DIFFERENTIAL PROCESSES GENERATED BY TWO INTERPOLATORS

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Abstract. We study pairs of interpolators, the differentials they generate and their associated commutator theorems. An essential part of our analysis is the study of the intrinsic symmetries of the process. Since we work without any compatibility or categorical assumption, our results are flexible enough to obtain and generalize most known results for commutators or translation operators, in particular those of Cwikel, Kalton, Milman, Rochberg [20] for differential methods and those of Carro, Cerdà and Soria [9] for compatible pairs of interpolators. We also consider stability issues extending the results in [13, 12, 17] from the complex method to general differential methods.

1. Introduction

We study pairs of interpolators, the derivations they generate and their associated commutator theorems. An essential part of our analysis is the study of the intrinsic symmetries of the process that allow us to jump from an ordered pair (Ψ, Φ) of interpolators to its reverse pair (Φ, Ψ). Our results are flexible enough to apply to a variety of interpolation methods: those of the kinds considered in [9] and [20], translation operators and the differential methods of [20]. In particular, we will obtain in Section 7 some general commutator results which exhibit a kind of symmetry. In the last section we generalize stability results in [13, 12, 17] from the complex method to general differential methods. Our approach will be free of categorical elements. We also avoid for most of the paper any compatibility condition in the sense of [9]. The reason to delay compatibility assumptions until 6.1 is that, in a sense, they obscure the symmetry between the results one obtains for an ordered pair (Ψ, Φ) of interpolators and its *reverse* pair (Φ, Ψ).

2. Preliminaries

The theory of twisted sums originated from the papers [22, 32] and was created by Kalton [25] and Kalton and Peck [27]. A brief comprehensive account can be found in [14]. A twisted sum of two quasi-Banach spaces Y, Z is a quasi-Banach space X which has a closed subspace isomorphic to Y such that the quotient X/Y is isomorphic to Z.

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A short exact sequence (of quasi-Banach spaces and linear continuous operators) is a diagram

$$0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0$$

in which the kernel of each arrow coincides with the image of the preceding one. Thus, the open mapping theorem yields that the middle space X is a twisted sum of Y and Z. The simplest short exact sequence is $0 \to Y \to Y \oplus Z \to Z \to 0$ with embedding $y \to (y, 0)$ and quotient map $(y, z) \to z$. Two short exact sequences $0 \to Y \to X_j \to Z \to 0$, $j = 1, 2$, are said to be *equivalent* if there exists an operator $T : X_1 \to X_2$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X_1 & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle T} & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X_2 & \longrightarrow & Z & \longrightarrow & 0. \end{array}$$

By the 3-lemma [14, p. 3] T must be an isomorphism. A short exact sequence is said to be *trivial*, or to *split*, if it is equivalent to $0 \to Y \to Y \oplus Z \to Z \to 0$. Observe that the short exact sequence splits if and only if the subspace Y of X is complemented.

Let M, N be closed subspaces of a Banach space Z, and let S_M denote the unit sphere of M. The *gap* $g(M, N)$ *between* M *and* N is defined by

$$g(M, N) = \max\big\{ \sup_{x \in S_M} \operatorname{dist}(x, N), \sup_{y \in S_N} \operatorname{dist}(y, M) \big\},$$

and the *minimum gap* $\gamma(M, N)$ *between* M *and* N is defined by

$$\gamma(M, N) = \inf_{u \in M \setminus N} \frac{\operatorname{dist}(u, N)}{\operatorname{dist}(u, M \cap N)}.$$

Note that $M + N$ is a closed subspace of Z if and only if $\gamma(M, N) > 0$ [28, Theorem IV.4.2].

Proposition 2.1. *Let M and N be closed subspaces of Z such that $M + N$ is closed, and let us denote $R = (1/2) \min\{\gamma(M, N), \gamma(N, M)\}$. If M_1 and N_1 are closed subspaces of Z and $g(M_1, M) + g(N_1, N) < R$, then*

(1) *$M \cap N = \{0\}$ implies $M_1 \cap N_1 = \{0\}$ and $M_1 + N_1$ is closed.*
(2) *$M + N = Z$ implies $M_1 + N_1 = Z$.*

In particular, if $Z = M \oplus N$ and $g(M_1, M) < R$ then $Z = M_1 \oplus N$; i. e., the property of a subspace being complemented is open with respect to the gap.

Proof. (1) Since $M \cap N = \{0\}$ and $M + N$ is closed, $\gamma(M, N) = \inf_{u \in S_M} \operatorname{dist}(u, N) > 0$.

Suppose that there exists $u \in M_1 \cap N_1$ with $\|u\| = 1$. Since $\operatorname{dist}(u, M) \leq g(M_1, M)$ and $\operatorname{dist}(u, N) \leq g(N_1, N)$, our hypothesis implies

$$\operatorname{dist}(u, M) + \operatorname{dist}(u, N) < \frac{1}{2}\gamma(M, N)$$

and $\operatorname{dist}(u, M \cap N) = 1$, contradicting [28, IV Lemma 4.4]. Hence $M_1 \cap N_1 = \{0\}$. A similar argument shows that $M_1 + N_1$ is closed. Indeed, otherwise for every $\varepsilon > 0$ we could find $u \in M_1$

and $v \in N_1$ with $\|u\| = \|v\| = 1$ and $\|u - v\| < \varepsilon$. Therefore $\mathrm{dist}(u, M) \leq g(M_1, M)$ and $\mathrm{dist}(u, N) \leq g(N_1, N) + \varepsilon$, and [28, IV Lemma 4.4] would imply

$$\frac{1}{2}\gamma(M, N) \leq g(M_1, M) + g(N_1, N) + \epsilon$$

for every $\epsilon > 0$, contradicting the hypothesis.

