INTERPOLATOR SYMMETRIES AND NEW KALTON-PECK SPACES

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Abstract. We study the six diagrams generated by three interpolators in an abstract Kalton-Montgomery complex like interpolation scheme. We will consider in detail the case of the first three Schechter interpolators associated to the usual Calderón complex interpolation method in two especially interesting cases: weighted ℓ_2 spaces –i.e., interpolation pairs $(\ell_2(w^{-1}), \ell_2(w))_\theta$ — and ℓ_p spaces —i.e., the interpolation pair $(\ell_\infty, \ell_1)_\theta$—, both at $\theta = 1/2$.

1. Introduction

The reader is addressed to the 'Basic notions' section below for all unexplained terms we use in this introduction. In the abstract part of the paper we will use an interpolation schema formed by a Kalton space, a couple (X_0, X_1) of quasi Banach spaces and a family of interpolators. However, in the more specific part of the paper we will consider the standard complex interpolation method generated from a Calderón space $\mathcal{C}$, the family of Schechter interpolators $\Delta_k(f) = f^{(k)}(1/2)/k!$ and a pair of Banach spaces, that will be either (ℓ_∞, ℓ_1) or weighted Hilbert spaces $(\ell_2(w^{-1}), \ell_2(w))$. See [3] and [31] for details. With those ingredients one can generate, following [30], the family of associated Rochberg spaces $\mathcal{R}_n(\theta) = \{(\Delta_{n-1}(f), \ldots, \Delta_0(f)) : f \in \mathcal{C}\}$. As it was shown in [7], the Rochberg spaces can be arranged forming commutative diagrams of exact sequences (where we have omitted the initial and final arrows $0 \to \cdot$ and $\cdot \to 0$)

$$\begin{array}{ccccc}
\mathcal{R}_k(\theta) & = & \mathcal{R}_k(\theta) & & \\
\downarrow & & \downarrow & & \\
\mathcal{R}_n(\theta) & \longrightarrow & \mathcal{R}_{n+m}(\theta) & \longrightarrow & \mathcal{R}_m(\theta) \\
\downarrow & & \downarrow & & \| \\
\mathcal{R}_{n-k}(\theta) & \longrightarrow & \mathcal{R}_{n+m-k}(\theta) & \longrightarrow & \mathcal{R}_m(\theta)
\end{array}$$

and each exact sequence $0 \longrightarrow \mathcal{R}_n(\theta) \longrightarrow \mathcal{R}_{n+m}(\theta) \longrightarrow \mathcal{R}_m(\theta) \longrightarrow 0$ is generated and corresponds to a quasilinear map $\Omega_{n,m}$, called the associated differential. The paper [11] considered the general situation of a Kalton space, a pair of Banach spaces and two interpolators (Ψ, Φ).

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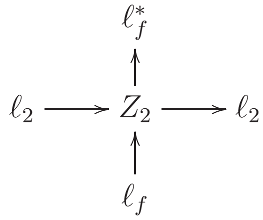
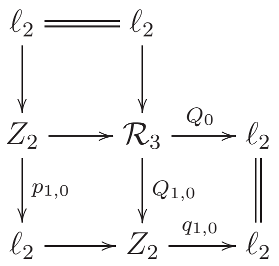
In that case, there exist only two exact sequences, $\Omega_{\Psi,\Phi}$ and $\Omega_{\Phi,\Psi}$ that generate two different "inverse" representations of the Rochberg space $\mathcal{R}_2$. For instance, in the particular case of the interpolation pair (ℓ_∞, ℓ_1) and the two classical interpolators Δ_1, Δ_0 at $1/2$ one obtains that $\mathcal{R}_1 = (\ell_\infty, \ell_1)_{1/2} = \ell_2$ while $\mathcal{R}_2$ is the celebrated Kalton-Peck space Z_2 and the two representations just mentioned are, as suggested in [4], (here ℓ_f is a certain Orlicz space)

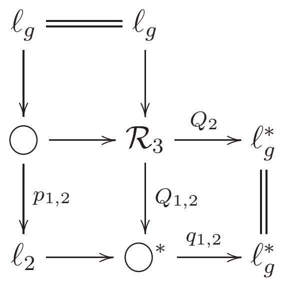


In the first part of this paper we will consider first the general situation of a finite family of interpolators on a Kalton space and will show that its study can be reduced to pairs of multi-component interpolators (Sections 2 and 3). Then, we will consider the general situation of three compatible interpolators and show that the six possible permutations of these three interpolators provide six possible commutative diagrams, hence six possible representations of the third Rochberg space $\mathcal{R}_3$ (Section 4). Sections 5 and 6 are devoted to studying the diagrams and spaces generated by the three interpolators $\Delta_2, \Delta_1, \Delta_0$ in two especially interesting cases: weighted Hilbert spaces and the Kalton-Peck case generated from the pair (ℓ_∞, ℓ_1). The first case is remarkable since it can be considered, by its simplicity (all sequences split and all spaces are isomorphic to Hilbert spaces), a template. The second case is the most interesting one since it involves the Kalton-Peck space Z_2. Moreover, the recently discovered self-duality of all associated Rochberg spaces yields unexpected symmetries in this case. More precisely, we will show that, denoting by $[abc]$ the diagram obtained from the permutation $(\Delta_a, \Delta_b, \Delta_c)$, the six diagrams are are:

[210]



[012]



[201]

$$\begin{array}{ccccc}
\ell_2 & = & \ell_2 & & \\
\downarrow & & \downarrow & & \\
\wedge & \longrightarrow & \mathcal{R}_3 & \xrightarrow{Q_1} & \ell_f^* \\
\downarrow{\scriptstyle p_{0,1}} & & \downarrow{\scriptstyle Q_{0,1}} & & \| \\
\ell_f & \longrightarrow & Z_2 & \xrightarrow{q_{0,1}} & \ell_f^*
\end{array}$$

[102]

$$\begin{array}{ccccc}
\ell_f & = & \ell_f & & \\
\downarrow & & \downarrow & & \\
\bigcirc & \longrightarrow & \mathcal{R}_3 & \xrightarrow{Q_2} & \ell_g^* \\
\downarrow{\scriptstyle p_{0,2}} & & \downarrow{\scriptstyle Q_{0,2}} & & \| \\
\ell_f & \longrightarrow & \wedge^* & \xrightarrow{q_{0,2}} & \ell_g^*
\end{array}$$

[120]

$$\begin{array}{ccccc}
\ell_f & = & \ell_f & & \\
\downarrow & & \downarrow & & \\
Z_2 & \longrightarrow & \mathcal{R}_3 & \xrightarrow{Q_0} & \ell_2 \\
\downarrow{\scriptstyle p_{2,0}} & & \downarrow{\scriptstyle Q_{2,0}} & & \| \\
\ell_f^* & \longrightarrow & \wedge^* & \xrightarrow{q_{2,0}} & \ell_2
\end{array}$$

[021]

$$\begin{array}{ccccc}
\ell_g & = & \ell_g & & \\
\downarrow & & \downarrow & & \\
\wedge & \longrightarrow & \mathcal{R}_3 & \xrightarrow{Q_1} & \ell_f^* \\
\downarrow{\scriptstyle p_{2,1}} & & \downarrow{\scriptstyle Q_{2,1}} & & \| \\
\ell_f^* & \longrightarrow & \bigcirc^* & \xrightarrow{q_{2,1}} & \ell_f^*
\end{array}$$

In this way, two new remarkable Kalton-Peck-Rochberg spaces appear, $\bigcirc$ and $\wedge$. In Sections 7 and 8 we will show that the spaces and maps in these six diagrams satisfy the following properties:

(1) All the spaces are super-reflexive and hereditarily ℓ_2.
(2) ℓ_f and ℓ_g are the Orlicz sequence spaces associated to the Orlicz functions $f(t) = t^2 \log^2 t$ and $g(t) = t^2(\log^2 t) \log^2 |\log t|$.
(3) The spaces $\mathcal{R}_3$ and $\wedge$ do not contain complemented copies of ℓ_2 and admit no unconditional basis, as it occurs with Z_2.
(4) The spaces $\wedge$ and $\bigcirc$ are not isomorphic to their dual spaces.
(5) All quotient maps, except perhaps $q_{2,1}$, are strictly singular.
(6) Every basic sequence in $\mathcal{R}_3$ contains a subsequence equivalent to the canonical basis of either ℓ_2, ℓ_f, ℓ_g.
(7) $\mathcal{R}_3$ does not contain complemented copies of Z_2.
(8) All spaces can be described as Fenchel-Orlicz spaces (or their duals).

2. Basic notions

A Banach space space Z is a *twisted sum of* Y *and* X if there exists an exact sequence $0 \to Y \to Z \to X \to 0$ (namely, a diagram formed by Banach spaces and continuous operators so that the kernel of each of them coincides with the image of the previous one), that corresponds [24, 5] to a quasi-linear map $X \longrightarrow Y$. We need to widen this notion as in [12], assuming that Y is continuously embedded in an "ambient" Banach space Σ_Y.

Definition 2.1. *A* quasi-linear map $\Omega : X \curvearrowright Y$ with ambient space Σ_Y *is a homogeneous map* $\Omega : X \longrightarrow \Sigma_Y$ *for which there is a constant* C *such that for all* $x_1, x_2 \in X$*,*

- $\Omega(x_1 + x_2) - \Omega(x_1) - \Omega(x_2) \in Y$ *and*

- $\|\Omega(x_1+x_2)-\Omega(x_1)-\Omega(x_2)\|_Y \leq C(\|x_1\|_X+\|x_2\|_X)$.

Given a quasi-linear map Ω as above, $Y\oplus_\Omega X=\{(\beta,x)\in\Sigma_Y\times X:\beta-\Omega(x)\in Y\}$ is a linear subspace of $\Sigma_Y\times X$ and $\|(\beta,x)\|_\Omega=\|\beta-\Omega(x)\|_Y+\|x\|_X$ defines a quasi-norm on $Y\oplus_\Omega X$.

The map $j:Y\longrightarrow Y\oplus_\Omega X$ given by $j(y)=(y,0)$ is an isometric embedding and the map $q:Y\oplus_\Omega X\longrightarrow X$ given by $q(\beta,x)=x$ takes the open unit ball of $Y\oplus_\Omega X$ onto that of X. They define an exact sequence

$$0\longrightarrow Y\xrightarrow{\ j\ } Y\oplus_\Omega X\xrightarrow{\ q\ } X\longrightarrow 0 \tag{1}$$

that shall be referred to as the *exact sequence generated by* Ω. Since X and Y are complete, $(Y\oplus_\Omega X,\|(\cdot,\cdot)\|_\Omega)$ is a quasi-Banach space [14, Lemma 1.5.b].

Definition 2.2. *A quasi-linear map* $\Omega:X\curvearrowright Y$ *with ambient space* Σ_Y *is said to be:*

- bounded *if there exists a constant* D *so that* $\Omega x\in Y$ *and* $\|\Omega x\|_Y\leq D\|x\|_X$ *for each* $x\in X$.
- trivial *if there exists a linear map* $L:X\longrightarrow\Sigma_Y$ *so that* $\Omega-L:X\longrightarrow Y$ *is bounded.*

Definition 2.3. *Let* Ω_1,Ω_2 *be quasilinear maps* $X\curvearrowright Y$ *with ambient space* Σ_Y. *The maps are said to be:*

- Boundedly equivalent *if* $\Omega_1-\Omega_2$ *is bounded. This means that* $\|(\cdot,\cdot)\|_{\Omega_1}$ *and* $\|(\cdot,\cdot)\|_{\Omega_2}$ *are equivalent quasi-norms.*
- Equivalent, *denoted* $\Omega_1\sim\Omega_2$, *if* $\Omega_1-\Omega_2$ *is trivial. This means that the two exact sequences they generate are equivalent in the standard homological sense, namely, there is an operator* T *making a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Y\oplus_{\Omega_1}X & \longrightarrow & X & \longrightarrow & 0\\ & & \| & & \downarrow{\scriptstyle T} & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & Y\oplus_{\Omega_2}X & \longrightarrow & X & \longrightarrow & 0. \end{array} \tag{2}$$

- *Two quasilinear maps* $\Omega_1:X_1\curvearrowright Y_1$ *and* $\Omega_2:X_2\curvearrowright Y_2$ *are said to be* isomorphically equivalent, *denoted* $\Omega_1\simeq\Omega_2$, *if there exist three isomorphisms* S,T,U *forming a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y_1 & \longrightarrow & Y_1\oplus_{\Omega_1}X_1 & \longrightarrow & X_1 & \longrightarrow & 0\\ & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle U} & & \\ 0 & \longrightarrow & Y_2 & \longrightarrow & Y_2\oplus_{\Omega_2}X_2 & \longrightarrow & X_2 & \longrightarrow & 0. \end{array} \tag{3}$$

The following notions of domain and range generalize the classical domain and range for Ω-operators obtained from an interpolation process [17, 18, 9], for centralizers on Banach L_∞-modules [4] or for G-centralizers in suitable G-Banach spaces [12].

Definition 2.4. *Let* $\Omega:X\curvearrowright Y$ *be a quasi-linear map with ambient space* Σ_Y. *The* domain *of* Ω *is the linear subspace* $\operatorname{Dom}\Omega=\{x\in X:\Omega x\in Y\}$ *endowed with the quasi-norm* $\|x\|_D=$

$\|\Omega x\| + \|x\|$. *The* range of Ω *is the linear subspace* $\operatorname{Ran}\Omega = \{\beta \in \Sigma_Y : \exists x \in X : \beta - \Omega x \in Y\}$ *endowed with the quasi-norm* $\|\omega\|_R = \inf\{\|\beta - \Omega x\| + \|x\| : \exists x \in X : \beta - \Omega x \in Y\}$.

