\mathcal{K}^cA$ is bounded on finitely supported sequences. Since

$$A\mathcal{K}(x)=A(\sum x_n\log\frac{\|x\|_2}{|x_n|}f_n)=\sum\left(x_{2n-1}\log\frac{\|x\|_2}{|x_{2n-1}|}+ix_{2n}\log\frac{\|x\|_2}{|x_{2n}|}\right)e_n$$

and

$$\mathcal{K}^cA(x)=\mathcal{K}^c\left(\sum(x_{2n-1}+ix_{2n})e_n\right)=\sum\left((x_{2n-1}+ix_{2n})\log\frac{\|x\|_2}{|x_{2n-1}+ix_{2n}|}\right)e_n$$

one gets

$$\begin{aligned}\|A\mathcal{K}(x)-\mathcal{K}^cA(x)\|^2 &= \left\|\sum\left(x_{2n-1}\log\frac{|x_{2n-1}+ix_{2n}|}{|x_{2n-1}|}+ix_{2n}\log\frac{|x_{2n-1}+ix_{2n}|}{|x_{2n}|}\right)e_n\right\|^2\\ &= \sum|x_{2n-1}+ix_{2n}|^2\left|\frac{x_{2n-1}}{x_{2n-1}+ix_{2n}}\right|^2\left|\log\frac{|x_{2n-1}+ix_{2n}|}{|x_{2n-1}|}\right|^2\\ &+ \sum|x_{2n-1}+ix_{2n}|^2\left|\frac{x_{2n}}{x_{2n-1}+ix_{2n}}\right|^2\left|\log\frac{|x_{2n-1}+ix_{2n}|}{|x_{2n}|}\right|^2\\ &\le 2\sum|x_{2n-1}+ix_{2n}|^2\\ &= 2\|x\|_2^2.\end{aligned}$$

□

Although every non trivial twisted Hilbert space does not have an unconditional basis [27], the next corollary shows a relation between the complex structures of $Z_2$ and the complexification space $Z_2 \oplus_{\mathbb{C}} Z_2$ that is analogous to that for Banach spaces with subsymmetric basis proved in [18].

**Corollary 4.2.**

(1) *For any complex structure $u$ on $Z_2$, the space $Z_2^u$ is isomorphic to a complemented subspace of $Z_2(\mathbb{C})$.*
(2) *For any complex structure $u$ on $Z_2$, the space $Z_2^u$ is $Z_2(\mathbb{C})$-complementably saturated and $\ell_2(\mathbb{C})$-saturated.*

*Proof.* The first assertion is a general fact: given a complex structure $u$ on some real space $X$, $X^u$ is complemented in $X \oplus_{\mathbb{C}} X$: indeed, $X \oplus_{\mathbb{C}} X = \{(x, ux), x \in X\} \oplus \{(x, -ux), x \in X\}$, the first summand being $\mathbb{C}$-linearly isomorphic to $X^{-u}$, and the second to $X^u$. The second assertion follows from results of Kalton-Peck [28]: complex structures on $Z_2$ inherit saturation properties of the complex $Z_2(\mathbb{C})$. □

**Proposition 4.3.** *Let $u$ be a complex structure on $Z_2$. If $Z_2^u$ is isomorphic to its square then it is isomorphic to $Z_2(\mathbb{C})$.*

*Proof.* The space $Z_2(\mathbb{C})$ is isomorphic to its square. Use the Corollary above and Pełczyński's decomposition method. □

## 5. Compatible complex structures on the hyperplanes of $Z_2$

As we mentioned in the introduction, our main motivation is the old open problem of whether $Z_2$ is isomorphic to its hyperplanes. Since $Z_2$ admits complex structures, showing that hyperplanes of $Z_2$ do not admit complex structures would show that $Z_2$ is not isomorphic to its hyperplanes. What we are going to show is that hyperplanes of $Z_2$ do not admit complex structures compatible, in a sense to be explained, with its twisted Hilbert structure. More precisely, we already know the existence of complex structures on $\ell_2$ that do not extend to complex structures (or even operators) on $Z_2$, and we shall show in this section that no complex structure on $\ell_2$ can be extended to a complex structure *on a hyperplane of $Z_2$*. An essential ingredient to prove this is [22, Proposition 8], which may be stated as:

**Proposition 5.1.** *Let $u, h$ be complex structures on, respectively, an infinite dimensional real Banach space $X$ and some hyperplane $\mathcal{H}$ of $X$. Then the operator $u_{|\mathcal{H}} - h$ is not strictly singular.*

This proposition may be understood as follows. It is clear that $u_{|\mathcal{H}}$ may not be *equal* to $h$. For then the induced quotient operator $\tilde{u}(x + \mathcal{H}) = ux + \mathcal{H}$ would be a complex structure on $X/\mathcal{H}$, which has dimension 1. The result above extends this observation to strictly singular perturbations.