(2) Let $M^\perp$ denote the annihilator of M in Z^*. Since $M + N = Z$ if and only if $M^\perp \cap N^\perp = \{0\}$ and $M^\perp + N^\perp$ is closed, $g(M, N) = g(M^\perp, N^\perp)$ and $\gamma(M, N) = \gamma(N^\perp, M^\perp)$ [28, Chapter IV], the result is a consequence of (1). □

We refer to [28, Chapter IV] for additional information on the gap between subspaces.

Given Banach spaces X and Y and a bounded operator $T : X \to Y$, it is usual in interpolation theory to endow $T(X)$ with the associated *quotient norm*, defined as follows:

$$\|Tx\|_T = \inf\{\|z\|_X : Tx = Tz, \quad z \in X\} = \mathrm{dist}(x, \ker T).$$

Since the operator T induces an isometry $x + \ker T \to Tx$ from $X/\ker T$ onto $(T(X), \|\cdot\|_T)$, the latter space is a Banach space.

3. Relations between the derivations generated by (Ψ, Φ) and (Φ, Ψ)

We adopt the language of [9], simpler than the one used in [20], but we omit the functor terminology because it is not necessary in our context. We consider a couple (X_0, X_1) of Banach spaces continuously embedded into their sum $\Sigma = X_0 + X_1$ and to this we will simply refer as *a couple*. This sum space admits a norm $\|\cdot\|_\Sigma$ making it a Banach space (see [2]). A linear operator $\tau : \Sigma \to \Sigma$ is said to *act on the couple* (X_0, X_1) if $\tau : X_i \to X_i$ is continuous for $i = 0, 1$.

An *abstract interpolation method for a given couple* (X_0, X_1) is generated by a Banach space $\mathcal{H}$, that we will call the *generalized Calderón space*, and a linear continuous operator $\Phi : \mathcal{H} \to \Sigma$, that we will call *interpolator on* $\mathcal{H}$, in such a way that, for every linear operator τ acting on the couple, there is a linear continuous operator $T : \mathcal{H} \to \mathcal{H}$ such that $\tau \circ \Phi = \Phi \circ T$.

We denote by X_Φ the space $\Phi(\mathcal{H})$ endowed with the quotient norm

$$\|x\|_\Phi = \inf\{\|f\|_\mathcal{H} : f \in \mathcal{H}, \Phi f = x\} \quad (x \in X_\Phi),$$

which is a Banach space. Fix $\varepsilon > 0$ for the rest of the paper. We will consider a homogeneous map $B_\Phi : X_\Phi \to \mathcal{H}$ such that $\Phi B_\Phi x = x$ and $\|B_\Phi(x)\|_\mathcal{H} \leq (1 + \varepsilon)\|x\|_\Phi$ for each $x \in X_\Phi$.

If $\tau : \Sigma \to \Sigma$ is a linear operator acting on the couple then we get an "interpolated linear operator" $\tau : X_\Phi \to X_\Phi$ since

$$\|\tau(\Phi f)\|_\Phi = \|\tau(\Phi B_\Phi \Phi f)\|_\Phi = \|\Phi(T B_\Phi \Phi f)\|_\Phi \leq \|\Phi\| \cdot \|T\|(1 + \varepsilon)\|\Phi(f)\|_\Phi.$$

Our main concern in this paper will be to study the situation in which we have two different interpolators Φ and Ψ acting on the same generalized Calderón space $\mathcal{H}$. Such situations arise in various settings such as those considered, for example, in [9] and [20] where one can even have a whole sequence of interrelated operators acting on the same space $\mathcal{H}$. A typical example of this situation, probably the first which was ever considered, occurs for a given couple (X_0, X_1), picking as $\mathcal{H}$ the Calderón space $\mathcal{F}(X_0, X_1)$ of continuous bounded functions from the closure of the unit strip in the complex plane $\mathbb{S} = \{z \in \mathbb{C} : 0 < Re(z) < 1\}$ to Σ, which are holomorphic on $\mathbb{S}$, so that the maps $t \mapsto f(it) \in X_0$ and $t \mapsto f(1+it) \in X_1$ are continuous and bounded, and endowed with the norm $\|f\|_{\mathcal{H}} = \sup_{t\in\mathbb{R}}\{\|f(it)\|_{X_0}, \|f(1+it)\|_{X_1}\}$. In this situation, we consider the interpolator $\Phi = \delta_\theta$, the evaluation map at $\theta \in \mathbb{S}$. Such choices of course mean that X_Φ is the complex interpolation space $[X_0, X_1]_\theta$. For the rest of this paper we will mostly consider θ real. In addition to δ_θ one has, for each $n \in \mathbb{N}$, the evaluation operators $\delta_\theta^{(n)}$ of the n^{th} derivative at θ. Each of those is also an interpolator on $\mathcal{F}(X_0, X_1)$. In this paper we will often be considering objects generated by pairs of interpolators (Ψ, Φ) which generalize the previously studied objects generated by the pair $(\delta_\theta', \delta_\theta)$.