Since $(\Omega x, x) \in Y \oplus_\Omega X$ for every $x \in X$, $\operatorname{span}\{\Omega x : x \in X\} \subset \operatorname{Ran}\Omega$. Note also that the map $i(z) = (0, z)$ is linear and isometric from $\operatorname{Dom}\Omega$ into $Y \oplus_\Omega X$, the map $p(\beta, x) = \beta$ is continuous and surjective from $Y \oplus_\Omega X$ onto $\operatorname{Ran}\Omega$, and the image of i coincides with the kernel of p. In particular, the image of i is closed. Thus, since $Y \oplus_\Omega X$ is complete, so are $\operatorname{Dom}\Omega$ and $\operatorname{Ran}\Omega$. Moreover we get another exact sequence

$$0 \longrightarrow \operatorname{Dom}\Omega \xrightarrow{\ i\ } Y \oplus_\Omega X \xrightarrow{\ p\ } \operatorname{Ran}\Omega \longrightarrow 0. \tag{4}$$

Definition 2.5. *Let* $\Omega : X \curvearrowright Y$ *be a quasilinear map with ambient space* Σ_Y. *The inverse map* $\Omega^{-1} : \operatorname{Ran}\Omega \to X$ *is defined by the identity* $B\beta = (\beta, \Omega^{-1}\beta)$, *where* B *is a bounded homogeneous selection for the quotient map in (4).*

Of course, the map Ω^{-1} is not unique, but two inverse maps for Ω are boundedly equivalent in the following sense:

Proposition 2.6. $\Omega^{-1} : \operatorname{Ran}\Omega \curvearrowright \operatorname{Dom}\Omega$ *is a quasilinear map with ambient space* X. *Two different inverses of* Ω *are boundedly equivalent.*

Proof. If $\alpha, \beta \in \operatorname{Ran}\Omega$ then $B(\alpha+\beta) - B\alpha - B\beta = (0, \Omega^{-1}(\alpha+\beta) - \Omega^{-1}\alpha - \Omega^{-1}\beta) \in Y \oplus_\Omega X$, hence $\Omega^{-1}(\alpha+\beta) - \Omega^{-1}\alpha - \Omega^{-1}\beta \in \operatorname{Dom}\Omega$. Moreover,

$$\begin{aligned} \|\Omega^{-1}(\alpha+\beta) - \Omega^{-1}\alpha - \Omega^{-1}\beta\|_D &= \|B(\alpha+\beta) - B\alpha - B\beta\|_\Omega \\ &\le C'\left(\|B(\alpha+\beta)\|_\Omega + \|B\alpha + B\beta\|_\Omega\right) \\ &\le C''(\|\alpha\|_R + \|\beta\|_R). \end{aligned}$$

Suppose that Ω_1^{-1} and Ω_2^{-1} are obtained using two different homogenous bounded selectors, each with norm smaller or equal to D. Then for each $\beta \in \operatorname{Ran}\Omega$ we have $(\beta, \Omega_1^{-1}\beta), (\beta, \Omega_2^{-1}\beta) \in Y \oplus_\Omega X$. Then $\Omega_1^{-1}\beta - \Omega_2^{-1}\beta \in \operatorname{Dom}\Omega$ and $\|\Omega_1^{-1}\beta - \Omega_2^{-1}\beta\|_D = \|(\Omega_1^{-1}\beta - \Omega_2^{-1}\beta, 0)\|_\Omega \le C(\Omega)2D\|\beta\|_R$. □

A similar result was mentioned in [4]. Observe that $\Omega : X \longrightarrow \operatorname{Ran}\Omega$ is a bounded map as well as $\Omega^{-1} : \operatorname{Ran}\Omega \longrightarrow X$. Hence $\Omega^{-1} \circ \Omega : X \longrightarrow X$ and $\Omega \circ \Omega^{-1} : \operatorname{Ran}\Omega \longrightarrow \operatorname{Ran}\Omega$ are bounded.
Classical quasilinear maps $\Omega : X \longrightarrow Y$ were introduced by Kalton [20] to provide a description of exact sequences of quasi Banach spaces. There are two natural situations in which quasi-linear maps, with the same meaning as in this paper, appear: one is when considering *centralizers* between Banach spaces X, Y that admit an L_∞-module structure, like in [22]. A second one is that of differentials $\Omega : X_\Phi \to \Sigma_Y$ generated by two interpolators Ψ, Φ; see [11].

3. Interpolators on Kalton spaces

Let (X_0, X_1) be an interpolation pair of Banach spaces as in [3, Section 2.3]. Both X_0 and X_1 are continuously embedded into their sum $\Sigma = X_0 + X_1$, endowed with its natural norm $\|x\|_\Sigma = \inf\{\|a\|_{X_0} + \|b\|_{X_1} : x = a + b, a \in X_0, b \in X_1\}$.

Definition 3.1. *We say that a continuous operator* $T : \Sigma \to \Sigma$ acts on the pair (X_0, X_1) *if* $T(X_i) \subset X_i$ *for* $i = 0, 1$.

By the closed graph theorem, each operator acting on the pair (X_0, X_1) is continuous on both spaces X_0 and X_1. Variants of the following notion of Kalton space were considered in [7, 10, 23].

Definition 3.2. *Let* U *be an open subset of* $\mathbb{C}$ *conformally equivalent to the unit disc* $\mathbb{D}$. *A* Kalton space for a pair (X_0, X_1) *of Banach spces is a Banach space* $\mathcal{F} \equiv (\mathcal{F}(U, \Sigma), \|\cdot\|_{\mathcal{F}})$ *of analytic functions* $F : U \to \Sigma$ *satisfying the following conditions:*

(a) *For each* $\theta \in U$*, the evaluation map* $\delta_\theta : \mathcal{F} \to \Sigma$ *is continuous.*
(b) *If* $\varphi : U \to \mathbb{D}$ *is a conformal equivalence and* $F : U \to \Sigma$ *is an analytic map, then* $F \in \mathcal{F}$ *if and only if* $\varphi \cdot F \in \mathcal{F}$. *In this case* $\|\varphi \cdot F\|_{\mathcal{F}} = \|F\|_{\mathcal{F}}$.

It is not difficult to show that the evaluation map of the n^{th}- derivative $\delta_\theta^{(n)} : \mathcal{F} \to \Sigma$ is continuous for each $n \in \mathbb{N}$.

Definition 3.3. *Let* $\mathcal{F}$ *be a Kalton space for* (X_0, X_1). *An* interpolator on $\mathcal{F}$ *is a continuous operator* $\Gamma : \mathcal{F} \to \Sigma$ *such that for every operator* $T : \Sigma \to \Sigma$ *acting on the pair there exists a continuous operator* $T_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}$ *satisfying* $T \circ \Gamma = \Gamma \circ T_{\mathcal{F}}$.

Given an interpolator Γ on $\mathcal{F}$, we denote by X_Γ the space $\Gamma(\mathcal{F})$ endowed with the quotient norm $\|x\|_\Gamma = \inf\{\|f\|_{\mathcal{F}} : f \in \mathcal{F}, \Gamma f = x\}$, which is a Banach space isometric to $\mathcal{F}/\ker\Gamma$. The next result implies that if Γ_1 and Γ_2 are interpolators on $\mathcal{F}$ and $\Gamma_1(\mathcal{F}) = \Gamma_2(\mathcal{F})$ then the spaces X_{Γ_1} and X_{Γ_2} are isomorphic.

Proposition 3.4. *Let* $T_1 : X_1 \to Y$ *and* $T_2 : X_2 \to Y$ *be continuous operators between Banach spaces with* $T_1(X_1) = T_2(X_2)$. *Then the quotients* $X_1/\ker T_1$ *and* $X_2/\ker T_2$ *are isomorphic.*

Proof. Let $\widehat{T_1} : X_1/\ker T_1 \to Y$ denote the injective operator induced by T_1. Then

$$\widehat{T_2}^{-1} \circ \widehat{T_1} : X_1/\ker T_1 \to X_2/\ker T_2$$

is a closed and bijective operator, which is continuous by the closed graph theorem. □

3.1. **Pairs of multi-component interpolators.** Here we extend the results of [11] for pairs of interpolators to a more general context. Let $\{\Gamma_i : i = 1, \dots, n+k\}$ be a finite family of interpolators on a Kalton space $\mathcal{F}(U, \Sigma)$. We consider the pair (Ψ, Φ) of *multi-component interpolators*

$$\Psi = \langle \Gamma_{k+n}, \dots \Gamma_{k+1} \rangle : \mathcal{F} \to \Sigma^n \quad \text{and} \quad \Phi = \langle \Gamma_k, \dots \Gamma_1 \rangle : \mathcal{F} \to \Sigma^k,$$

defined by $\Psi(f) = (\Gamma_{k+n} f, \dots \Gamma_{k+1} f)$ and $\Phi(f) = (\Gamma_k f, \dots \Gamma_1 f)$, and we will show that for this pair (Ψ, Φ) we can repeat most of the arguments and constructions in [11] for a pair of interpolators.

We denote $X_\Phi \equiv X_{\langle \Gamma_k, \dots, \Gamma_1 \rangle} = \langle \Gamma_k, \dots, \Gamma_1 \rangle(\mathcal{F})$, and observe that $\langle \Psi, \Phi \rangle$ is also a multi-interpolator mapping $\mathcal{F}$ into $\Sigma^n \times \Sigma^k$. Proceeding as in [11], we obtain the following commutative diagram with exact rows and columns (we have excluded the arrows $0 \to$ and $\to 0$):

$$
\begin{array}{ccccc}
\ker\Psi \cap \ker\Phi & = & \ker\langle\Psi,\Phi\rangle & & \\
\downarrow & & \downarrow & & \\
\ker\Phi & \longrightarrow & \mathcal{F} & \xrightarrow{\Phi} & X_\Phi \\
\Psi\downarrow & & \downarrow\langle\Psi,\Phi\rangle & & \| \\
\Psi(\ker\Phi) & \xrightarrow{\imath} & X_{\langle\Psi,\Phi\rangle} & \xrightarrow{\rho} & X_\Phi
\end{array} \tag{5}
$$

where $\Psi(\ker\Phi)$ is endowed with the quotient norm $\|\cdot\|_{\Psi|_{\ker\Phi}}$, and the maps $\imath$ and ρ are defined by $\imath\Psi g = (\Psi g, 0)$ and $\rho(\Psi f, \Phi f) = \Phi f$. In particular, we have an exact sequence

$$0 \longrightarrow \Psi(\ker\Phi) \xrightarrow{\imath} X_{\langle\Psi,\Phi\rangle} \xrightarrow{\rho} X_\Phi \longrightarrow 0. \tag{6}$$

Definition 3.5. *A family $\{\Phi_i : i \in I\}$ of interpolators on $\mathcal{F}(X_0, X_1)$ is* consistent *if for each operator T acting on the pair (X_0, X_1), there exists a continuous operator $T_{\mathcal{F}}$ acting on $\mathcal{F}$ so that $T \circ \Phi_i = \Phi_i \circ T_{\mathcal{F}}$ for every $i \in I$.*

Let $B_\Phi : X_\Phi \to \mathcal{F}$ be an homogeneous bounded selection for the quotient map $\Phi : \mathcal{F} \to X_\Phi$. We denote $\|B_\Phi\| = \sup\{\|B_\Phi x\|_{\mathcal{F}} : \|x\|_\Phi = 1\}$.

Proposition 3.6. *If the family $\{\Gamma_{n+k}, \dots, \Gamma_1\}$ is consistent and $T : \Sigma \to \Sigma$ is an operator acting on the pair then $T_\Phi(\Phi f) = \Phi(T_{\mathcal{F}} f)$ defines a continuous operator acting on X_Φ with $\|T_\Phi\| \leq \|T_{\mathcal{F}}\| \cdot \|B_\Phi\|$.*

Proof. Indeed, $\|T_\Phi(\Phi f)\|_\Phi = \|T_\Phi(\Phi B_\Phi \Phi f)\|_\Phi = \|\Phi(T_{\mathcal{F}} B_\Phi \Phi f)\|_\Phi \leq \|T_{\mathcal{F}}\| \cdot \|B_\Phi\| \cdot \|\Phi f\|_\Phi$, because Φ has norm one as an operator from $\mathcal{F}$ into X_Φ. □

Other cases follow from here. For instance, if $g \in \ker\Phi$ then $T_{\mathcal{F}} g \in \ker\Phi$, because $\Phi T_{\mathcal{F}} g = T\Phi g$. Thus, given two interpolators (Ψ, Φ), also $\Psi(\ker\Phi)$ is invariant under T_Ψ.

Definition 3.7. *The* differential associated to (Ψ, Φ) *is the map $\Omega_{\Psi,\Phi} : X_\Phi \to \Sigma^n$ given by $\Omega_{\Psi,\Phi} = \Psi \circ B_\Phi$.*

Proposition 3.8. *$\Omega_{\Psi,\Phi} : X_\Phi \curvearrowright \Psi(\ker\Phi)$ is a quasilinear map with ambient space Σ^n.*

Proof. Indeed, if $x, y \in X_\Phi$ then $B_\Phi(x+y) - B_\Phi(x) - B_\Phi(y) \in \ker\Phi$, and

$$
\begin{aligned}
\|\Omega_{\Psi,\Phi}(x+y) - \Omega_{\Psi,\Phi}x - \Omega_{\Psi,\Phi}y\|_{\Psi|_{\ker\Phi}} &\leq \|\Psi\|\|B_\Phi(x+y) - B_\Phi(x) - B_\Phi(y)\|_{\mathcal{F}} \\
&\leq 2C\|\Psi\|\|B_\Phi\|(\|x\|_\Phi + \|y\|_\Phi). \quad \square
\end{aligned}
$$

Definition 3.9. *The* derived space *associated to the quasi-linear map $\Omega_{\Psi,\Phi}$ is*

$$d\Omega_{\Psi,\Phi} := \Psi(\ker\Phi) \oplus_{\Omega_{\Psi,\Phi}} X_\Phi = \{(w,x) \in \Sigma^n \times X_\Phi : w - \Omega_{\Psi,\Phi}x \in \Psi(\ker\Phi)\},$$

endowed with the quasi norm $\|(w,x)\|_{\Omega_{\Psi,\Phi}} = \|w - \Omega_{\Psi,\Phi}x\|_{\Psi|_{\ker\Phi}} + \|x\|_\Phi$.

We get an exact sequence

$$0 \longrightarrow \Psi(\ker \Phi) \xrightarrow{\ j\ } d\Omega_{\Psi,\Phi} \xrightarrow{\ q\ } X_\Phi \longrightarrow 0 \tag{7}$$

with inclusion $jw = (w, 0)$ and quotient map $q(w, x) = x$.