The following result appears proved in [7]:

**Lemma 5.2.** *Assume one has a commutative diagram:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{q} & C & \longrightarrow & 0 \\ & & \downarrow t & & \downarrow T & & \| & & \\ 0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

*If both $q$ and $t$ are strictly singular then $T$ is strictly singular.*

Consequently, if one has an exact sequence $0 \to Y \to X \xrightarrow{q} Z \to 0$ with strictly singular quotient map $q$ and an operator $t : Y \to Y'$ admitting two extensions $T_i : X \to X'$, $(i = 1, 2)$, so that one has commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X & \xrightarrow{q} & Z & \longrightarrow & 0 \\ & & {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle T_i} & & \| & & \\ 0 & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

then necessarily $T_1 - T_2$ is strictly singular simply because $(T_2 - T_1)|_Y = 0$ and thus one has the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X & \xrightarrow{q} & Z & \longrightarrow & 0 \\ & & {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle T_1-T_2} & & \| & & \\ 0 & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & Z & \longrightarrow & 0. \end{array}$$

From this we get:

**Proposition 5.3.** *Let $0 \to Y \to X \xrightarrow{q} Z \to 0$ be an exact sequence of real Banach spaces with strictly singular quotient map. Let $u$ be a complex structure on $Y$ and let $\mathcal{H}$ be a hyperplane of $X$ containing $Y$. Then $u$ does not extend simultaneously to a complex structure on $X$ and to a complex structure on $\mathcal{H}$.*

*Proof.* Let $\gamma$ be a complex structure on $X$ such that $\gamma_{|_Y} = u$. Assume that $\gamma^{\mathcal{H}}$ is a complex structure on $\mathcal{H}$ extending also $u$, then it follows from the previous argument that $\gamma_{|_{\mathcal{H}}} - \gamma^{\mathcal{H}}$ is strictly singular; which contradicts Proposition 5.1. □

And therefore, since the Kalton-Peck map is singular:

**Corollary 5.4.** *Let $u$ be any complex structure on $\ell_2$ and let $\mathcal{H}$ be a hyperplane of $Z_2$ containing the canonical copy of $\ell_2$ into $Z_2$. Assume $u$ extends to a complex structure on $Z_2$. Then it cannot extend to a complex structure on $\mathcal{H}$.*

We shall show that the conclusion in the corollary always holds. We shall actually prove this in a slightly more general setting by isolating the notion of complex structure on an hyperplane of a twisted sum *compatible* with the twisted sum.

**Definition 5.1.** *Let $0 \to Y \to X \to Z \to 0$ be an exact sequence with associated quasi-linear map $\Omega$. We will say that a complex structure $u$ on an hyperplane $\mathcal{H}$ of $X$ is compatible with $\Omega$, or simply compatible, if $u(\mathcal{H} \cap Y) \subset Y$.*

In other words, $u$ is compatible if $u$ restricts to a complex structure on $\mathcal{H} \cap Y$. Of course if the hyperplane $\mathcal{H}$ contains $Y$ then this is simply restricting to a complex structure on $Y$. Our purpose is now to show:

**Theorem 5.5.** *No complex structure on a hyperplane of $Z_2$ is compatible with Kalton-Peck map $\mathcal{K}$.*

We first need the next perturbation lemma, based on essentially well-known ideas in the theory of twisted sums.