Let (Ψ, Φ) be a pair of interpolators on $\mathcal{H}$, let $\langle\Psi, \Phi\rangle : \mathcal{H} \to \Sigma \times \Sigma$ be the map defined by $\langle\Psi, \Phi\rangle f = (\Psi f, \Phi f)$, and let $X_{\Psi,\Phi}$ denote the space $\langle\Psi, \Phi\rangle(\mathcal{H}) = \{(\Psi(f), \Phi(f)) : f \in \mathcal{H}\}$, endowed with the quotient norm. One thus has the following commutative diagram with exact rows and columns:

$$(1)\qquad \begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \ker\Psi \cap \ker\Phi & = & \ker\langle\Psi,\Phi\rangle & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \ker\Phi & \longrightarrow & \mathcal{H} & \xrightarrow{\Phi} & X_\Phi & \longrightarrow & 0 \\
 & & {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle\langle\Psi,\Phi\rangle} & & \| & & \\
0 & \longrightarrow & \Psi(\ker\Phi) & \xrightarrow{\imath} & X_{\Psi,\Phi} & \xrightarrow{\rho} & X_\Phi & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $\Psi(\ker\Phi)$ is endowed with the obvious quotient norm that we will call, when necessary, $\|\cdot\|_{\Psi(\ker\Phi)}$; the maps $\imath, \rho$ are defined by $\imath\Psi g = (\Psi g, 0)$ and $\rho(\Psi f, \Phi f) = \Phi f$, respectively, and the sequence

$$(2)\qquad 0 \longrightarrow \Psi(\ker\Phi) \xrightarrow{\imath} X_{\Psi,\Phi} \xrightarrow{\rho} X_\Phi \longrightarrow 0$$

is exact.

Definition 3.1. *The* derivation *associated to the pair* (Ψ, Φ) *is the map* $\Omega_{\Psi,\Phi} : X_\Phi \to \Sigma$ *given by* $\Omega_{\Psi,\Phi} = \Psi B_\Phi$.

The derivation $\Omega_{\Psi,\Phi}$ generates the so-called *derived space*

$$d\Omega_{\Psi,\Phi} = \{(w, x) \in \Sigma \times X_\Phi : w - \Omega_{\Psi,\Phi}x \in \Psi(\ker\Phi)\},$$

endowed with the quasi-norm $\|(w, x)\|_{\Omega_{\Psi,\Phi}} = \|w - \Omega_{\Psi,\Phi}x\|_{\Psi(\ker\Phi)} + \|x\|_\Phi$.

Remark 1. $\|(\cdot,\cdot)\|_{\Omega_{\Psi,\Phi}}$ is a quasi-norm because $B_\Phi(x+y)-B_\Phi(x)-B_\Phi(y)\in\ker\Phi$, hence

$$\|\Omega_{\Psi,\Phi}(x+y)-\Omega_{\Psi,\Phi}(x)-\Omega_{\Psi,\Phi}(y)\|_\Phi \leq 2(1+\varepsilon)\|\Psi:\ker\Phi\to\Psi(\ker\Phi)\|\left(\|x\|_\Phi+\|y\|_\Phi\right). \tag{3}$$

Note also that $\|(\cdot,\cdot)\|_{\Omega_{\Psi,\Phi}}$, as well as $\Omega_{\Psi,\Phi}$ depends on the choice of B_Φ. However, all spaces $d\Omega_{\Psi,\Phi}$ obtained with no matter which choice of ε and B_Φ are the same, and endowed with equivalent quasi-norms.

Therefore, one has a short exact sequence

$$0 \longrightarrow \Psi(\ker\Phi) \xrightarrow{\jmath} d\Omega_{\Psi,\Phi} \xrightarrow{\eta} X_\Phi \longrightarrow 0 \tag{4}$$

with inclusion $\jmath w=(w,0)$ and quotient map $\eta(w,x)=x$.

Proposition 3.2. *The exact sequences (2) and (4) are equivalent. In particular $X_{\Psi,\Phi}$ is isomorphic to $d\Omega_{\Psi,\Phi}$.*

Proof. We will show that the operator s given by $(w,x)\to(w,x)$ makes commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Psi(\ker\Phi) & \xrightarrow{\jmath} & d\Omega_{\Psi,\Phi} & \xrightarrow{\eta} & X_\Phi & \longrightarrow & 0\\ & & \| & & \downarrow s & & \| & & \\ 0 & \longrightarrow & \Psi(\ker\Phi) & \xrightarrow{\imath} & X_{\Psi,\Phi} & \xrightarrow{\rho} & X_\Phi & \longrightarrow & 0\end{array} \tag{5}$$