Proposition 3.10. *One has $X_{\langle\Psi,\Phi\rangle} = d\Omega_{\Psi,\Phi}$ with equivalent quasi norms.*

Proof. One just has to prove that the formal identity $(w, x) \to (w, x)$ is continuous and makes the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Psi(\ker \Phi) & \xrightarrow{\ \imath\ } & d\Omega_{\Psi,\Phi} & \xrightarrow{\ \rho\ } & X_\Phi & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & \Psi(\ker \Phi) & \xrightarrow{\ j\ } & X_{\langle\Psi,\Phi\rangle} & \xrightarrow{\ q\ } & X_\Phi & \longrightarrow & 0 \end{array} \tag{8}$$

commutative. Indeed, let $(w, x) \in d\Omega_{\Psi,\Phi}$. Since $w - \Omega_{\Psi,\Phi}x \in \Psi(\ker \Phi)$, $w - \Omega_{\Psi,\Phi}x = \Psi f$ for some $f \in \ker \Phi$ with $\|f\|_{\mathcal{F}} \leq C\|w - \Omega_{\Psi,\Phi}x\|_{\Psi|_{\ker \Phi}}$. Thus $w = \Omega_{\Psi,\Phi}x + \Psi f = \Psi(B_\Phi x + f)$ and therefore $(w, x) = (\Psi(B_\Phi x + f), \Phi(B_\Phi x + f)) \in X_{\Psi,\Phi}$ with

$$\|(w, x)\|_{X_{\langle\Psi,\Phi\rangle}} \quad \leq \quad \|B_\Phi x + f\| \leq \|B\|_\Phi\|x\| + C\|w - \Omega_{\Psi,\Phi}x\| \leq \max\{\|B_\Phi\|, C\}\|(w, x)\|_{d\Omega_{\Psi,\Phi}}.$$

Conversely, $(\Psi f, \Phi f) \in X_{\Psi,\Phi}$ implies $\Phi f \in X_\Phi$ and $\Psi f - \Omega_{\Psi,\Phi}\Phi f = \Psi(f - B_\Phi \Phi f) \in \Psi(\ker \Phi)$, hence $(\Psi f, \Phi f) \in d\Omega_{\Psi,\Phi}$. □

We obtain now the domain, range and inverse of $\Omega_{\Psi,\Phi}$.

Proposition 3.11. *One has the following identities (with equivalence of norms in (1) and (2)):*

(1) $\mathrm{Dom}\,(\Omega_{\Psi,\Phi}) = \Phi(\ker \Psi)$.
(2) $\mathrm{Ran}\,(\Omega_{\Psi,\Phi}) = X_\Psi$.
(3) $\Omega_{\Phi,\Psi} = (\Omega_{\Psi,\Phi})^{-1}$.

Proof. (1) If $x \in \mathrm{Dom}\,(\Omega_{\Psi,\Phi})$ then $x \in X_\Phi$ and $\Psi B_\Phi x \in \Psi(\ker \Phi)$. Thus $\Psi B_\Phi x = \Psi g$ for some $g \in \ker \Phi$, hence $B_\Phi x - g \in \ker \Psi$ and $x = \Phi(B_\Phi x - g) \in \Phi(\ker \Psi)$. Conversely, if $y \in \Phi(\ker \Psi)$ then $(0, y) \in X_{\Psi,\Phi}$, thus $y \in X_\Phi$ and $\Omega_{\Psi,\Phi}y \in \Psi(\ker \Phi)$, hence $y \in \mathrm{Dom}\,(\Omega_{\Psi,\Phi})$.

(2) If $w \in \mathrm{Ran}\,(\Omega_{\Psi,\Phi})$ then there exists $x \in X_\Phi$ such that $w - \Omega_{\Psi,\Phi}x \in \Psi(\ker \Phi) \subset X_\Psi$. Since $\Omega_{\Psi,\Phi}x \in X_\Psi$, we get $w \in X_\Psi$. Conversely, if $w \in X_\Psi$ then $w = \Psi f$ for some $f \in \mathcal{F}$. Since $(\Psi f, \Phi f) \in X_{\Psi,\Phi}$, $w = \Psi f \in \mathrm{Ran}\,(\Omega_{\Psi,\Phi})$.

(3) If we consider the natural exact sequence

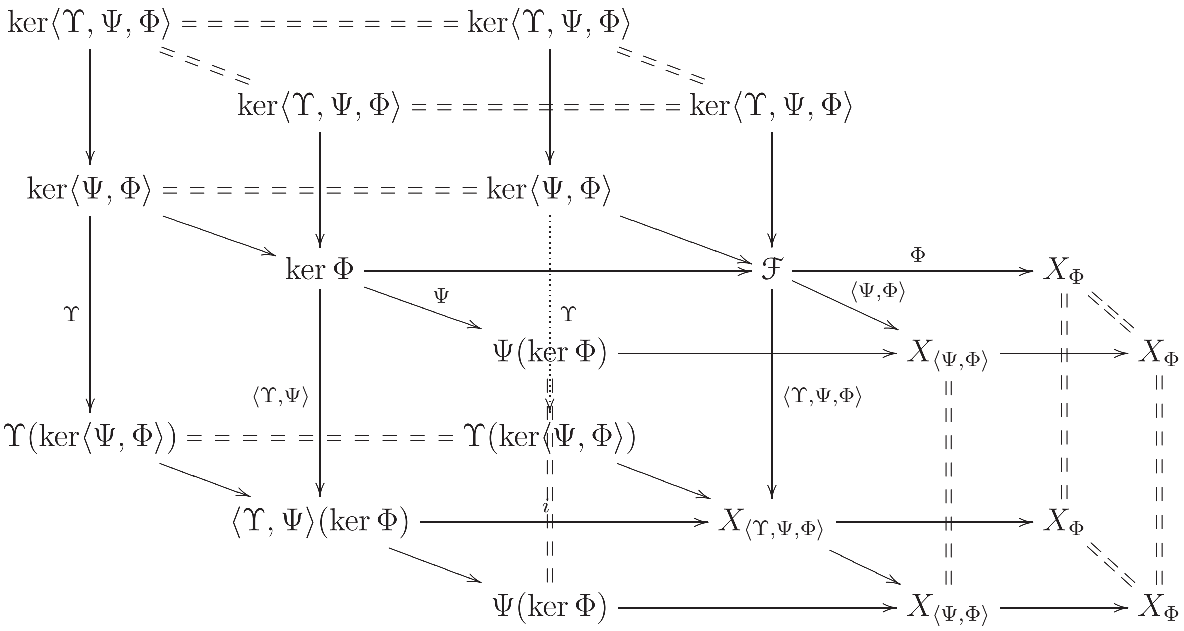
$$0 \longrightarrow \mathrm{Dom}\,(\Omega_{\Psi,\Phi}) = \Phi(\ker \Psi) \xrightarrow{\ i\ } X_{\Psi,\Phi} \xrightarrow{\ p\ } \mathrm{Ran}\,(\Omega_{\Psi,\Phi}) = X_\Psi \longrightarrow 0, \tag{9}$$

then $\langle\Psi, \Phi\rangle B_\Psi$ is a bounded homogeneous selection for p, and $\langle\Psi, \Phi\rangle B_\Psi y = (y, \Omega_{\Phi,\Psi}y)$ for each $y \in X_\Psi$. □

4. Diagrams generated by three interpolators

For an (ordered) triple of interpolators (Υ, Ψ, Φ) on a Kalton space $\mathcal{F}$, we introduce the associated diagram as follows: pick first the pair (Ψ, Φ) to get diagram (5) and then incorporate the action of Υ; one obtains the following three-dimensional diagram:

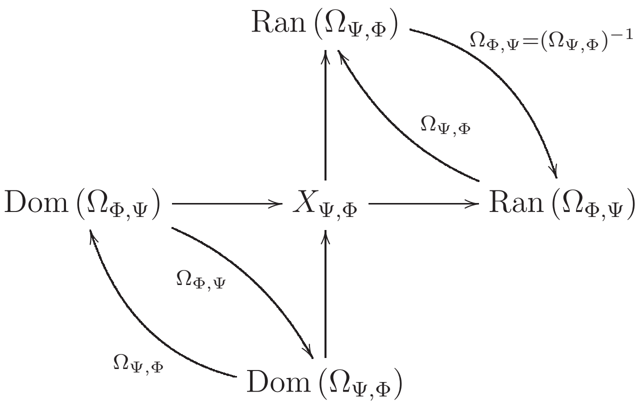
(10)



The new and interesting part of this diagram is the bottom face of the cube

(11)
$$\begin{array}{ccccc}
\Upsilon(\ker\langle\Psi,\Phi\rangle) & = & \Upsilon(\ker\langle\Psi,\Phi\rangle) & & \\
\downarrow & & \downarrow & & \\
\langle\Upsilon,\Psi\rangle(\ker\Phi) & \longrightarrow & \langle\Upsilon,\Psi,\Phi\rangle(\mathcal{F}) & \longrightarrow & \Phi(\mathcal{F}) \\
\downarrow & & \downarrow & & \| \\
\Psi(\ker\Phi) & \longrightarrow & \langle\Psi,\Phi\rangle(\mathcal{F}) & \longrightarrow & \Phi(\mathcal{F})
\end{array}$$

Let us show the interest of this construction. When only two interpolators (Ψ, Φ) are considered the situation was envisioned in [4], and completely described and studied in [11]: only two "bottom faces" exist, and thus everything can be resumed in the diagram



so that the only two new spaces are the Domain and Range of the original quasilinear maps and no new twisted sum spaces. When three interpolators (Υ, Ψ, Φ) are considered there are six possible permutations of the interpolators and therefore six possible "bottom face" diagrams. This makes several new domain and range spaces as well as new twisted sum spaces appear. We will consider an interpolation pair (X_0, X_1) and take as Kalton space the Calderón space $\mathcal{C}$ associated to the interpolation pair (see [3]); we consider the first three Schechter interpolators: for each $k \in \mathbb{N} \cup \{0\}$, $\Delta_k(f) = f^{(k)}(1/2)/k!$ defines an interpolator on $\mathcal{C}$ taking values in $X_0 + X_1$, and $\{\Delta_k : k \in \mathbb{N} \cup \{0\}\}$ is a consistent family. Indeed, if T is an operator acting on the couple, then $T_{\mathcal{C}} f = T \circ f$ defines an operator on $\mathcal{C}$ such that $\Delta_k T_{\mathcal{C}} = T\Delta_k$ for each k. Note also that $\ker\langle\Delta_b, \Delta_c\rangle = \ker \Delta_b \cap \ker \Delta_c$. We will focus on $k = 0, 1, 2$ in the following two situations: first, the sampler case of a pair $(\ell_2(w^{-1}), \ell_2(w))$ of weighted Hilbert spaces at $1/2$ and then, the most important case: the interpolation pair (ℓ_∞, ℓ_1) at $1/2$.

The diagram $[abc]$**.** Let (a, b, c) be a permutation of $(0, 1, 2)$. We denote by $[abc]$ the bottom face of the cube generated by the triple $(\Delta_a, \Delta_b, \Delta_c)$:

$$
[abc] \qquad \begin{array}{ccccc}
\Delta_a(\ker \Delta_b \cap \ker \Delta_c) & = & \Delta_a(\ker \Delta_b \cap \ker \Delta_c) & & \\
\downarrow j & & \downarrow k & & \\
\langle\Delta_a, \Delta_b\rangle(\ker \Delta_c) & \xrightarrow{l} & \langle\Delta_a, \Delta_b, \Delta_c\rangle(\mathcal{C}) & \xrightarrow{s} & \Delta_c(\mathcal{C}) \\
\downarrow q & & \downarrow r & & \| \\
\Delta_b(\ker \Delta_c) & \xrightarrow{i} & \langle\Delta_b, \Delta_c\rangle(\mathcal{C}) & \xrightarrow{p} & \Delta_c(\mathcal{C})
\end{array} \tag{12}
$$

where the maps are given by

- $j(\Delta_a h) = (\Delta_a h, 0)$, $\quad k(\Delta_a h) = (\Delta_a h, 0, 0)$, $\quad h \in \ker \Delta_b \cap \ker \Delta_c$;
- $l(\Delta_a g, \Delta_b g) = (\Delta_a g, \Delta_b g, 0)$, $\quad q(\Delta_a g, \Delta_b g) = \Delta_b g$, $\quad i(\Delta_b g) = (\Delta_b g, 0)$, $\quad g \in \ker \Delta_c$;
- $s(\Delta_a f, \Delta_b f, \Delta_c f) = \Delta_c f$, $\ r(\Delta_a f, \Delta_b f, \Delta_c f) = (\Delta_b f, \Delta_c f)$, $\ p(\Delta_b f, \Delta_c f) = \Delta_c f$, $\quad f \in \mathcal{C}$.

The quasi-linear maps. We simplify the notation for the quasi-linear maps as follows:

$$\Omega_{a,b} = \Omega_{\Delta_a,\Delta_b}; \quad \Omega_{a,\langle b,c\rangle} = \Omega_{\Delta_a,\langle\Delta_b,\Delta_c\rangle} \quad \text{and} \quad \Omega_{\langle a,b\rangle,c} = \Omega_{\langle\Delta_a,\Delta_b\rangle,\Delta_c}.$$

It easily follows from Proposition 3.11 that

(1) the **central column** of $[abc]$ is generated by $\Omega_{a,\langle b,c\rangle}$,
(2) the **central row** of $[abc]$ is generated by $\Omega_{\langle a,b\rangle,c}$,
(3) the **lower row** of $[abc]$ is generated by $q\circ\Omega_{\langle a,b\rangle,c}\simeq\Omega_{b,c}$, since $q\circ\langle\Delta_a,\Delta_b\rangle=\Delta_b$.
(4) the **left column** of $[abc]$ is generated by $\Omega_{a,\langle b,c\rangle}\circ i$.

Some symmetries: isomorphic equivalence of quasi-linear maps. The following equivalences are obvious, or can be derived from Proposition 3.11:

$$\begin{aligned}\Omega_{\langle b,c\rangle,a}&\simeq\Omega_{\langle c,b\rangle,a}\\ \Omega_{a,\langle b,c\rangle}&\simeq\Omega_{a,\langle c,b\rangle}\\ (\Omega_{a,\langle b,c\rangle})^{-1}&\simeq\Omega_{\langle b,c\rangle,a}\\ (\Omega_{\langle a,b\rangle,c})^{-1}&\simeq\Omega_{c,\langle a,b\rangle}\\ (\Omega_{a,b})^{-1}&\simeq\Omega_{b,a}\end{aligned}$$

The first three Rochberg spaces [7, 30] obtained from the first three interpolators Δ_2, Δ_1 and Δ_0 applied to a suitable pair (X^*,X) having a common unconditional basis are:

- $\Delta_0(\mathcal{C})$ is the interpolation space $(X^*,X)_{1/2}=\ell_2$.
- $\langle\Delta_1,\Delta_0\rangle(\mathcal{C})$ is the Rochberg derived space $\mathcal{R}_2$, in this case the twisted Hilbert space $\ell_2\oplus_{\Omega_{1,0}}\ell_2$. In the particular case $X=\ell_1$ this is the celebrated Kalton-Peck space Z_2 (see [24] and [7]).
- $\langle\Delta_2,\Delta_1,\Delta_0\rangle(\mathcal{C})$ is the third Rochberg derived space, that we will denote $\mathcal{R}_3$.