**Lemma 5.6.** *Let $Y$, $X$, $Z$ be Banach spaces. Let $\Omega : Z \to Y$ be a quasi-linear, and let $N : X \to Z$ be a nuclear operator of the form $N = \sum_n s_n x_n^*(\cdot) z_n$, $x_n^*$ normalized in $X^*$, $z_n$ normalized in $Z$, $s = (s_n)_n \in \ell_1$. Then $\Omega N : X \to Y$ is $C$-trivial for $C = Z(\Omega)\|s\|_1$. In the case when $X = Y$ and $Z$ is $B$-convex then there is a constant $c_Z$, depending only on $Z$, such that also $N\Omega : Z \to Y$ is $c_Z Z(\Omega)\|s\|_1$-trivial.*

*Proof.* The operator $N : X \to Z$ factorizes as $N = zDf$ where $f : X \to \ell_\infty$ is $f(x) = (x^*_m(x))_m$, $D : \ell_\infty \to \ell_1$ is $D(\xi) = (\xi_m s_m)_m$ and $z : \ell_1 \to Z$ is $z(\xi) = \sum_m \xi_m z_m$. The fact that $N : X \to Z$ factorizes through $\ell_1$ guarantees that $\Omega N$ is trivial on $X$. Indeed if $L$ is the linear map defined on $c_{00}$ by $L(\xi) = \sum_m \xi_m \Omega(z_m)$, then $\|(\Omega z - L)(\xi)\| \leq Z(\Omega)\|\xi\|_1$. Therefore for $x \in X$,

$$\|\Omega N(x) - LDf(x)\| = \|(\Omega z - L)Df(x)\| \leq Z(\Omega)\|Df(x)\|_1 \leq Z(\Omega)\|s\|_1\|x\|.$$

We conclude by noting that $LDf$ is linear. When $X = Y$ the fact that $N$ factorizes through $\ell_\infty$ guarantees that $N\Omega$ is trivial. Indeed according to [26, Proposition 3.3] each coordinate of the map $f\Omega : Z \to \ell_\infty$ may be approximated by a linear map $\ell_n : Z \to \mathbb{R}$ with constant $c_Z Z(\Omega)$, for some constant $c_Z$; therefore $f\Omega$ is at finite distance $c_Z Z(\Omega)$ from a linear map $L : Z \to \ell_\infty$, and $N\Omega = zDf\Omega$ is at distance $c_Z Z(\Omega)\|s\|_1$ to $zDL$. □

We need some extra work. Given a quasi-linear map $\Omega : Z \to Y$ and a closed subspace $W = [u_n] \subset Z$ generated by a basic sequence, we define the following quasi-linear map, associated to $W$, on finite combinations $x = \sum \lambda_n u_n$,

$$\Omega_W(\sum \lambda_n u_n) = \Omega(\sum \lambda_n u_n) - \sum \lambda_n \Omega u_n.$$

Since the difference $\Omega_{|W} - \Omega_W$ is linear, we have that the quasi-linear maps $\Omega_{|W}$ and $\Omega_W$ are equivalent.

Assume that $T$ is an isomorphism on $Z$, then $(Tu_n)$ is again a basic sequence and $TW = [Tu_n]$, so the meaning of $\Omega_{TW}$ is clear. One thus has $[T, \Omega]_{|W} \equiv T\Omega_W - \Omega_{TW}T_{|W}$.

The following proposition will be useful.

**Proposition 5.7.** *For every operator $T : \ell_2 \to \ell_2$, and for every block subspace $W$ of $X$, the commutator $[\mathcal{K}, T]$ is trivial on some block subspace of $W$. In particular, $[\mathcal{K}, T]$ is not singular.*

*Proof.* Assume that $T$ is an isomorphism on a block subspace $W = [u_n]$. By Lemma 5.6 we may replace $T$ by a nuclear perturbation to assume also that $\{Tu_n\}$ is a block basis (jumping to a subsequence if necessary). It is not hard to check (see also [8, Lemma 3]) that $\mathcal{K}_W$ is a linear perturbation of the Kalton-Peck map of $W$; i.e. for $x = \sum \lambda_n u_n$ one has (identities are up to a linear map):

$$\mathcal{K}_W(x) \equiv \sum \lambda_n \log \frac{\|x\|}{|\lambda_n|} u_n$$

and

$$\mathcal{K}_{TW}(Tx) \equiv \sum \lambda_n \log \frac{\|Tx\|}{|\lambda_n|} Tu_n.$$

Thus,

$$[\mathcal{K}, T]_{|W}(x) \equiv \sum \lambda_n \log \frac{\|x\|}{|\lambda_n|} Tu_n - \sum \lambda_n \log \frac{\|Tx\|}{|\lambda_n|} Tu_n = T(\sum \lambda_n \log \frac{\|x\|}{\|Tx\|} u_n).$$

Let $c \geq 1$ be such that $c^{-1}\|x\| \leq \|Tx\| \leq c\|x\|$ on $W$; since $[u_n]$ is unconditional

$$\left\| T\left(\sum \lambda_n \log \frac{\|x\|}{\|Tx\|} u_n\right) \right\| \leq \|T\| \left\| \sum \lambda_n \log \frac{\|x\|}{\|Tx\|} u_n \right\| \leq (\|T\| \log c)\|x\|.$$

So the commutator $[\mathcal{K}, T]$ is trivial on $W$.