Let $(w,x)\in d\Omega_{\Psi,\Phi}$. Since $w-\Omega_{\Psi,\Phi}x\in\Psi(\ker\Phi)$, $w-\Omega_{\Psi,\Phi}x=\Psi f$ for some $f\in\ker\Phi$ with $\|f\|_{\mathcal{H}}\leq(1+\varepsilon)\|w-\Omega_{\Psi,\Phi}x\|_q$. Thus $w=\Omega_{\Psi,\Phi}x+\Psi f=\Psi(B_\Phi x+f)$ and therefore $(w,x)=(\Psi(B_\Phi x+f),\Phi(B_\Phi x+f))\in X_{\Psi,\Phi}$ with

$$\|(w,x)\| \;\leq\; \|B_\Phi x+f\|\leq(1+\varepsilon)\|x\|+(1+\varepsilon)\|w-\Omega_{\Psi,\Phi}x\|\leq(1+\varepsilon)\|(w,x)\|_{\Omega_{\Psi,\Phi}}. \qquad \square$$

The space $X_{\Psi,\Phi}$ would correspond, in the case of complex interpolation described above, to the second Rochberg derived space $\mathcal{R}_2$ [33]. See also [7] for a comprehensive approach. A forerunner for Rochberg $\mathcal{R}_2$ space is the space $B_0^{(2)}$ introduced in [34, p.323]. We thank the referee for this information.

We study now the domain and range spaces associated to $\Omega_{\Psi,\Phi}$. The paper [9] contains a similar analysis for compatible couples (see Section 6.1).

Definition 3.3. *Let* (Ψ,Φ) *be a pair of interpolators on* $\mathcal{H}$. *The* domain *and* range *of* $\Omega_{\Psi,\Phi}$ with respect to the short exact sequence (4) *are defined as follows:*

$$\mathrm{Dom}(\Omega_{\Psi,\Phi})=\{x\in X_\Phi:\Omega_{\Psi,\Phi}(x)\in\Psi(\ker\Phi)\}$$

endowed with the quasi-norm $\|x\|_{\mathrm{Dom}(\Omega_{\Psi,\Phi})}=\|\Omega_{\Psi,\Phi}x\|_{\Psi(\ker\Phi)}+\|x\|_\Phi$ *and*

$$\mathrm{Ran}(\Omega_{\Psi,\Phi})=\{w\in\Sigma:\exists x\in X_\Phi,\ w-\Omega_{\Psi,\Phi}(x)\in\Psi(\ker\Phi)\}$$

endowed with the quasi-norm $\|w\|_{\mathrm{Ran}(\Omega_{\Psi,\Phi})}=\inf\{\|w-\Omega_{\Psi,\Phi}(x)\|_{\Psi(\ker\Phi)}+\|x\|_\Phi : x\in X_\Phi,\ w-\Omega_{\Psi,\Phi}(x)\in\Psi(\ker\Phi)\}$.

We have followed the definitions in [3]. Observe that $\mathrm{Ran}(\Omega_{\Psi,\Phi})$ is different from the set $Rang(\Omega_A)$ considered in [9, Definition 10], which is not a vector space in general.

Proposition 3.4. *The maps $Jx = (0, x)$ and $Q(w, y) = w$ define a short exact sequence*

$$0 \longrightarrow \mathrm{Dom}(\Omega_{\Psi,\Phi}) \xrightarrow{J} X_{\Psi,\Phi} \xrightarrow{Q} \mathrm{Ran}(\Omega_{\Psi,\Phi}) \longrightarrow 0. \tag{6}$$

with $\|Jx\|_{\Omega_{\Psi,\Phi}} = \|x\|_{\mathrm{Dom}}$ and $\|w\|_{\mathrm{Ran}} = \inf\{\|(w, x)\|_{\Omega_{\Psi,\Phi}} : (w, x) \in d\Omega_{\Psi,\Phi}\}$.

Proof. Note that $x \in \mathrm{Dom}(\Omega_{\Psi,\Phi})$ if and only if $(0, x) \in d\Omega_{\Psi,\Phi}$, and $\|x\|_{\mathrm{Dom}(\Omega_{\Psi,\Phi})} = \|(0, x)\|_{\Omega_{\Psi,\Phi}}$. Therefore J is an isometric linear operator. The image of J is closed: if a sequence $(0, x_n)$ in $\mathrm{Im}(J)$ converges to $(y, x) \in d\Omega_{\Psi,\Phi}$, then $\lim_n x - x_n = 0$ and $\lim_n y - \Omega_{\Psi,\Phi}(x - x_n) = 0$ in X_Φ. Since $\Omega_{\Psi,\Phi} : X_\phi \to \Sigma$ is continuous at 0 (because B_Φ is so) and the inclusion $X_\Phi \to \Sigma$ is continuous, $y = 0$.

The map Q is well-defined and surjective: $(w, x) \in d\Omega_{\Psi,\Phi}$ implies $w - \Omega_{\Psi,\Phi}x \in X_\Phi$, hence $w \in \mathrm{Ran}(\Omega_{\Psi,\Phi})$. Moreover, $w \in \mathrm{Ran}(\Omega_{\Psi,\Phi})$ implies the existence of $x \in X_\Phi$ such that $w - \Omega_{\Psi,\Phi} \in X_\Phi$, hence $(w, x) \in d\Omega_{\Psi,\Phi}$.