5. The case of weighted Hilbert spaces.

This test case is rather interesting and provides some insight about what occurs in other situations. Let w be a weight sequence (we will understand as in [27, 4.e.1] a non-increasing sequence of positive numbers such that $\lim w_n=0$ and $\sum w_n=\infty$). We set $w_0=w^{-1}$ and $w_1=w$ and let us consider the interpolation pair $(\ell_2(w^{-1}),\ell_2(w))$, whose complex interpolation space at $1/2$ is ℓ_2. A homogeneous bounded selector for Δ_0 is $B(x)(z)=w^{2z-1}x$ since $\Delta_0B(x)=x$, and therefore $B(x)'(z)=2w^{2z-1}\log w\cdot x$ and $\Omega_{1,0}x=\Delta_1Bx=2\log w\cdot x$. The Rochberg $\mathcal{R}_2$ derived space will be

$$Z_2(w)=\{(y,x):x\in\ell_2,\quad y-2\log w\cdot x\in\ell_2\}$$

from where $\operatorname{Dom}\Omega_{1,0}=\{x\in\ell_2:2\log w\cdot x\in\ell_2\}=\ell_2(\log w)=\{(0,x)\in Z_2(w)\}$ and $\operatorname{Ran}\Omega_{1,0}=\ell_2((\log w)^{-1})$ so that $(\Omega_{1,0})^{-1}x=\frac{1}{2\log w}x$ and thus $\operatorname{Dom}(\Omega_{1,0})^{-1}=\{x\in\ell_2((\log w)^{-1}):(\log w)^{-1}\cdot x\in\ell_2(\log w)\}=\ell_2=\operatorname{Ran}(\Omega_{1,0})^{-1}$, as we already know.

Next, $B(x)''(z)=4w^{2z-1}\log^2 w\cdot x$, and thus $\Delta_2B(x)=2\log^2 w\cdot x$. Therefore $\Omega_{\langle 2,1\rangle,0}$ is the linear map

$$\Omega_{\langle 2,1\rangle,0}(x)=(\Delta_2B(x),\Delta_1B(x))=\big(2\log^2 w\cdot x,2\log w\cdot x\big)$$

with domain $\operatorname{Dom}\Omega_{\langle 2,1\rangle,0} = \{x \in \ell_2 : (2\log^2 w \cdot x, 2\log w \cdot x) \in Z_2(w)\} = \ell_2(\log^2 w)$ since one must have $2\log w \cdot x \in \ell_2$ and $2\log^2 w \cdot x - 4\log^2 \cdot w = -2\log^2 w \cdot x \in \ell_2$. Therefore we have some parts of the first two diagrams [210] and [012]

$$
\begin{array}{ccccc}
\ell_2 & = & \ell_2 & & \\
\downarrow & & \downarrow & & \\
Z_2(w) & \longrightarrow & \mathcal{R}_3 & \longrightarrow & \ell_2 \\
\downarrow & & \downarrow & & \| \\
\ell_2 & \longrightarrow & Z_2(w) & \longrightarrow & \ell_2
\end{array}
\qquad\qquad
\begin{array}{ccccc}
\ell_2(\log^2 w) & = & \ell_2(\log^2 w) & & \\
\downarrow & & \downarrow & & \\
\bigcirc & \longrightarrow & \mathcal{R}_3 & \longrightarrow & \ell_2(\log^{-2} w) \\
\downarrow & & \downarrow & & \| \\
\blacksquare & \longrightarrow & \bigcirc^* & \longrightarrow & \ell_2(\log^{-2} w)
\end{array}
$$

We need to know now who are $\bigcirc = \operatorname{Dom}\Omega_{2,\langle 1,0\rangle}$ and $\blacksquare = \bigcirc/\ell_2(\log^2 w)$. To get the first of those spaces we need to know $\Omega_{2,\langle 1,0\rangle}$. Recall from the standard diagram

$$
\begin{array}{ccccc}
\ker a & \longrightarrow & \mathcal{C} & \xrightarrow{a} & \ell_2 \\
{\scriptstyle b}\downarrow & & \downarrow{\scriptstyle (b,a)} & & \| \\
\ell_2 & \longrightarrow & Z_2(w) & \longrightarrow & \ell_2
\end{array}
$$

that if A, B are homogeneous bounded selectors for a and b then

$$W(y,x) = B(y - \Omega_{b,a}x) + Ax$$

is a selector for (b,a) and therefore $\Omega_{c,(b,a)} = cW$. With this info at hand, we need a selector W for $\langle \Delta_1, \Delta_0\rangle$ to then obtain $\Omega_{2,\langle 1,0\rangle} = \Delta_2 W$. Now, the selector for Δ_0 is $Bx(z) = w^{2z-1}x$ as we already know, and the selector for $\Delta_1 : \ker\delta_0 \to \ell_2$ is $\frac{1}{\varphi'(1/2)}\varphi B$ where φ is a conformal mapping with $\varphi(1/2) = 0$. Therefore, $W(y,x) = \frac{\varphi}{\varphi'(1/2)}B(y - \Omega_{1,0}x) + Bx$, and elementary calculations yield

$$
\begin{aligned}
\Omega_{2,\langle 1,0\rangle}(y,x) &= \frac{1}{2}W(y,x)''(1/2) = \Omega_{1,0}(y - \Omega_{1,0}x) + \frac{\varphi''(1/2)}{2\varphi'(1/2)}(y - \Omega_{1,0}x) + \frac{1}{2}Bx''(1/2) \\
&= 2\log w \cdot (y - 2\log w \cdot x) + \frac{\varphi''(1/2)}{2\varphi'(1/2)}(y - 2\log w \cdot x) + 2\log^2 w \cdot x
\end{aligned}
$$

Setting $d = \frac{\varphi''(1/2)}{2\varphi'(1/2)}$ one gets $\Omega_{2,\langle 1,0\rangle}(y,x) = (2\log w + d)y - (2\log^2 w + 2d\log w)x$. This yields $\operatorname{Dom}\Omega_{2,\langle 1,0\rangle} = \{(y,x) \in Z_2(w) : (2\log w + d)y - (2\log^2 w + 2d\log w)x \in \ell_2\}$ and then $\operatorname{Dom}\Omega_{2,\langle 1,0\rangle}|_{\operatorname{Dom}\Omega_{1,0}} = \{(0,x) \in Z_2(w) : (2\log^2 w + 2d\log w)x \in \ell_2\} = \ell_2(\log^2 w)$. And since

$dy - 2d\log wx \in \ell_2$ when $(y,x) \in Z_2(w)$ one gets

$$\begin{aligned}
\bigcirc &= \{(y,x) \in Z_2(w) : (2\log w + d)y - (2\log^2 w + 2d\log w)x \in \ell_2\}\\
&= \{(y,x) \in Z_2(w) : \log wy - \log^2 wx \in \ell_2\}\\
&= \{(y,x) \in Z_2(w) : \log w(y - \log wx) \in \ell_2\}\\
&= \{(y,x) : x \in \ell_2 \quad \text{and} \quad y - \log wx \in \ell_2(\log w)\}.
\end{aligned}$$

By obvious reasons we will call this space $\bigcirc = Z_{\ell_2(\log w)}(w)$. It is clear that $\bigcirc$ is a twisting $0 \longrightarrow \ell_2(\log w) \longrightarrow Z_{\ell_2(\log w)}(w) \longrightarrow \ell_2(\log w) \longrightarrow 0$ of $\ell_2(\log w)$ obtained with the *same* quasilinear map $\Omega x = 2\log wx$. This is a bonus effect of working with weighted spaces in which all maps are linear. On the other hand, $\blacksquare$ is the domain of $\Delta_2\Omega^{-1}_{2,\langle 1,0\rangle}$. We will show later in Proposition 6.6 that $\Delta_0(\ker\Delta_2) = \Delta_0(\ker\Delta_1) \Longrightarrow \Delta_1(\ker\Delta_2) = \Delta_1(\ker\Delta_0)$, which in this case yields $\operatorname{Dom}(\Omega) = \ell_2(\log w) \Longrightarrow \blacksquare = \ell_2$. Thus, giving the analogous meaning as before to the space $Z_{\ell_2((\log w)^{-1})}(w)$, diagrams [210] and [012] are

$$\begin{array}{ccccc}
\ell_2 & = & \ell_2 & & \\
\downarrow & & \downarrow & & \\
Z_2(w) & \longrightarrow & \mathcal{R}_3 & \longrightarrow & \ell_2\\
\downarrow & & \downarrow & & \|\\
\ell_2 & \longrightarrow & Z_2(w) & \longrightarrow & \ell_2
\end{array}
\qquad
\begin{array}{ccccc}
\ell_2(\log^2 w) & = & \ell_2(\log^2 w) & & \\
\downarrow & & \downarrow & & \\
Z_{\ell_2(\log w)}(w) & \longrightarrow & \mathcal{R}_3 & \longrightarrow & \ell_2(\log^{-2} w)\\
\downarrow & & \downarrow & & \|\\
\ell_2 & \longrightarrow & Z_{\ell_2((\log w)^{-1})}(w) & \longrightarrow & \ell_2(\log^{-2} w)
\end{array}$$

The other relevant new space appears in [201]

$$\begin{array}{ccccc}
\ell_2 & = & \ell_2 & & \\
\downarrow & & \downarrow & & \\
\wedge & \longrightarrow & \mathcal{R}_3(w) & \longrightarrow & \ell_2(\log^{-1} w)\\
\downarrow & & \downarrow & & \|\\
\ell_2(\log w) & \longrightarrow & Z_2(w) & \longrightarrow & \ell_2(\log^{-1} w)
\end{array}$$

that we can identify as the pullback space $\wedge = \{((y,0,x) \in \mathcal{R}_3\}$ generated with the map $\Omega_{2,\langle 1,0\rangle}|_{\operatorname{Dom}\Omega_{1,0}} x = -(2\log^2 w + 2d\log w)x$. We thus get that [102] and [201] are

$$
\begin{array}{ccccc}
\ell_2(\log w) & = & \ell_2(\log w) & & \\
\downarrow & & \downarrow & & \\
Z_{\ell_2(\log w)}(w) & \longrightarrow & \mathcal{R}_3 & \longrightarrow & \ell_2(\log^{-2} w) \\
\downarrow & & \downarrow & & \| \\
\ell_2(\log w) & \longrightarrow & \wedge^* & \longrightarrow & \ell_2(\log^{-2} w)
\end{array}
\qquad
\begin{array}{ccccc}
\ell_2 & = & \ell_2 & & \\
\downarrow & & \downarrow & & \\
\wedge & \longrightarrow & \mathcal{R}_3 & \longrightarrow & \ell_2(\log^{-1} w) \\
\downarrow & & \downarrow & & \| \\
\ell_2(\log w) & \longrightarrow & Z_2 & \longrightarrow & \ell_2(\log^{-1} w)
\end{array}
$$

The vertical sequence on the left is defined by $\Omega x = 2 \log wx$ because this is the derivation associated to the interpolation couple $(\ell_2(w^{-1} \log w), \ell_2(w \log w))_{1/2} = \ell_2(\log w)$. It is moreover easy to check the diagram for inverse mappings

$$
\begin{array}{ccccc}
 & & \ell_2 & & \\
 & & \uparrow & & \\
\ell_2(\log w) & \longrightarrow & \bigcirc & \longrightarrow & \ell_2(\log w) \\
 & & \uparrow & & \\
 & & \ell_2(\log^2 w) & &
\end{array}
$$

since $\operatorname{Dom} \Omega = \{x \in \ell_2(\log w) : \log wx \in \ell_2(\log w)\} = \{x \in \ell_2(\log w) : \log^2 wx \in \ell_2\} = \ell_2(\log^2 w)$ one gets that [021] and [120] are

$$
\begin{array}{ccccc}
\ell_2(\log^2 w) & = & \ell_2(\log^2 w) & & \\
\downarrow & & \downarrow & & \\
\wedge & \longrightarrow & \mathcal{R}_3 & \longrightarrow & \ell_2(\log^{-1} w) \\
\downarrow & & \downarrow & & \| \\
\ell_2(\log^{-1} w) & \longrightarrow & \bigcirc^* & \longrightarrow & \ell_2(\log^{-1} w)
\end{array}
\qquad
\begin{array}{ccccc}
\ell_f & = & \ell_f & & \\
\downarrow & & \downarrow & & \\
Z_2 & \longrightarrow & \mathcal{R}_3 & \longrightarrow & \ell_2 \\
\downarrow & & \downarrow & & \| \\
\ell_2(\log^{-1} w) & \longrightarrow & \wedge^* & \longrightarrow & \ell_2
\end{array}
$$

An interesting question which we left out of the scope of this paper is to investigate the case of weighted ℓ_p-spaces, or even weighted versions of a given space with an unconditional basis. In this context the first thing we lose is duality: Z_p is not isomorphic to Z_p^*, so there are more spaces and the hidden symmetries probably disappear.

6. The case of Kalton-Peck spaces

Things are not as simple here as in the weighted case. We will however profit from some symmetries (some obvious, some hidden) to complete the six diagrams as the reader can see in Section 1. Peculiarities about the structure of the spaces will be considered in the next section.

First of all, let us observe that all the spaces in the diagrams admit symmmetric Schauder decompositions:

Proposition 6.1. *The unit vector basis (e_n) is a symmetric basis for the three Banach spaces $\Delta_c(\mathcal{C})$, $\Delta_b(\ker\Delta_c)$ and $\Delta_a(\ker\Delta_b\cap\ker\Delta_c)$. Similarly, $\langle\Delta_a,\Delta_b\rangle(\ker\Delta_c)$ and $\langle\Delta_a,\Delta_b\rangle(\mathcal{C})$ admit a symmetric two-dimensional decomposition and $\langle\Delta_a,\Delta_b,\Delta_c\rangle(\mathcal{C})$ admits a symmetric three-dimensional decomposition.*

Proof. Let X be one of the first three spaces and let P_n denote the natural projection onto the subspace generated by $\{e_1,\dots,e_n\}$. Since P_n is a norm-one operator on ℓ_∞ and ℓ_1, (P_n) is a bounded sequence of operators on X by Proposition 3.6. Clearly (e_n) is contained in X and generates a dense subspace. Since for each $x\in\operatorname{span}\{e_n : n\in\mathbb{N}\}$, P_nx converges to x in X, it does for each $x\in X$. Thus (e_n) is a Schauder basis for X, and considering the operators associated to permutations of the basis, Proposition 3.6 implies that the basis is symmetric. The remaining results are proved in a similar way, using the operators induced by P_n in each of the remaining spaces. □

The next result shows that some of the spaces in the diagrams coincide. Note that algebraic equality implies isomorphism by Proposition 3.4.