It follows immediately that if $[\mathcal{K}, T]$ is singular on some $W$, then $T$ is strictly singular on $W$. But since $[\mathcal{K}, T] \equiv -[\mathcal{K}, I-T]$, the previous implication means that both $T$ and $I-T$ are strictly singular on $W$, which is a contradiction. $\square$

**Proposition 5.8.** *Let $X$ be a space with a basis and $\Omega : X \to X$ be a singular quasi-linear map, with the property that for any operator $T$ on $X$ and for every block subspace $W$ of $X$, $[T, \Omega]$ is trivial on some block subspace of $W$. Let $T, U : X \to X$ be bounded linear operators such that $(T, U)$ is compatible with $\Omega$. Then $T-U$ is strictly singular.*

*Proof.* Assume that the restriction of $T-U$ to some block subspace $W$ is an isomorphism into, then passing to a subspace we may assume that $\Omega T - T\Omega$ is trivial on $W$. Since $\Omega U - T\Omega$ is trivial by hypothesis and $\Omega U - \Omega T - \Omega(U-T)$ is trivial, we obtain that $\Omega(T-U)$ is trivial on $W$. This means that $\Omega$ is trivial on $(T-U)W$, which is not possible because $\Omega$ is singular. So $T-U$ must be strictly singular. $\square$

**Corollary 5.9.** *If $T, U$ are bounded linear operators on $\ell_2$ such that $(T, U)$ is compatible with $\mathcal{K}$, then $T-U$ is compact.*

We are ready to obtain:

**Proof of Theorem 5.5** Let $\mathcal{H}$ be an hyperplane of $Z_2$ admitting a compatible complex structure $u$. Starting from the representation

$$0 \longrightarrow \ell_2 \longrightarrow Z_2 \stackrel{q}{\longrightarrow} \ell_2 \longrightarrow 0$$

there are two possibilities: either $\mathcal{H}$ contains (the natural copy of) $\ell_2$ or not. If $\mathcal{H}$ contains $\ell_2$ then $u_0 : \ell_2 \to \ell_2$, defined by $u_0(x) = u(x)$ is also a complex structure. This induces a complex structure $\overline{u} : \mathcal{H}/\ell_2 \to \mathcal{H}/\ell_2$ yielding a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_2 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{H}/\ell_2 & \longrightarrow & 0 \\ & & {\scriptstyle u_0}\downarrow & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \overline{u}} & & \\ 0 & \longrightarrow & \ell_2 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{H}/\ell_2 & \longrightarrow & 0. \end{array}$$

Observe that if $j : \mathcal{H}/\ell_2 \to \ell_2$ is the canonical embedding then the quasi-linear map associated to the above exact sequence is $\mathcal{K}j$; and the commutative diagram above means that the couple $(u_0, \overline{u})$ is compatible with $\mathcal{K}j$. Since $j(\mathcal{H}/\ell_2)$ is an hyperplane of $\ell_2$, we can extend $j\overline{u}j^{-1}$ to an operator $\tau : \ell_2 \to \ell_2$, and the couple $(u_0, \tau)$ is compatible with $\mathcal{K}$: the maps $u_0\mathcal{K}$ and $\mathcal{K}\tau$ are equivalent since so are their restrictions $u_0\mathcal{K}j$ and $\mathcal{K}\tau j = \mathcal{K}j\overline{u}$ to a one codimensional subspace. By Corollary 5.9, $u_0 - \tau$ is compact.

On the other hand, we have a complex structure $u_0$ on $\ell_2$ and another $j\overline{u}j^{-1}$ on an hyperplane of $\ell_2$, which means, according to Proposition 5.1, that $u_0 j - j\overline{u}$ cannot be strictly singular; hence $u_0 - \tau$ cannot be strictly singular either; yielding a contradiction.