Also it is clear that $\mathrm{Im}(J) = \ker(Q)$ and that $\|w\|_{\mathrm{Ran}(\Omega_{\Psi,\Phi})}$ satisfies the required equality. □

The following result is in [3].

Corollary 3.5. *The spaces $\mathrm{Dom}(\Omega_{\Psi,\Phi})$ and $\mathrm{Ran}(\Omega_{\Psi,\Phi})$, endowed with their respective quasi-norms, are complete.*

The short exact sequences (4) and (6) thus provide two different representations of the same derived space $d\Omega_{\Psi,\Phi}$ as a twisted sum.

Definition 3.6. *Given a pair (Ψ, Φ) of interpolators let us call (Φ, Ψ) its reverse pair. Given a derivation $\Omega_{\Psi,\Phi}$ we will call $\Omega_{\Phi,\Psi}$ its reverse derivation.*

It is not necessary to call $X_{\Phi,\Psi}$ the reverse of $X_{\Psi,\Phi}$ since these two spaces are actually isomorphic (by a simple permutation). Subsections 5.2, 5.3 and 5.4 provide examples of reverse pairs and derivations. We show now that sequences (3) and (5) correspond to the reverse derivations and display the symmetry relations between them.

Proposition 3.7. *Let (Ψ, Φ) be a pair of interpolators on $\mathcal{H}$. Then*

1. $\mathrm{Dom}(\Omega_{\Psi,\Phi}) = \Phi(\ker \Psi)$.
2. $\mathrm{Ran}(\Omega_{\Psi,\Phi}) = X_\Psi$.
3. *The derivation associated to the short exact sequence (6) is $\Omega_{\Phi,\Psi}$.*

Proof. (1) If $x \in \mathrm{Dom}(\Omega_{\Psi,\Phi})$ then $x \in X_\Phi$ and $\Psi B_\Phi x \in \Psi(\ker \Phi)$. Thus $\Psi B_\Phi x = \Psi g$ for some $g \in \ker \Phi$, hence $B_\Phi x - g \in \ker \Psi$ and $x = \Phi(B_\Phi x - g) \in \Phi(\ker \Psi)$. Conversely, if $y \in \Phi(\ker \Psi)$ then $y \in X_\Phi$ and there is $f \in \ker \Psi$ such that $y = \Phi(f)$. We have $B_\Phi(y) - f \in \ker \Phi$, so $\Psi(B_\Phi(y) - f) = \Omega_{\Psi,\Phi}(y) \in \Psi(\ker \Phi)$. Hence $y \in \mathrm{Dom}(\Omega_{\Psi,\Phi})$.

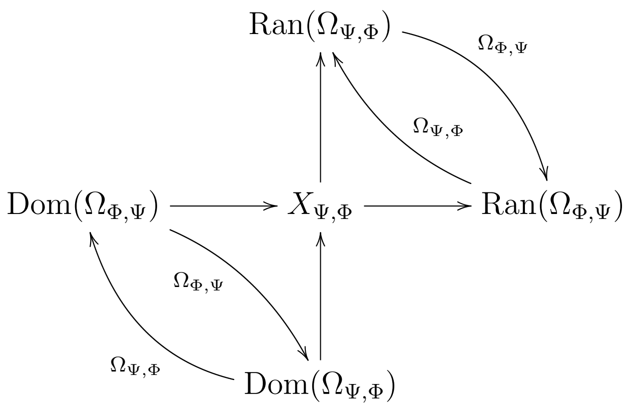
(2) If $w \in \mathrm{Ran}(\Omega_{\Psi,\Phi})$ then there exists $x \in X_\Phi$ such that $w - \Omega_{\Psi,\Phi}x \in \Psi(\ker \Phi) \subset X_\Psi$. Since $\Omega_{\Psi,\Phi}x \in X_\Psi$, we get $w \in X_\Psi$. Conversely, if $w \in X_\Psi$ then $w = \Psi f$ for some $f \in \mathcal{H}$. Since $(\Psi f, \Phi f) \in X_{\Psi,\Phi}$, Proposition 3.4 implies $w = \Psi f \in \mathrm{Ran}(\Omega_{\Psi,\Phi})$.

(3) We have to show that $X_{\Phi,\Psi} = \{(w, x) \in \Sigma \times X_\Psi : w - \Omega_{\Phi,\Psi}x \in \Phi(\ker \Psi)\}$. If $f \in \mathcal{H}$, then $\Psi f \in X_\Psi$ and $\Phi f - \Omega_{\Phi,\Psi}\Psi f = \Phi(f - B_\Psi \Psi f) \in \Phi(\ker \Psi)$ because $f - B_\Psi \Psi f \in \ker \Psi$.

Conversely, if $x \in X_\Psi$ and $w - \Omega_{\Phi,\Psi}x \in \Phi(\ker \Psi)$ then $x = \Psi h$ and $w - \Phi B_\Psi x = \Phi g$ with $h \in \mathcal{H}$ and $g \in \ker \Psi$. Thus $\Phi(g + B_\Psi x) = w$ and $\Psi(g + B_\Psi x) = \Psi B_\Psi x = x$, hence $(w, x) \in X_{\Phi,\Psi}$. □

Clearly $\Omega_{\Psi,\Phi}$ sends $\mathrm{Dom}(\Omega_{\Psi,\Phi})$ into $\Psi(\ker \Phi)$, which coincides with $\mathrm{Dom}(\Omega_{\Phi,\Psi})$ by Proposition 3.7, and thus the general situation is described by the diagram.