Proposition 6.2. *The following equalities hold:*

(1) $\Delta_2(\ker\Delta_1\cap\ker\Delta_0)=\Delta_1(\ker\Delta_0)=\Delta_0(\mathcal{C})$,
(2) $\langle\Delta_2,\Delta_1\rangle(\ker\Delta_0)=\langle\Delta_1,\Delta_0\rangle(\mathcal{C})$,
(3) $\Delta_1(\ker\langle\Delta_0,\Delta_2\rangle)=\Delta_0(\ker\Delta_1)$.

Proof. Let $\varphi:\mathbb{S}\to\mathbb{D}$ be a conformal equivalence such that $\varphi(1/2)=0$. Since $\varphi'(1/2)\neq 0$, we can define $\phi=\varphi'(1/2)^{-1}\cdot\varphi$.

(1) For each $g\in\ker\Delta_0$ there is $f\in\mathcal{C}$ such that $g=\phi\cdot f$, hence $\Delta_1 g=\Delta_0 f$, and we get $\Delta_1(\ker\Delta_0)\subset\Delta_0(\mathcal{C})$. Conversely, if $f\in\mathcal{C}$ then $g=\phi\cdot f\in\ker\Delta_0$ and $\Delta_0 f=\Delta_1 g$, so the second equality is proved. The first equality can be proved in a similar way. It was proved in [7, Theorem 4] that $j(x_1,x_0)=(x_1,x_0,0)$ and $q(y_2,y_1,y_0)=y_0$ define an exact sequence

$$0 \longrightarrow \langle\Delta_1,\Delta_0\rangle(\mathcal{C}) \xrightarrow{\ j\ } \langle\Delta_2,\Delta_1,\Delta_0\rangle(\mathcal{C}) \xrightarrow{\ q\ } \Delta_0(\mathcal{C}) \longrightarrow 0,$$

and (2) follows from $\langle\Delta_2,\Delta_1,\Delta_0\rangle(\ker\Delta_0)=\ker q$ and $\langle\Delta_1,\Delta_0,0\rangle(\mathcal{C})=\operatorname{Im} j$.

(3) Note that $y\in\Delta_0(\ker\Delta_1)$ if and only if $(0,y)\in\langle\Delta_1,\Delta_0\rangle(\mathfrak{C})=\langle\Delta_2,\Delta_1\rangle(\ker\Delta_0)$; equivalently, $y\in\Delta_1(\ker\Delta_0\cap\ker\Delta_2)=\Delta_1(\ker\langle\Delta_0,\Delta_2\rangle)$. □

Next we identify the corner spaces as Orlicz sequence spaces.

Proposition 6.3. $\Delta_0(\ker\Delta_1)=\ell_f$ *and* $\Delta_0(\ker\Delta_1\cap\ker\Delta_2)=\ell_g$, *the Orlicz sequence spaces associated to the Orlicz functions* $f(t)=t^2(\log t)^2$ *and* $g(t)=t^2(\log t)^2(\log|\log t|)^2$.

Proof. The first equality was essentially proved in [24]. With our notation,

$$\Delta_0(\ker\Delta_1)=\operatorname{Dom}\Omega_{1,0}=\{x\in\ell_2:\Omega_{1,0}x\in\ell_2\}$$

and $\Omega_{1,0} : \ell_2 \to \ell_\infty$ is given by $\Omega_{1,0} = 2x \log(|x|/\|x\|_2)$. Thus

$$\Delta_0(\ker \Delta_1) = \{x \in \ell_2 : x \log |x| \in \ell_2\} = \ell_f.$$

Similarly, since $\Delta_0(\ker \Delta_1 \cap \ker \Delta_2) = \mathrm{Dom}\, \Omega_{\langle 2,1\rangle,0}$ and $\Omega_{\langle 2,1\rangle,0} : \ell_2 \to \ell_\infty \times \ell_\infty$ is given by

$$\Omega_{\langle 2,1\rangle,0} x = \left(2x \log^2 \frac{|x|}{\|x\|_2}, 2x \log \frac{|x|}{\|x\|_2}\right)$$

(see [7]), $\Delta_0(\ker \Delta_1 \cap \ker \Delta_2) = \{x \in \ell_2 : (2x \log^2 |x|, 2x \log |x|) \in Z_2\}$. Hence $x \in \Delta_0(\ker \Delta_1 \cap \ker \Delta_2)$ if and only if $x \in \ell_2$, $2x \log |x| \in \ell_2$ and

$$2x \log^2 |x| - \Omega_{1,0}(x \log |x|) = 2x \log^2 |x| - 4x \log |x| \log \frac{|x \log |x||}{\|x \log |x|\|_2} \in \ell_2.$$

Since $\log |x \log |x|| = \log |x| + \log | \log |x||$, we conclude that
$\Delta_0(\ker \Delta_1 \cap \ker \Delta_2) = \{x \in \ell_2 : x \log |x| \cdot \log | \log |x|| \in \ell_2\} = \ell_g$. □

The second equality in the following result was observed in [4].

Proposition 6.4. $\Delta_2(\ker \Delta_0) = \Delta_1(\mathcal{C}) = \ell_f^*$.

Proof. For the first equality, $\langle \Delta_1, \Delta_0\rangle(\mathcal{C}) = \langle \Delta_2, \Delta_1\rangle(\ker \Delta_0)$ by Proposition 6.2. Thus

$$\begin{aligned} x \in \Delta_1(\mathcal{C}) \;&\Leftrightarrow\; (x, f(1/2)) = (f'(1/2), f(1/2)) \text{ for some } f \in \mathcal{F} \\ &\Leftrightarrow\; (x, g'(1/2)) = (g''(1/2), g'(1/2)) \text{ for some } g \in \ker \Delta_0 \\ &\Leftrightarrow\; x \in \Delta_2(\ker \Delta_0). \end{aligned}$$

For the second equality, since $Z_2 = \langle \Delta_1, \Delta_0\rangle(\mathcal{C})$, we have a natural exact sequence

$$0 \longrightarrow \Delta_0(\ker \Delta_1) = \ell_f \xrightarrow{\;i\;} Z_2 \xrightarrow{\;p\;} \Delta_1(\mathcal{C}) \longrightarrow 0 \tag{13}$$

with $i(x) = (0, x)$ and $p(y, x) = y$. Moreover there is a bijective isomorphism $U_2 : Z_2 \to Z_2^*$ given by $U_2(y, x) = (-x, y)$ [24]. Since $i^* U_2 = p$, we get $\Delta_1(\mathcal{C}) = \ell_f^*$. □

The following three results were unexpected for us since, at first glance, the first two spaces look incomparable. We do not know which of these results are true "in general" (they are true in the weighted case considered earlier):

Proposition 6.5. $\Delta_0(\ker \Delta_2) = \Delta_0(\ker \Delta_1) = \ell_f$.

Proof. The second equality is proved in Proposition 6.3. Moreover, the map $\Omega_{2,0} : \ell_2 \to \ell_\infty$ is given by $\Omega_{2,0} = 2x \log^2(|x|/\|x\|)$. Thus

$$\Delta_0(\ker \Delta_2) = \mathrm{Dom}\, \Omega_{2,0} = \{x \in \ell_2 : x \log^2 |x| \in \Delta_2(\ker \Delta_0) = \ell_f^*\}.$$

Since $\ell_f = \{x \in \ell_2 : x \log |x| \in \ell_2\}$, $\ell_f^* = \{x \in \ell_\infty : x \log^{-1} |x| \in \ell_2\}$ [27, Example 4.c.1]. Then

$$x \in \Delta_0(\ker \Delta_2) \Leftrightarrow x \in \ell_2 \;\text{ and }\; \frac{x \log^2 |x|}{\log(|x| \log^2 |x|)} = \frac{x \log^2 |x|}{\log |x| + 2 \log | \log x|} \in \ell_2.$$

Thus $x \in \Delta_0(\ker \Delta_2)$ if and only if $x \log |x| \in \ell_2$; equivalently $x \in \ell_f$. □

It was proved in [6] that there exists a bijective isomorphism $U_3 : \mathcal{R}_3 \to \mathcal{R}_3^*$ defined by $U_3(x_2, x_1, x_0) = (x_0, -x_1, x_2)$; more precisely, given (x_2, x_1, x_0) and (y_2, y_1, y_0) in $\mathcal{R}_3$ one has

$$\langle U_3(x_2, x_1, x_0), (y_2, y_1, y_0)\rangle = \langle x_0, y_2\rangle - \langle x_1, y_1\rangle + \langle x_2, y_0\rangle.$$

Proposition 6.6. $\Delta_2(\ker \Delta_1) = \Delta_2(\ker \Delta_0) = \ell_f^*$.

Proof. The second equality is proved in Proposition 6.4, and we derive the first equality from Proposition 6.5 by constructing an isomorphism from $\Delta_2(\ker \Delta_1)$ onto $\Delta_0(\ker \Delta_2)^*$ that takes e_n to e_n for every $n \in \mathbb{N}$. Recall that if M and N are closed subspaces of X with $N \subset M$ then $(M/N)^* \simeq N^\perp/M^\perp$. Thus, with the natural identifications we get

$$\Delta_0(\ker \Delta_2) \simeq \frac{\langle \Delta_1, \Delta_0\rangle(\ker \Delta_2)}{\Delta_1(\ker \Delta_0 \cap \ker \Delta_2)} \implies \Delta_0(\ker \Delta_2)^* \simeq \frac{(\Delta_1(\ker \Delta_0 \cap \ker \Delta_2))^\perp}{(\langle \Delta_1, \Delta_0\rangle(\ker \Delta_2))^\perp}$$

and

$$\Delta_2(\ker \Delta_1) \simeq \frac{\langle \Delta_0, \Delta_2\rangle(\ker \Delta_1)}{\Delta_0(\ker \Delta_2 \cap \ker \Delta_1)},$$

and we conclude that U_3 induces an isomorphism from $\Delta_2(\ker \Delta_1)$ onto $\Delta_0(\ker \Delta_2)^*$ by showing that U_3 takes $\langle \Delta_0, \Delta_2\rangle(\ker \Delta_1)$ onto $(\Delta_1(\ker \Delta_0 \cap \ker \Delta_2))^\perp$ and $\Delta_0(\ker \Delta_2 \cap \ker \Delta_1)$ onto $(\langle \Delta_1, \Delta_0\rangle(\ker \Delta_2))^\perp$. Indeed, $\Delta_1(\ker \Delta_0 \cap \ker \Delta_2)$ can be identified with the subspace of the vectors $(0, y, 0)$ in $\mathcal{R}_3$. Then $(\Delta_1(\ker \Delta_0 \cap \ker \Delta_2))^\perp$ is the subspace of the vectors $(x, 0, z)$ in $\mathcal{R}_3^*$, which coincides with $U_3(\langle \Delta_0, \Delta_2\rangle(\ker \Delta_1))$, and similarly $\langle \Delta_1, \Delta_0\rangle(\ker \Delta_2)^\perp = U_3(\Delta_0(\ker \Delta_2 \cap \ker \Delta_1))$, and it is clear that the induced isomorphism takes e_n to e_n for every $n \in \mathbb{N}$. □

Propositions 6.5 and 6.6 yield:

Proposition 6.7. $\Delta_1(\ker \Delta_2) = \Delta_1(\ker \Delta_0) = \ell_2$.

Proof. Proposition 6.5 implies $\ker \Delta_0 + \ker \Delta_1 = \ker \Delta_0 + \ker \Delta_2$, from which we get

$$\Delta_1(\ker \Delta_0) = \Delta_1(\ker \Delta_0 + \ker \Delta_2) \supset \Delta_1(\ker \Delta_2),$$

while Proposition 6.6 implies $\ker \Delta_2 + \ker \Delta_0 = \ker \Delta_2 + \ker \Delta_1$. Thus

$$\Delta_1(\ker \Delta_2) = \Delta_1(\ker \Delta_2 + \ker \Delta_0) \supset \Delta_1(\ker \Delta_0),$$

and the result is proved. □

6.1. **Construction of the diagrams.** The six diagrams are special cases of Diagram (12). Of course, $\langle \Delta_a, \Delta_b, \Delta_c\rangle(\mathcal{C}) \simeq \mathcal{R}_3$ for each (a, b, c).

Diagram [210]**:** By Proposition 6.2, $\Delta_2(\ker\Delta_1 \cap \ker\Delta_0) = \Delta_1(\ker\Delta_0) = \Delta_0(\mathcal{C}) \simeq \ell_2$ and $\langle\Delta_2, \Delta_1\rangle(\ker\Delta_0) = \langle\Delta_1, \Delta_0\rangle(\mathcal{C}) \simeq Z_2$. We thus get

$$\begin{array}{ccccc}
\ell_2 & = & \ell_2 & & \\
\downarrow & & \downarrow & & \\
Z_2 & \longrightarrow & \mathcal{R}_3 & \longrightarrow & \ell_2 \\
\downarrow & & \downarrow & & \| \\
\ell_2 & \longrightarrow & Z_2 & \longrightarrow & \ell_2
\end{array}$$

The two quasilinear maps generating the two middle sequences are $\Omega_{\langle 2,1\rangle,0}$ and $\Omega_{2,\langle 1,0\rangle}$; both can be found explicitly in [7] and at the appropriate places in this paper.

Diagram [012]**:** Let us denote $\bigcirc = \langle\Delta_0, \Delta_1\rangle(\ker\Delta_2)$ as before. By Propositions 6.3 and 6.7, $\Delta_0(\ker\Delta_1 \cap \ker\Delta_2) = \ell_g$ and $\Delta_1(\ker\Delta_2) = \ell_2$. So we have the spaces in the left column. The next result provides the spaces in the lower row.

Proposition 6.8.

(a) $\Delta_2(\mathcal{C})$ *is isomorphic to* ℓ_g^*.

(b) $\langle\Delta_1, \Delta_2\rangle(\mathcal{C})$ *is isomorphic to* $\bigcirc^*$.