If, on the other hand, $\mathcal{H}$ does not contain $\ell_2$, then necessarily $\ell_2 + \mathcal{H} = Z_2$ and thus $\mathcal{H}/(\ell_2 \cap \mathcal{H}) \simeq (\mathcal{H} + \ell_2)/\ell_2 \simeq Z_2/\ell_2 \simeq \ell_2$ and one has the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_2 \cap \mathcal{H} & \longrightarrow & \mathcal{H} & \longrightarrow & \ell_2 & \longrightarrow & 0 \\ & & {\scriptstyle i}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \ell_2 & \longrightarrow & Z_2 & \longrightarrow & \ell_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & \mathbb{R} & = & \mathbb{R}. & & & & \end{array}$$

which means that there is a version $\mathcal{K}'$ of $\mathcal{K}$ whose image is contained in $\ell_2 \cap \mathcal{H}$; i.e., $\mathcal{K} \equiv i\mathcal{K}'$.

Let again $u$ be a complex structure on $\mathcal{H}$ inducing a complex structure $u_0$ on $\ell_2 \cap \mathcal{H}$, and therefore a complex structure $\overline{u}$ on $\ell_2$ so that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_2 \cap \mathcal{H} & \longrightarrow & \mathcal{H} & \longrightarrow & \ell_2 & \longrightarrow & 0 \\ & & {\scriptstyle u_0}\downarrow & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \overline{u}} & & \\ 0 & \longrightarrow & \ell_2 \cap \mathcal{H} & \longrightarrow & \mathcal{H} & \longrightarrow & \ell_2 & \longrightarrow & 0. \end{array}$$

Therefore the couple $(u_0, \overline{u})$ is compatible with $\mathcal{K}'$. Extend $u_0$ to an operator $\tau$ on $\ell_2$ to get a couple $(\tau, \overline{u})$ compatible with $\mathcal{K}$, which yields that $\tau - \overline{u}$ is compact. On the other hand, $u_0$ is a complex structure on $\ell_2 \cap \mathcal{H}$, while $\overline{u}$ is a complex structure on $\ell_2$ which, by Proposition 5.1 implies that $u_0 - \overline{u}_{|\ell_2 \cap \mathcal{H}}$ cannot be strictly singular, so $\tau - \overline{u}$ cannot be strictly singular either, a contradiction.□

## 6. Open problems

Of course, the main question left open in this paper is

**Problem 1.** *Do hyperplanes of $Z_2$ admit complex structures?*

And the second could be

**Problem 2.** *Is there a twisted Hilbert space not admitting a complex structure?*

Complex and compatible complex structures could be different objects. One could daydream about to what extent all complex structures on $Z_2$ are compatible with some nice representation.

Passing to more specific issues, observe that Proposition 5.7 depends on the explicit form of the Kalton-Peck map. Is this result true for arbitrary singular quasi-linear maps? Is Corollary 5.9 true in the general case? Precisely

**Problem 3.** *Assume $\Omega : \ell_2 \to \ell_2$ is a singular quasi-linear map, and that $T, U$ are bounded linear operators on $\ell_2$ such that $(T, U)$ is compatible with $\Omega$. Does it follow that $T - U$ is compact?*

Or, else, keeping in mind that $Z_2$ admits many representations as a twisted sum space, even of non-Hilbert spaces:

**Problem 4.** *Given an exact sequence $0 \to H \to Z_2 \to H \to 0$, with $H$ Hilbert, is it true that given $T, U$ bounded linear operators on $H$ such that $(T, U)$ is compatible with the sequence then $T - U$ must be compact?*

Or (this formulation is due to G. Godefroy):

**Problem 5.** *Let $\Omega$ be a (singular) centralizer on $\ell_2$. Do hyperplanes of $\ell_2 \oplus_\Omega \ell_2$ admit compatible complex structures?*

("centralizer" in this context denotes a quasi-linear map induced on $\ell_2$ by complex interpolation of Köthe spaces, e.g, with 1-unconditional bases). A few other questions specific to Kalton-Peck space remain unsolved: the first one may give a finer understanding of complex structures on $Z_2$.

**Problem 6.** *Assume that $(T, U)$ is compatible with Kalton-Peck map $\mathcal{K}$. Does there exist a compact perturbation $V$ of $T$ (and therefore of $U$) such that $(V, V)$ is compatible with $\mathcal{K}$?*

Two more related although more general problems can be formulated. We conjecture a positive solution to the following problem:

**Problem 7.** *Show that $Z_2$ admits unique complex structure.*

The answer could follow from Proposition 4.3. In connection with this we do not know the answer to:

**Problem 8.** *Find a complex space which is isomorphic to its square as a real space but not as a complex space.*

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