(7)



Cabello denotes in [3] $\Omega = \Omega_{\Psi,\Phi}$ and $\mho = \Omega_{\Phi,\Psi}$. He then claim that "the roles of $\mho$ and Ω are perfectly symmetric" in [3, p. 48] and refers to the Kalton-Peck case obtained by complex interpolation in which the pair $\Psi = \delta'_\theta$ and $\Phi = \delta_\theta$, in which $\mathrm{Dom}(\Omega_{\Phi,\Psi}) = X_\Phi = \mathrm{Ran}(\Omega_{\Phi,\Psi})$. It is not hard to check that $\Omega_{\Phi,\Psi}\Omega_{\Psi,\Phi}$ is bounded see below Theorem 4.1 (4') and (5'), therefore, thinking in boundedly equivalent terms we could also denote $\Omega_{\Phi,\Psi} = \Omega^{-1}_{\Psi,\Phi}$. We will not pursue this line in this paper.

4. The bounded splitting theorem

We study now the bounded splitting of the induced sequences. Recall that the short exact sequence

$$0 \longrightarrow \Psi(\ker \Phi) \longrightarrow d\Omega_{\Psi,\Phi} \longrightarrow X_\Phi \longrightarrow 0 \tag{8}$$

generated by the derivation $\Omega_{\Psi,\Phi}$ boundedly splits if $\Omega_{\Psi,\Phi} : X_\Phi \to \Psi(\ker \Phi)$ is bounded. The sequence splits if there is a linear map $L : X_\Phi \to \Sigma$ such that $\Omega_{\Psi,\Phi} - L : X_\Phi \to \Psi(\ker \Phi)$ is bounded. The following result extends and completes [12, Theorem 3.16].

Theorem 4.1. *For a pair (Ψ, Φ) of interpolators, the following conditions are equivalent:*

(1) $\mathcal{H} = \ker \Phi + \ker \Psi$.
(2) $X_\Phi = \Phi(\ker \Psi)$.
(3) $X_\Psi = \Psi(\ker \Phi)$.
(4) $\mathrm{Dom}(\Omega_{\Psi,\Phi}) = X_\Phi$.
(5) $\mathrm{Dom}(\Omega_{\Phi,\Psi}) = X_\Psi$.

The above conditions are also equivalent to their "topological" counterparts:

(1') $\mathcal{H} = \ker\Phi + \ker\Psi$ *and there exists* $C > 0$ *such that for every* $f \in \mathcal{H}$ *we can find* $g \in \ker\Psi$ *and* $h \in \ker\Phi$ *with* $f = g + h$, $\|g\| \leq C\|f\|$ *and* $\|h\| \leq C\|f\|$.
(2') $X_\Phi = \Phi(\ker\Psi)$ *with equivalent norms.*
(3') $X_\Psi = \Psi(\ker\Phi)$ *with equivalent norms.*
(4') $\Omega_{\Psi,\Phi}$ *is bounded from* X_Φ *to* $\Psi(\ker\Phi)$.
(5') $\Omega_{\Phi,\Psi}$ *is bounded from* X_Ψ *to* $\Phi(\ker\Psi)$.

Proof. (1) $\Leftrightarrow$ (2) because $\Phi(\mathcal{H}) = \Phi(\ker\Psi)$ if and only if $\mathcal{H} = \ker\Phi + \ker\Psi$. Similarly (1) $\Leftrightarrow$ (3). Clearly (3) $\Rightarrow$ (4) and, by Proposition 3.7, (4) $\Rightarrow$ (3). Similarly (2) $\Leftrightarrow$ (5).

(1) $\Rightarrow$ (1') Since $\mathcal{H} = \ker\Phi + \ker\Psi$, the map $\Psi : \ker\Phi \to X_\Psi$ is open. Thus there exists $c > 0$ such that, for every $f \in \mathcal{H}$, we can find $g \in \ker\Phi$ with $\|g\| \leq c\|f\|$ and $\Psi f = \Psi g$. Then $f - g \in \ker\Psi$, $\|f - g\| \leq (1 + c)\|f\|$ and $f = g + (f - g)$.

(2) $\Rightarrow$ (2') follows from $\operatorname{dist}(f, \ker\Phi) \leq \operatorname{dist}(f, \ker\Phi \cap \ker\Psi)$ and the open mapping theorem, and the proof of (3) $\Rightarrow$ (3') is similar.