Proof. (a) By Proposition 6.3, $\ell_g = \Delta_0(\ker\Delta_1 \cap \ker\Delta_2)$ which is isomorphic to a closed subspace of $\mathcal{R}_3$, namely $\{(x_2, x_1, x_0) \in \mathcal{R}_3 : x_2 = x_1 = 0\}$. Hence $\ell_g^* \simeq \mathcal{R}_3^*/\left(\Delta_0(\ker\Delta_1 \cap \ker\Delta_2)\right)^{\perp}$. Since $\left(\Delta_0(\ker\Delta_1 \cap \ker\Delta_2)\right)^{\perp} = U_3\left(\langle\Delta_0, \Delta_1\rangle(\ker\Delta_2)\right)$ then

$$\Delta_2(\mathcal{C}) \simeq \frac{\mathcal{R}_3}{\langle\Delta_0, \Delta_1\rangle(\ker\Delta_2)} \simeq \ell_g^*.$$

(b) The space $\bigcirc = \langle\Delta_0, \Delta_1\rangle(\ker\Delta_2)$ is isomorphic to $\{(x_2, x_1, x_0) \in \mathcal{R}_3 : x_2 = 0\}$, a closed subspace of $\mathcal{R}_3$. Hence $\bigcirc^* \simeq \mathcal{R}_3^*/\left(\langle\Delta_0, \Delta_1\rangle(\ker\Delta_2)\right)^{\perp}$. Since $\left(\langle\Delta_0, \Delta_1\rangle(\ker\Delta_2)\right)^{\perp} = U_3\left(\Delta_0(\ker\Delta_1 \cap \ker\Delta_2)\right)$ then

$$\langle\Delta_1, \Delta_2\rangle(\mathcal{C}) \simeq \frac{\mathcal{R}_3}{\Delta_0(\ker\Delta_1 \cap \ker\Delta_2)} \simeq \bigcirc^*. \quad \square$$

We thus get the diagram:

$$\begin{array}{ccccc}
\ell_g & = & \ell_g & & \\
\downarrow & & \downarrow & & \\
\bigcirc & \longrightarrow & \mathcal{R}_3 & \longrightarrow & \ell_g^* \\
\downarrow & & \downarrow & & \| \\
\ell_2 & \longrightarrow & \bigcirc^* & \longrightarrow & \ell_g^*
\end{array}$$

Diagram [201]**:** $\Omega_{2,\langle 0,1\rangle} \simeq \Omega_{2,\langle 1,0\rangle}$ gives the central column (coincides with that of [210]), and Propositions 6.4 and 6.5 give the lower row. Thus, denoting $\wedge = \langle \Delta_2, \Delta_0\rangle(\ker \Delta_1)$ we get

$$\begin{array}{ccccc}
\ell_2 & = & \ell_2 & & \\
\downarrow & & \downarrow & & \\
\wedge & \longrightarrow & \mathcal{R}_3 & \longrightarrow & \ell_f^* \\
\downarrow & & \downarrow & & \| \\
\ell_f & \longrightarrow & Z_2 & \longrightarrow & \ell_f^*
\end{array}$$

Arguing as in the proof of Proposition 6.8, we get.

Proposition 6.9. $\langle \Delta_2, \Delta_0\rangle(\mathcal{C})$ *is isomorphic to* $\wedge^* = \langle \Delta_2, \Delta_0\rangle(\ker \Delta_1)^*$.

Proof. Since $\wedge = \langle \Delta_2, \Delta_0\rangle(\ker \Delta_1)$ is a subspace of $\mathcal{R}_3$, $\wedge^* \simeq \mathcal{R}_3^*/\left(\langle \Delta_2, \Delta_0\rangle(\ker \Delta_1)\right)^{\perp}$. Since $\left(\langle \Delta_2, \Delta_0\rangle(\ker \Delta_1)\right)^{\perp} = U_3\left(\Delta_1(\ker \Delta_2 \cap \ker \Delta_0)\right)$ then

$$\langle \Delta_2, \Delta_0\rangle(\mathcal{C}) \simeq \mathcal{R}_3/\left(\Delta_1(\ker \Delta_2 \cap \ker \Delta_0)\right) \simeq \wedge^*. \quad \square$$

Diagram [120]**:** $\Omega_{\langle 1,2\rangle,0} \simeq \Omega_{\langle 2,1\rangle,0}$ gives the central row, and $\Delta_1(\ker \Delta_2 \cap \ker \Delta_0) = \ell_f$ and $\Delta_2(\ker \Delta_0) = \ell_f^*$ by Propositions 6.2, 6.3 and 6.6. Since $\wedge^* \simeq \langle \Delta_2, \Delta_0\rangle(\mathcal{C})$ by Proposition 6.9 and $\Delta_0(\mathcal{C}) = \ell_2$, we get

$$\begin{array}{ccccc}
\ell_f & = & \ell_f & & \\
\downarrow & & \downarrow & & \\
Z_2 & \longrightarrow & \mathcal{R}_3 & \longrightarrow & \ell_2 \\
\downarrow & & \downarrow & & \| \\
\ell_f^* & \longrightarrow & \wedge^* & \longrightarrow & \ell_2
\end{array}$$

Diagram [021]**:** $\Omega_{0,\langle 2,1\rangle} \simeq \Omega_{0,\langle 1,2\rangle}$ gives the central column and $\Omega_{\langle 0,2\rangle,1} \simeq \Omega_{\langle 2,0\rangle,1}$ gives the central row. Since $\Delta_2(\ker \Delta_1) = \ell_f^*$ by Proposition 6.6, we get

$$\begin{array}{ccccc}
\ell_g & = & \ell_g & & \\
\downarrow & & \downarrow & & \\
\wedge & \longrightarrow & \mathcal{R}_3 & \longrightarrow & \ell_f^* \\
\downarrow & & \downarrow & & \| \\
\ell_f^* & \longrightarrow & \bigcirc^* & \longrightarrow & \ell_f^*
\end{array}$$

Diagram [102]**:** $\Omega_{1,\langle 0,2\rangle} \simeq \Omega_{1,\langle 2,0\rangle}$ gives the central column, and $\Omega_{\langle 1,0\rangle,2} \simeq \Omega_{\langle 0,1\rangle,2}$ gives the central row. Moreover, $\Delta_0(\ker \Delta_2) \simeq \ell_f$ by Proposition 6.5. So we get

$$\begin{array}{ccccc}
\ell_f & = & \ell_f & & \\
\downarrow & & \downarrow & & \\
\bigcirc & \longrightarrow & \mathcal{R}_3 & \longrightarrow & \ell_g^* \\
\downarrow & & \downarrow & & \| \\
\ell_f & \longrightarrow & \wedge^* & \longrightarrow & \ell_g^*
\end{array}$$

7. Structure of the new spaces

In this section we describe interesting isomorphic properties of the spaces appearing in the diagrams. Recall that a Banach space X is said to be *hereditarily* ℓ_2 if every infinite dimensional subspace of X contains a subspace isomorphic to ℓ_2. Observe that being hereditarily ℓ_2 is not inherited by quotients. In fact, every separable reflexive space is a quotient of a reflexive hereditarily ℓ_2 space [2, Theorem 6.2].

Proposition 7.1. *All the spaces appearing in the diagrams are hereditarily* ℓ_2.

Proof. Each infinite dimensional subspace of a reflexive Orlicz sequence space contains a copy of ℓ_p for $p \in [\alpha, \beta]$, being α (resp. β) the lower (resp. upper) Boyd index of the space [26, Proposition I.4.3, Theorem I.4.6]. Since $\mathcal{R}_3$ has type $2-\varepsilon$ and cotype $2+\varepsilon$ for each $\varepsilon > 0$, the same happens with ℓ_f and ℓ_g and their dual spaces, hence their Boyd indices are 2 and these spaces are hereditarily ℓ_2. The remaining spaces are hereditarily ℓ_2 too because this is a three-space property [14]. □

Recall from [25, Corollary 13] that if M is an Orlicz function satisfying the Δ_2-condition and $2 \leq q < \infty$ then the space ℓ_M has cotype q if and only if there exists $K > 0$ such that $M(tx) \geq Kt^q M(x)$ for all $0 \leq t, x \leq 1$. Consequently:

Corollary 7.2. *The spaces* ℓ_f *and* ℓ_g *have cotype* 2 *and their dual spaces* ℓ_f^* *and* ℓ_g^* *have type* 2.

We need one more technical result:

Proposition 7.3. *Let* X *be a Banach space.*

(a) *If* X *has type* 2 *then every subspace isomorphic to* ℓ_2 *is complemented.*
(a) *If* X *has an unconditional basis and cotype* 2 *then every subspace of* X *isomorphic to* ℓ_2 *contains an infinite dimensional subspace complemented in* X.

Proof. (a) is a consequence of Maurey's extension theorem; see [19, Corollary 12.24]. (b) The following argument is similar to the proof of [29, Theorem 3.1] for subspaces of L_p, $1 < p < 2$, with an unconditional basis. Let (e_n) be an unconditional basis of X, let (x_k) be a normalized block basis of (e_n), and take a sequence (c_j) of scalars and a successive sequence (B_k) of intervals of integers so that $x_k = \sum_{i\in B_k} c_i e_i$. We consider the sequence of projections (P_k) in X defined by $P_k e_j = e_j$ if $j \in B_k$, and $P_k e_j = 0$ otherwise. Let Q_k be a norm-one projection on $\operatorname{span}\{e_j : j \in B_k\}$ onto the one-dimensional subspace generated by x_k. We claim that $Px = \sum_{k=1}^{\infty} Q_k P_k x$ defines a projection on X onto the closed subspace generated by (x_k). If $x \in X$ then $\sum_{k=1}^{\infty} P_k x$ is unconditionally converging and $\|\sum_{k=1}^{\infty} P_k x\| \leq D\|x\|$ for some $D > 0$. Moreover, since X has cotype 2, $(\sum_{k=1}^{\infty} \|P_k x\|^2)^{1/2} \leq E\|\sum_{k=1}^{\infty} P_k x\|$ for some $E > 0$. We write $Q_k P_k x = s_k x_k$ for each k. Then

$$\left(\sum_{k=1}^{\infty} |s_k|^2\right)^{1/2} \leq \left(\sum_{k=1}^{\infty} \|P_k x\|^2\right)^{1/2} \leq E \cdot D\|x\|.$$

Hence $\sum_{k=1}^{\infty} Q_k P_k x$ converges, and it is easy to check that P is the required projection. □

Corollary 7.4. *Each infinite dimensional subspace of one of the spaces ℓ_f, ℓ_g, ℓ_f^* and ℓ_g^* contains a complemented copy of ℓ_2.*

In the next result, observe that $Z_2 \simeq Z_2^*$. Hence X is (isomorphic to) a subspace of Z_2 if and only if X^* is a quotient of Z_2.

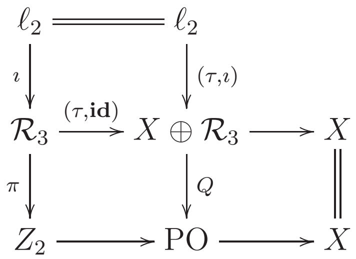
Proposition 7.5. *None of the spaces $\bigcirc$, $\bigcirc^*$, $\wedge$ and $\wedge^*$ is (isomorphic to) a subspace or a quotient of Z_2.*

Proof. It was proved in [24, Theorem 5.4] that every normalized basic sequence in Z_2 has a subsequence equivalent to the basis of one of the spaces ℓ_2 or ℓ_f. Thus none of the four spaces is a subspace of Z_2 because $\bigcirc$ and $\wedge$ contain a copy of ℓ_g and $\bigcirc^*$ and $\wedge^*$ contain a copy of ℓ_f^*, as we can see in the diagrams. □

Next we extend to $\mathcal{R}_3$ some of the fundamental structure results for Z_2 from [24]:

Proposition 7.6. *An operator $\tau : \mathcal{R}_3 \to X$ either is strictly singular or an isomorphism on a complemented copy of $\mathcal{R}_3$.*

Proof. Since the quotient map in the sequence $0 \to \ell_2 \to \mathcal{R}_3 \to Z_2 \to 0$ is strictly singular (see [7]) an operator $\tau : \mathcal{R}_3 \to X$ is strictly singular if and only if $\tau|_{\ell_2}$ is strictly singular. So, let τ be a non-strictly singular operator. Let us assume first that $\tau|_{\ell_2}$ is an embedding so that we can assume that $\|\tau(y, 0)\| \geq \|y\|$ for all $y \in \ell_2$. Observe the commutative diagram:



- The composition $Q\,(\tau, \mathbf{id})$ is strictly singular since it factors through π.
- $Q\,(\tau, \mathbf{id}) = Q(\tau, 0) + Q\,(0, \mathbf{id})$.

- $Q\,(0, \mathbf{id})$ is an embedding since

$$\begin{aligned}\|Q(0,z)\| &= \inf_{y\in\ell_2} \|(0,z)-(\tau,\imath)(y)\| = \inf_{y\in\ell_2}\|(-\tau y, z-y)\|\\ &= \inf_{y\in\ell_2}\left\{\|\tau(y,0)\|+\|z-y\|\right\} \geq \|y\|+\|z\|-\|y\| = \|z\|.\end{aligned}$$

Thus, $Q(\tau, 0)$, being the difference (or sum) between a strictly singular operator and an embedding, has to have closed range and finite dimensional kernel [27, Proposition 2.c.10] and therefore it must be an isomorphism on some finite codimensional subspace of $\mathcal{R}_3$, and the same happens to τ. All subspaces of $\mathcal{R}_3$ with codimension 3 are isomorphic to $\mathcal{R}_3$ and thus we are done.

In the general case, we assume that $\tau|_U$ is an embedding for some subspace $U = [u_n : n \in \mathbb{N}]$ of ℓ_2 generated by normalized disjointly supported blocks u_n of the canonical basis. Define the operator $\tau_U : \Sigma \to \Sigma$ given by $\tau_U(e_n) = u_n$. It was shown by Kalton [21] that if $S_U : Z_2 \longrightarrow Z_2$ is the operator $S_U(e_n, 0) = (u_n, 0)$ and $S_U(0, e_n) = (\Omega_{1,0}u_n, u_n)$ then there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_2 & \longrightarrow & Z_2 & \longrightarrow & \ell_2 & \longrightarrow & 0\\ & & \downarrow{\scriptstyle \tau_U} & & \downarrow{\scriptstyle S_U} & & \downarrow{\scriptstyle \tau_U} & & \\ 0 & \longrightarrow & \ell_2 & \longrightarrow & Z_2 & \longrightarrow & \ell_2 & \longrightarrow & 0\end{array} \tag{14}$$

Observe that we can describe the operator S_U as given by the matrix $S_U = \begin{pmatrix} u & 2u\log u\\ 0 & u\end{pmatrix}$. The theory developed in [12, Proposition 7.1] explains why the upper-right entry of the matrix has to be $2u\log u$. Analogously, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_2 & \longrightarrow & \mathcal{R}_3 & \longrightarrow & \ell_2 & \longrightarrow & 0\\ & & \downarrow{\scriptstyle S_U} & & \downarrow{\scriptstyle R_U} & & \downarrow{\scriptstyle \tau_U} & & \\ 0 & \longrightarrow & Z_2 & \longrightarrow & \mathcal{R}_3 & \longrightarrow & \ell_2 & \longrightarrow & 0\end{array} \tag{15}$$

with the operator R_U emerging from the theory developed in [12, Proposition 7.1] and given by

$$R_U = \begin{pmatrix} u & 2u\log u & 2u\log^2 u\\ 0 & u & 2u\log u\\ 0 & 0 & u\end{pmatrix}$$

Since τ_U is an into isometry, so are S_U and R_U. Thus, $R_U[\mathcal{R}_3]$ is an isometric copy of $\mathcal{R}_3$. Let us show it is complemented. With that purpose, consider $\mathcal{R}_3^U$ the space $\mathcal{R}_3$ constructed with each block u_n in place of e_n; namely, Z_2^U is the twisted sum space $U \oplus_{\Omega_{1,0}^U} U$ constructed with $\Omega_{1,0}^U(u) = 2\sum \lambda_n \log \frac{|u|}{\|u\|}$ for $u \in U$ and then $\mathcal{R}_3^U$ is the space $\mathcal{R}_2^U \oplus_{\Omega^U_{\langle 2,1\rangle,0}} U$ with the corresponding definition for $\Omega^U_{\langle 2,1\rangle,0}$. We can in this way understand R_U as an operator $R_U' : \mathcal{R}_3^U \to \mathcal{R}_3$ in the obvious form: $R_U'(u_n,0,0) = R_U(e_n,0,0)$, $R_U'(0,u_n,0) = R_U(0,e_n,0)$ and $R_U'(0,0,u_n) = R_U(0,0,e_n)$. Consider the diagram

$$\begin{array}{ccc} \mathcal{R}_3^U & \xrightarrow{R'_U} & \mathcal{R}_3 \\ {\scriptstyle D_U}\downarrow & & \downarrow{\scriptstyle D} \\ (\mathcal{R}_3^U)^* & \xleftarrow[{R'_U}^*]{} & \mathcal{R}_3 \end{array}$$

Here D_U is the obvious isomorphism between $\mathcal{R}_3^U$ and $(\mathcal{R}_3^U)^*$ induced by D. The diagram is commutative: for normalized blocks $u_i, u_j, u_k, u_l, u_m, u_n$ one has

$$R'_U(u_a, u_b, u_c) = (u_a + 2u_b \log u_b + 2u_c \log^2 u_c, u_b + 2u_c \log u_c, u_c)$$

while

$$D\left(u_i + 2u_j \log u_j + 2u_k \log^2 u_k, u_j + 2u_k \log u_k, u_k\right)\left(u_l + 2u_m \log u_m + 2u_n \log^2 u_n, u_m + 2u_n \log u_n, u_n\right)$$

is

$$(u_i+2u_j \log u_j+2u_k \log^2 u_k)u_n-(u_j+2u_k \log u_k)(u_m+2u_n \log u_n)+u_k(u_l+2u_m \log u_m+2u_n \log^2 u_n)$$

namely

$$\delta_{in}+2\delta_{jn} \log u+2\delta_{kn} \log^2 u-\delta_{jm}-2\delta_{jn} \log u-2\delta_{km} \log u-4\delta_{kn} \log^2 u+\delta_{kl}+2\delta_{km} \log u+2\delta_{kn} \log^2 u$$

which is $\delta_{in} - \delta_{jm} + \delta_{kl}$. Thus

$$\begin{aligned} {R'_U}^* D R'_U(u_i, u_j, u_k)(u_l, u_m, u_n) &= D R'_U(u_i, u_j, u_k)\left(R'_U(u_l, u_m, u_n)\right) \\ &= \langle R'_U(u_i, u_j, u_k), R'_U(u_l, u_m, u_n)\rangle \\ &= \delta_{in} - \delta_{jm} + \delta_{kl} \\ &= D_U(u_i, u_j, u_k)(u_l, u_m, u_n) \end{aligned}$$

Therefore, $D_U^{-1} {R'_U}^* D$ is a projection onto the range of R_U, as desired, and one can repeat the same argument as before working now with $\tau|_U$ instead of $\tau|_{\ell_2}$. □

Corollary 7.7. *Every operator from $\mathcal{R}_3$ into a twisted Hilbert space is strictly singular. In particular, $\mathcal{R}_3$ does not contain complemented copies of either Z_2 or ℓ_2.*

Proof. That $\mathcal{R}_3$ cannot be a subspace of a twisted Hilbert space was proved in [6]. □

Corollary 7.8. *The six representations of $\mathcal{R}_3$ as a twisted sum in the diagrams are non-trivial.*

Proof. Since $\mathcal{R}_3$ contains no complemented copy of ℓ_2 and $\mathcal{R}_3 \simeq \mathcal{R}_3^*$ [6], by Corollary 7.4 the exact sequences $Z_2 \to \mathcal{R}_3 \to \ell_2$, $\wedge \to \mathcal{R}_3 \to \ell_f^*$ and $\bigcirc \to \mathcal{R}_3 \to \ell_g^*$ have strictly singular quotient map, while $\ell_2 \to \mathcal{R}_3 \to Z_2$, $\ell_f \to \mathcal{R}_3 \to \wedge^*$ and $\ell_g \to \mathcal{R}_3 \to \bigcirc^*$ have strictly cosingular embedding. Of course, the second part is a dual result of the first one. □

We now improve those results. In [24, Theorem 5.4] it is proved that every normalized basic sequence in Z_2 admits a subsequence equivalent to the basis of one of the spaces ℓ_2 or ℓ_f. We obtain the corresponding results for $\mathcal{R}_3$:

Theorem 7.9. *Every normalized basic sequence in $\mathcal{R}_3$ admits a subsequence equivalent to the basis of one of the spaces ℓ_2, ℓ_f, ℓ_g.*

Proof. Let $(y_n, x_n, z_n)_n$ be a normalized basic sequence in $\mathcal{R}_3$. If $\|z_n\| \longrightarrow 0$ we can assume that $\sum \|z_n\| < \infty$ and thus that, up to a perturbation, (y_n, x_n) is a basic sequence in Z_2 that therefore admits a subsequence equivalent to the basis of either ℓ_2 or ℓ_f [24, Theorem 5.4]. If, $\|z_n\| \geq \varepsilon$ then we can assume after perturbation that there is a block basic sequence (u_n) in ℓ_2 such that $\sum \|z_n - u_n\| < \infty$. Since

$$\begin{aligned}(y_n, x_n, z_n) &= (y_n, x_n, z_n) - (\Omega_{\langle 2,1\rangle,0} u_n, u_n) + (\Omega_{\langle 2,1\rangle,0} u_n, u_n)\\ &= ((y_n, x_n) - \Omega_{\langle 2,1\rangle,0} u_n, z_n - u_n) + (\Omega_{\langle 2,1\rangle,0} u_n, u_n)\end{aligned}$$

and $z_n - u_n \to 0$ we can assume that $((y_n, x_n) - \Omega_{\langle 2,1\rangle,0} u_n, z_n - u_n)$ admits a subsequence equivalent to the basis of either ℓ_2 or ℓ_f. We conclude showing that $(\Omega_{\langle 2,1\rangle,0} u_n, u_n)$ is equivalent to the canonical basis of ℓ_g. And thus the plan is to show that $\sum (x_n \Omega_{\langle 2,1\rangle,0} u_n, \sum x_n u_n)$ converges in $\mathcal{R}_3$ if and only if $(x_n) \in \ell_g$. In order to show that, we simplify the notation: let x be a scalar sequence, let $u = (u_n)$ be the sequence of blocks and let us denote $xu = \sum x_n u_n$. Showing that $(x\Omega_{\langle 2,1\rangle,0} u, xu)$ converges in $\mathcal{R}_3$ is the same as showing that its norm is finite. Recall that for a positive normalized z one has $\Omega_{\langle 2,1\rangle,0}(z) = (2z \log^2 z, 2z \log z)$. Since

$$\|(x\Omega_{\langle 2,1\rangle,0} u, xu)\|_{\mathcal{R}_3} = \|(x\Omega_{\langle 2,1\rangle,0} u - \Omega_{\langle 2,1\rangle,0}(xu)\|_{Z_2} + \|xu\|_2 = \|(x\Omega_{\langle 2,1\rangle,0} u - \Omega_{\langle 2,1\rangle,0}(xu)\|_{Z_2} + \|xu\|_2$$

assuming $\|u_n\| = 1$ for all n and $\|xu\| = 1$ then

$$\begin{aligned}x\Omega_{\langle 2,1\rangle,0} u - \Omega_{\langle 2,1\rangle,0}(xu) &= \big(x2u \log^2 u, 2x \log u\big) - \big(2xu \log^2(xu), 2xu \log(xu)\big)\\ &= \big(2xu(\log^2 u - \log^2 xu), 2xu(\log u - \log(ux))\big)\\ &= \big(2xu(\log^2 u - (\log^2 x + \log^2 u + 2\log x \log u), -2xu \log x)\big)\\ &= \big(-2xu(\log^2 x + 2\log x \log u), -2xu \log x)\big)\end{aligned}$$

and therefore one gets

$$\begin{aligned}&\|(x\Omega_{\langle 2,1\rangle,0} u - \Omega_{\langle 2,1\rangle,0}(xu)\|_{Z_2} = \|\big(-2xu(\log^2 x + 2\log x \log u), -2xu \log x)\big)\|_{Z_2}\\ =\;& \|-2xu(\log^2 x + 2\log x \log u) + 4xu \log x \log(2xu \log x)\|_2 + \|2xu \log x\|_2\\ =\;& \|2xu\left(\log^2 x + 2\log 2 \log x + 2\log x \log\log x\right)\|_2 + \|2xu \log x\|_2.\end{aligned}$$

That means that the sequence x satisfies $x(\log|x|)\log|\log|x|| \in \ell_2$; namely, $x \in \ell_g$. □

Let us obtain a few consequences. First, about the structure of $\mathcal{R}_3$:

Proposition 7.10. *$\mathcal{R}_3$ has no complemented subspace with an unconditional basis.*

Proof. If (x_n) were an unconditional basic sequence in $\mathcal{R}_3$ generating a complemented subspace, it would admit a subsequence (x_{n_k}) equivalent to the basis of one of the spaces ℓ_2, ℓ_f, ℓ_g by Theorem 7.9. Since this subsequence would generate a complemented subspace of $\mathcal{R}_3$, we would conclude that $\mathcal{R}_3$ contains a complemented copy of ℓ_2, by Corollary 7.4, which cannot happen. □

Then about the structure of its subspaces:

Lemma 7.11. *None of the spaces $\wedge, \wedge^*, \bigcirc, \bigcirc*$ is either a subspace of a quotient of Z_2. In particular, none of the spaces $\wedge, \bigcirc$ is isomorphic to Z_2.*

Proof. We need to from [6] the estimates for the cotype constants of $\mathcal{R}_3$ that show that $\mathcal{R}_3$, ℓ_g and ℓ_g^* are not subspaces or quotients of Z_2. This, and diagrams [012], [021] and [102]

$$\begin{array}{ccccc} \ell_g & = & \ell_g & & \\ \downarrow & & \downarrow & & \\ \bigcirc & \longrightarrow & \mathcal{R}_3 & \longrightarrow & \ell_g^* \\ \downarrow & & \downarrow & & \| \\ \ell_2 & \longrightarrow & \bigcirc^* & \longrightarrow & \ell_g^* \end{array} \qquad \begin{array}{ccccc} \ell_g & = & \ell_g & & \\ \downarrow & & \downarrow & & \\ \wedge & \longrightarrow & \mathcal{R}_3 & \longrightarrow & \ell_f^* \\ \downarrow & & \downarrow & & \| \\ \ell_f^* & \longrightarrow & \bigcirc^* & \longrightarrow & \ell_f^* \end{array} \qquad \begin{array}{ccccc} \ell_f & = & \ell_f & & \\ \downarrow & & \downarrow & & \\ \bigcirc & \longrightarrow & \mathcal{R}_3 & \longrightarrow & \ell_g^* \\ \downarrow & & \downarrow & & \| \\ \ell_f & \longrightarrow & \wedge^* & \longrightarrow & \ell_g^* \end{array}$$

yield the result. □

Moreover

Proposition 7.12. *The spaces $\wedge$ and $\bigcirc$ are not isomorphic to their dual spaces.*

Proof. Both $\wedge$ and $\bigcirc$ are subspaces of $\mathcal{R}_3$, hence Theorem 7.9 applies. But $\wedge^*$ and $\bigcirc^*$ contain a copy of ℓ_f^*, as we can see in the diagrams, while the canonical basis of ℓ_f^* (or any of its subsequences) is not equivalent to those of ℓ_2, ℓ_f or ℓ_g. □

Proposition 7.13. *$\wedge$ is not isomorphic to either $\bigcirc$ of $\bigcirc^*$.*

Proof. The idea for the proof is to show that every weakly null sequence in $\wedge$ contains a subsequence equivalent to the canonical basis of either ℓ_2 or ℓ_g, so that $\wedge$ cannot contain either ℓ_f or ℓ_f^* and therefore it cannot be isomorphic to either $\bigcirc$ or $\bigcirc^*$. Why it is so is essentially contained in the displayed proof of Theorem 7.9, taking into account that the elements of $\wedge$ have the form $(y, 0, z)$. Our interest is now in showing that when (u_n) are blocks in ℓ_2 (actually in ℓ_f) and $\sum(x_n y_n, 0, u_n)$ converges in $\mathcal{R}_3$ then $x = (x_n)$ is in either ℓ_2 or ℓ_g. Using the same notation as then, since $\|(xy, 0, xu)\|_{\mathcal{R}_3} = \left\|(xy, 0) - \Omega_{\langle 2,1\rangle,0}(xu)\right\|_{Z_2} + \|xu\|_{\ell_2}$, and since $(xy, 0)$ and xu converge when $x \in \ell_2$, our only concern is when $\Omega_{\langle 2,1\rangle,0}(xu)$ converges in Z_2. But this means that $x \in \operatorname{Dom} \Omega_{\langle 2,1\rangle,0} = \ell_g$. □

Proposition 7.14. *The spaces $\wedge$ and $\wedge^*$ do not contain ℓ_2 complemented. Consequently, they do not have an unconditional basis.*

Proof. Consider the diagram [120]

$$\begin{array}{ccccc} \ell_f & = & \ell_f & & \\ \downarrow & & \downarrow & & \\ Z_2 & \longrightarrow & \mathcal{R}_3 & \xrightarrow{Q_0} & \ell_2 \\ \downarrow & & \downarrow{\scriptstyle Q_{2,0}} & & \| \\ \ell_f^* & \longrightarrow & \wedge^* & \longrightarrow & \ell_2 \end{array}$$

Its lower sequence comes defined by $\triangle(x) = x\log^2 x$, obtained from the composition $\Omega_{\langle 2,1\rangle,0}x = (x\log^2 x, x\log x)$ with the projection onto the first coordinate. Let u be a sequence of disjoint blocks of the canonical basis of ℓ_2 and let $x \in \ell_2$.