(3') $\Rightarrow$ (4') is a consequence of $\|\Omega_{\Psi,\Phi} f\|_\Psi = \|\Psi B_\Phi f\|_\Psi \leq \|\Psi\|(1 + \varepsilon)\|f\|_\Phi$, (4') $\Rightarrow$ (3') follows from Proposition 3.7, and the proof of (2') $\Leftrightarrow$ (5') is similar. $\square$

In particular, Theorem 4.1 shows that the sequence

$$0 \longrightarrow \Psi(\ker\Phi) \longrightarrow d\Omega_{\Psi,\Phi} \longrightarrow X_\Phi \longrightarrow 0 \tag{9}$$

boundedly splits precisely when $\mathcal{H} = \ker\Phi + \ker\Psi$, which also happens if an only if

$$0 \longrightarrow \Phi(\ker\Psi) \longrightarrow d\Omega_{\Psi,\Phi} \longrightarrow X_\Psi \longrightarrow 0 \tag{10}$$

boundedly splits. Thus, it also shows the symmetry between the results for a pair of interpolators and for its reverse.

Corollary 4.2. $\Omega_{\Psi,\Phi}$ *is bounded if and only if its reverse* $\Omega_{\Phi,\Psi}$ *is bounded.*

We however could not decide whether it is true that $\Omega_{\Psi,\Phi}$ is trivial if and only if so is $\Omega_{\Phi,\Psi}$.

5. Examples

Several relevant examples in the literature admit a formulation in the scheme of pairs we have just presented. They include Cwikel, Kalton, Milman, Rochberg differential methods [20], the compatible and almost compatible pairs of interpolators of Carro, Cerdà and Soria [9], see Section 6.1, the translation operators considered by Cwikel, Jawerth, Milman and Rochberg [19] and, of course, the complex and real methods.

5.1. **Differential methods.** The so-called differential methods of Cwikel, Kalton, Milman and Rochberg [20] correspond to our schema of two interpolators (Ψ, Φ), and this is the content of [20, Section 5]. With the same notation used there (see [20] for precise definitions): $\overline{B} = (X_0, X_1)$, $\mathbf{X} = (\mathcal{X}_0, \mathcal{X}_1)$ is a couple of (Laurent compatible) pseudolattices, and

$$\mathcal{J}(\mathbf{X}, \overline{B}) = \{(b_n)_{n\in\mathbb{Z}} : b_n \in X_0 \cap X_1, (e^{jn} b_n)_{n\in\mathbb{Z}} \in \mathcal{X}_j(B_j), j = 0, 1\}$$

is endowed with the norm $\|(b_n)\| = \max_{j=0,1} \|(e^{jn}b_n)\|_{X_j(B_j)}$. Fix the open annulus $\mathbb{A} = \{z \in \mathbb{C} : 1 < |z| < e\}$, so that Laurent compatibility allows one to identify the elements of $\mathfrak{J}(\mathbf{X}, \overline{B})$ with certain analytic functions $f : \mathbb{A} \to X_0 + X_1$. For $s \in \mathbb{A}$, two interpolators $\Phi_s : \mathfrak{J}(\mathbf{X}, \overline{B}) \to \Sigma$ and $\Psi_s : \mathfrak{J}(\mathbf{X}, \overline{B}) \to \Sigma$ are given by

$$\Phi_s((b_n)) = \sum s^n b_n \quad \text{and} \quad \Psi_s((b_n)) = \sum n s^{n-1} b_n.$$

As it is observed in [20], the identification of the space $\mathfrak{J}(\mathbf{X}, \overline{B})$ with certain analytic functions $f : \mathbb{A} \to X_0 + X_1$ implies that the pair of interpolators (Ψ_s, Φ_s) adopt the form $\Phi_s(f) = f(s)$ and $\Psi_s(f) = f'(s)$. It is carefully shown in [20] that these methods subsume most versions of the real and complex interpolation methods [20, Section 4]; and also the method of compatible and almost-compatible pairs of interpolators of Carro, Cerdà and Soria [9] to which we will return later (see [20, Section 5]); in particular, the differential condition [20, Def. 3.4] is there to get almost-compatible pairs of interpolators, while an additional condition (the left-shift maps boundedly $\mathfrak{J}(\mathbf{X}, \overline{B})$ into itself) is required to make the pair of interpolators compatible.

5.2. **Translation operators.** Consider an ordered pair (Φ_θ, Φ_ν) of evaluation interpolators both associated to a differential interpolation method as above in which the left-shift operator $(b_n)_{n\in\mathbb{Z}} \to (b_{n-1})_{n\in\mathbb{Z}}$ is bounded on $\mathcal{X}_j(B_j), j = 0, 1$. The associated derivation $\Phi_\theta B_{\Phi_\nu}$ is the translation map $\mathcal{R}_{\theta,\nu}$ considered in [20, 19]. In this case the reverse derivation associated to the reverse pair (Φ_ν, Φ_θ) is obviously $\mathcal{R}_{\nu,\theta}$. Observe that $\mathcal{R}_{\nu,\theta} : X_\theta \to X_\nu$ is clearly bounded, and therefore the induced short exact sequence $0 \to X_\nu \to dR_{\nu,\theta} \to X_\theta \to 0$ splits. Slightly less obvious is that also the natural sequence generated by $\mathcal{R}_{\nu,\theta}$, namely,

$$0 \longrightarrow \Phi_\nu(\ker \Phi_\theta) \longrightarrow X_{\nu,\phi} \longrightarrow X_\theta \longrightarrow 0$$

splits: this is consequence of Theorem 4.1 and

Lemma 5.1. $\mathfrak{J}(\boldsymbol{X}, \overline{B}) = \ker \Phi_\nu + \ker \Phi_\theta$.