$$\begin{aligned} \triangle(xu) &= xu\log^2(xu) = xu\left(\log x + \log u)^2\right) \\ &= xu\left(\log^2 x + \log^2 u + 2\log x\log u\right) \\ &= xu\log^2 x + xu\log^2 u + 2xu\log x\log u \end{aligned}$$

Observe that the second term $x \to xu\log^2 u$ is linear while the third term $x \to 2x\log xu\log u$ is, up to a weight, the Kalton-Peck map relative to the subspace $[u]$ generated by w and thus it is, up to a linear map, the map $x \to \Omega_{1,0}(x)$ (see [8]). This map is bounded when considered with value sin its range ℓ_f^*. All this yields that $\triangle|_{[w]}$ is, up to a linear plus a bounded map, $\triangle$. Therefore the exact sequence $0 \to \ell_f^* \to \wedge^* \to \ell_2 \to 0$ is singular, thus its dual sequence $0 \to \ell_2 \to \wedge \stackrel{Q}{\to} \ell_f \to 0$, which is the left column in diagram [201], is cosingular. Assume that $\wedge$ contains a subspace A isomorphic to ℓ_2 and complemented by some projection P. Since Q is strictly singular, there exist an infinite dimensional subspace $A' \subset \ell_2$ and a nuclear operator $K : A' \to \wedge$ such that $I - K : A' \to A$ is an embedding. Passing to a further subspace if necessary we may assume that the nuclear norm $\|K\|_n$ is strictly smaller than 1. Let N be a nuclear operator on $\wedge$ extending K with $\|N\|_n < 1$. Then $\mathbf{1}_\wedge - N$ is invertible and $(\mathbf{1}_\wedge - N)^{-1} = \sum_{k\geq 0} N^k$, and it is easily seen that

$$(\mathbf{1}_\wedge - N) \circ P \circ (\mathbf{1}_\wedge - N)^{-1}$$

is a projection on $\wedge$, hence of ℓ_2 onto A'. This cannot be since the sequence is cosingular. Since $\wedge$ is reflexive, it immediately follows that $\wedge^*$ cannot contain ℓ_2 complemented. As for the second part, since $\wedge$ is a subspace of $\mathcal{R}_3$, the argument in the proof of Corollary 7.10 also proves the result. □

Now we consider the strict singularity of the quotient maps in the six diagrams.

Proposition 7.15. *The following maps are strictly singular:*

(1) Q_0, Q_1, Q_2, $Q_{1,0}$, $Q_{0,1}$, $Q_{2,0}$, $Q_{0,2}$, $Q_{1,2}$ *and* $Q_{2,1}$.
(2) $p_{1,0}$, $p_{0,1}$, $p_{2,0}$, $p_{0,2}$, $p_{2,1}$ *and* $p_{1,2}$.
(3) $q_{1,0}$, $q_{0,1}$, $q_{2,0}$, $q_{0,2}$.

Proof. (1) That Q_0, Q_1, Q_2, $Q_{1,0}$ and $Q_{0,1}$ are strictly singular is a consequence of Proposition 7.6, because ℓ_2, ℓ_f^*, ℓ_g^* and Z_2 do not contain $\mathcal{R}_3$. The lower part in the diagram [120]

$$\begin{array}{ccccc} Z_2 & \longrightarrow & \mathcal{R}_3 & \xrightarrow{Q_0} & \ell_2 \\ \downarrow{\scriptstyle p_{2,0}} & & \downarrow{\scriptstyle Q_{2,0}} & & \| \\ \ell_f^* & \longrightarrow & \wedge^* & \xrightarrow{q_{2,0}} & \ell_2 \end{array}$$

plus the technique used before shows that $Q_{2,0}$, hence $Q_{0,2}$, is strictly singular. Therefore, its restriction $p_{0,2}$ is strictly singular too. Consider then the two inverse representations

$$\begin{array}{ccccc} & & \ell_2 & & \\ & & \uparrow{\scriptstyle p_{1,2}} & & \\ \ell_f & \longrightarrow & \bigcirc & \xrightarrow{p_{0,2}} & \ell_f \\ & & \uparrow & & \\ & & \ell_g & & \end{array}$$

The restriction of $p_{1,2}$ to ℓ_f is the canonical inclusion of ℓ_f into ℓ_2, which is strictly singular due to the criterion [27, Theorem 4.a.10] asserting that given two Orlicz spaces ℓ_M, ℓ_N for which the canonical inclusion $\jmath : \ell_M \to \ell_N$ is continuous then $\jmath$ is strictly singular if and only if for each $B > 0$ there is a sequence $\tau_1, \ldots, \tau_n$ in $(0, 1]$ such that

$$\sum M(\tau_i t) \geq B \sum N(\tau_i t)$$

for all $t \in [0, 1]$. This yields that the canonical inclusions $\ell_g \to \ell_f \to \ell_2$ are strictly singular by straightforward calculations. Thus, also $p_{0,2}$ is strictly singular and consequently the lower part of diagram [102]

$$\begin{array}{ccccc} \bigcirc & \longrightarrow & \mathcal{R}_3 & \xrightarrow{Q_2} & \ell_g^* \\ \downarrow{\scriptstyle p_{0,2}} & & \downarrow{\scriptstyle Q_{0,2}} & & \| \\ \ell_f & \longrightarrow & \wedge^* & \xrightarrow{q_{0,2}} & \ell_g^* \end{array}$$

yields that $Q_{0,2}$, hence $Q_{2,0}$ too, is strictly singular. (2) the maps are restrictions of $Q_{1,0}$, $Q_{0,1}$, $Q_{2,0}$ and $Q_{0,2}$. (3) follows from Corollary 7.4 because Z_2 and $\wedge^*$ contain no complemented copy of ℓ_2. $\square$

We have been unable to prove that $q_{1,2}$ and $q_{2,1}$ are strictly singular, from where it would follow that $\bigcirc$ and $\bigcirc^*$ do not have an unconditional basis.

8. The Fenchel-Orlicz approach

A rich theory [1, 32] contemplates Z_2 as a Fenchel-Orlicz space, with the meaning described next. Recall that a function $\varphi : \mathbb{C}^n \to [0,\infty)$ is a Young function if it is convex, $\varphi(0) = 0$, $\lim_{t\to\infty} \varphi(tx) = \infty$ and $\varphi(zx) = \varphi(x)$ for every $z \in \mathbb{C}$ of norm 1 and for every $x \neq 0$. A Young function φ generates the Fenchel-Orlicz space

$$\ell_\varphi = \{(x^j)_{j\geq 1} \subset \mathbb{C}^n : \exists \rho > 0 \text{ such that } \sum \varphi\Big(\frac{1}{\rho}x^j\Big) < \infty\}$$

endowed with the norm $\|(x^j)_{j\geq 1}\|_\varphi = \inf\{\rho > 0 : \sum \varphi(\frac{1}{\rho}x^j) \leq 1\}$ For $n = 1$ we obtain Orlicz spaces. In general, the Rochberg spaces $\mathcal{R}_n$ generated by the sequence of interpolators (Δ_n) are Fenchel-Orlicz spaces in a natural way (see [16]), i.e., for $\theta \in (0,1)$ fixed and $n \geq 2$ there is a Young function $\varphi_n : \mathbb{C}^n \to [0,\infty)$ such that the identity is an isomorphism between $\mathcal{R}_n$ and ℓ_{φ_n}. At this point, what we need to know is that the spaces $\ell_2, Z_2, \mathcal{R}_3$ are Fenchel-Orlicz spaces in the following way:

- ℓ_2 is $\ell_{\phi^{(1)}}$, the Orlicz space generated by the Orlicz function $\phi^{(1)}(x) = f_0(x) = |x_0|^2$.
- Z_2 is $\ell_{\phi^{(2)}}$, the Fenchel-Orlicz space generated by the quasi-Young function

$$\phi^{(2)}(x_1, x_0) = |x_1 - x_0 \log|x_0||^2 + |x_0|^2.$$

  Keep track that
  - $\phi^{(2)}(x_1, 0) = |x_1|^2$, so $\ell_2 = \{(x,y) \in \ell_{\phi^{(2)}} : y = 0\}$.
  - $\phi^{(2)}(0, x_0) = |x_0 \log|x_0||^2 + |x_0|^2 \sim f$, so $\ell_f = \text{Dom}\,\mathsf{KP} = \{(x,y) \in \ell_{\phi^{(2)}} : x = 0\}$.
- $\mathcal{R}_3$ is $\ell_{\phi^{(3)}}$, the Fenchel-Orlicz space generated by the quasi-Young function

$$\phi^{(3)}(x_2, x_1, x_0) = \phi^{(2)}(x_1, x_0) + \phi^{(1)}(x_2 - g_{(x_1,x_0)}[2])$$

  where the reader should recall that $h[i]$ stands for $\frac{h^{(i)}(1/2)}{i!}$ and $g_x(z) = |x|^{2z-1}x$, so that $g'_x(z) = 2|x|^{2z-1}x \log|x|$, and thus
  $g_x[1] = g'_x(1/2) = 2x \log|x|$. Now, we set $g_{(x_1,x_0)} = g_{x_0} + \frac{\varphi}{k_2} g_{x_1 - g_{x_0}[1]}$, with $\varphi : \mathbb{S} \to \mathbb{D}$ a conformal map such that $\varphi(\frac{1}{2}) = 0$ and k_2 is adjusted so that $g_{(x_1,x_0)}[1] = x_1$.

One therefore has

$$
\begin{aligned}
g_{(x_1,x_0)}(z) &= g_{x_0}(z) + \frac{\varphi(z)}{k_2} g_{x_1-2x_0\log|x_0|}(z) \\
&= |x_0|^{2z-1}x_0 + \frac{\varphi(z)}{k_2}|x_1 - 2x_0\log|x_0||^{2z-1}(x_1 - 2x_0\log|x_0|) \\
g'_{(x_1,x_0)}(z) &= g'_{x_0}(z) + \left(\frac{\varphi}{k_2} g_{x_1-g_{x_0}[1]}\right)'(z) \\
&= 2|x_0|^{2z-1}x_0\log|x_0| \\
&+ \frac{\varphi(z)}{k_2}2|x_1 - 2x_0\log|x_0||^{2z-1}(x_1 - 2x_0\log|x_0|)\log\left(|x_1 - 2x_0\log|x_0||\right) \\
&+ \frac{\varphi'(z)}{k_2}|x_1 - 2x_0\log|x_0||^{2z-1}(x_1 - 2x_0\log|x_0|)
\end{aligned}
$$

$$
\begin{aligned}
g''_{(x_1,x_0)}(z) &= g''_{x_0}(z) + \left(\frac{\varphi(z)}{k_2} g_{x_1-g_{x_0}[1]}(z)\right)''(z) \\
&= 4|x_0|^{2z-1}x_0\log^2|x_0| + \frac{\varphi(z)}{k_2} whatever \\
&+ \frac{\varphi'(z)}{k_2}4|x_1 - 2x_0\log|x_0||^{2z-1}(x_1 - 2x_0\log|x_0|)\log\left(|x_1 - 2x_0\log|x_0||\right) \\
&+ \frac{\varphi''(z)}{k_2}|x_1 - 2\log|x_0||^{2z-1}(x_1 - 2x_0\log|x_0|)
\end{aligned}
$$

hence

$$
\begin{aligned}
g_{(x_1,x_0)}[2] &= \frac{1}{2}g''_{(x_1,x_0)}(1/2) \\
&= 2x_0\log^2|x_0| + \frac{\varphi'(1/2)}{k_2}2(x_1 - 2x_0\log|x_0|)\log(|x_1 - 2x_0\log|x_0||) \\
&+ \frac{\varphi''(1/2)}{2k_2}(x_1 - 2x_0\log|x_0|)
\end{aligned}
$$

In particular

- $g_{(0,0)}[2] = 0$
- $g_{(x_1,0)}[2] = \frac{\varphi'(1/2)}{k_2}2x_1\log|x_1| + \frac{\varphi''(1/2)}{k_2}x_1$
- $g_{(0,x_0)}[2] = 2x_0\log^2|x_0| + \frac{\varphi'(1/2)}{k_2}2(-2x_0\log|x_0|)\log|2x_0\log|x_0|| + \frac{\varphi''(1/2)}{k_2}(-2x_0\log|x_0|)$

Some notation will render things easier. For nonempty $A \subset [n]$, let $i_a : \mathbb{R}^{|A|} \to \mathbb{R}^n$ be the natural inclusion induced by A (for example, if $A = \{2\} \subset [2]$, then $i_A(z) = (z, 0, 0)$). Given a Young function φ on $\mathbb{R}^{n+1}$ we set $\varphi_A = \varphi \circ i_A$ and it is easy to see that φ_A is a Young function.

In conclusion:

(1) $\phi_2^{(3)}(x_2) = \phi^{(3)}(x_2, 0, 0) = \phi^{(2)}(0, 0) + \phi^{(1)}(x_2) = |x_2|^2$ generates the space ℓ_2.

(2) $\phi^{(3)}_1(x_1) = \phi^{(3)}(0, x_1, 0) = \phi^{(2)}(x_1, 0)+\phi^{(1)}(g_{(x_1,0)}[2]) = |x_1|^2+|\frac{\varphi'(1/2)}{k_2}2x_1 \log|x_1|+\frac{\varphi''(1/2)}{k_2}x_1|^2$ generates the space ℓ_f.

(3) $\phi^{(3)}_0(x_0) = \phi^{(3)}(0, 0, x_0) = \phi^{(2)}(0, x_0) + |g_{(0,x_0)[2]}|^2$ generates the space ℓ_g.

(4) $\phi^{(3)}_{2,1}(x_2, x_1) = \phi^{(3)}(x_2, x_1, 0) = \phi^{(2)}(x_1, 0)+\phi^{(1)}(x_2-g_{(x_1,0)}) = |x_1|^2+|x_2-\frac{\varphi'(1/2)}{k_2}2x_1 \log|x_1|-\frac{\varphi''(1/2)}{k_2}x_1|^2$ generates the space Z_2.

(5) $\phi^{(3)}_{2,0}(x_2, x_0) = \phi^{(3)}(x_2, 0, x_0) = \phi^{(2)}(0, x_0) + |x_2 - g_{(0,x_0)}[2]|^2$ generates the space $\wedge$.

(6) $\phi^{(3)}_{1,0}(x_1, x_0) = \phi^{(3)}(0, x_1, x_0) = \phi^{(2)}(x_1, x_0) + |g_{(x_1,x_0)}[2]|^2$ generates the space $\bigcirc$.

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