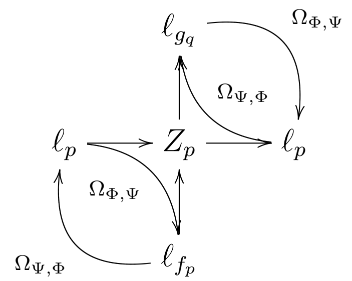
Proof. Pick $f \in \mathfrak{J}(\mathbf{X}, \overline{B})$ and set $f = \frac{z-\nu}{\theta-\nu} f + \frac{\theta-z}{\theta-\nu} f$. It is only required to check hat both $\frac{z-\nu}{\theta-\nu} f \in \mathfrak{J}(\mathbf{X}, \overline{B})$ and $\frac{\theta-z}{\theta-\nu} f \in \mathfrak{J}(\mathbf{X}, \overline{B})$. This essentially means checking that whenever $f \in \mathfrak{J}(\mathbf{X}, \overline{B})$ then also $zf \in \mathfrak{J}(\mathbf{X}, \overline{B})$, and this follows from the left-shift assumption. □

Corollary 5.2. $\Phi_\nu(\ker \Phi_\theta) = X_\nu$ *with equivalence of norms.*

5.3. **Kalton-Peck maps.** The arguably simplest case, that of the couple (ℓ_∞, ℓ_1) and the interpolators $\Psi = \delta'_\theta$ and $\Phi = \delta_\theta$ from complex interpolation yield $X_\Phi = \ell_p$ for $p = \theta^{-1}$ with associated derivation

$$\Omega_{\Psi,\Phi}(x) = p\, x \log \frac{|x|}{\|x\|_p}$$

—usually called the Kalton-Peck map— and derived space $X_{\Psi,\Phi} = Z_p$, the so-called Kalton-Peck spaces [27]. According to [27, Lemma 5.3 (c)], $\mathrm{Dom}(\Omega_{\Psi,\Phi}) = \ell_{f_p}$ is the Orlicz sequence space generated by $f_p(t) = t^p |\log t|^p$, while $\mathrm{Ran}(\Omega_{\Psi,\Phi})$ can be obtained by duality as the Orlicz space $\ell^*_{f_p} = \ell_{f^*_p}$ generated by the Orlicz conjugate function f^*_p of f_p [19]. According to [30, Ex. 4.c.1], the Orlicz function f_p^* is equivalent to $g_q(t) = t^q|\log t|^{p(1-q)} = t^q|\log t|^{-q}$ at 0, for $pq = p + q$. Thus, the two sequences generated by $\Omega_{\Psi,\Phi}$ and $\Omega_{\Phi,\Psi}$ form the particular case of diagram (7):



While Ω can be described, up to a bounded perturbation, as $\Omega x = 2x \log \frac{|x|}{\|x\|_2}$, the general formula for $\mho$ seems too be complicated to compute. We can provide however some hints.

Claim. *Fix* $p = q = 2$. One has

$$\mho\left(\sum_{j=1}^{n} e_j\right) = \frac{-1}{2\log\sqrt{n}} \sum^{n} e_j$$

for large n. Indeed, set $x_n = \sum_{j=1}^n e_j$. By the very definition of Orlicz norm, and taking into account that Orlicz functions are defined on some neighborhood of 0, one has $n = \|x_n\|_{g_2}^2 \log^2 \|x_n\|_{g_2}$. Let W be Lambert's function on $(0, \infty)$, so that $t = W(te^t)$ for every $t > 0$. One has $t = W(t)e^{W(t)}$ and thus

$$W(\sqrt{n}) = W(\|x_n\|_{g_2} \log \|x_n\|_{g_2}) = \log \|x_n\|_{g_2}$$

which yields $\|x_n\|_{g_2} = e^{W(\sqrt{n})} = \frac{\sqrt{n}}{W(\sqrt{n})}$. Now, to calculate $\mho(x_n)$ we must find functions $f_n \in \mathcal{F}(\ell_\infty, \ell_1)$ such that $f_n'(\frac{1}{2}) = x_n$ and $\|f_n\| \leq C\|x_n\|_{g_2}$ for n big enough and some constant C independent of n. For $a = \sum a_j e_j \in c_{00}$ if

$$f_a(z) = \|a\|_2 \sum \frac{a_j}{|a_j|}\Big(\frac{|a_j|}{\|a\|_2}\Big)^{2z} e_j$$

then $f_a \in \mathcal{F}(\ell_\infty, \ell_1)$, $f_a(\frac{1}{2}) = a$, and $\|f_a\| = \|a\|_2$. Set $f_n = \frac{-1}{2\log\sqrt{n}} f_{x_n}$. Then $f_n'(\frac{1}{2}) = x_n$ and $\|f_n\| = \frac{1}{2}\frac{\sqrt{n}}{\log\sqrt{n}}$. Now, W is asymptotic to $\log(x) - \log(\log(x)) = \log \frac{x}{\log(x)}$ (see [16]) so that

$$\frac{\sqrt{n}}{\log\sqrt{n}} \sim e^{W(\sqrt{n})} = \frac{\sqrt{n}}{W(\sqrt{n})}.$$

Thus, g_n are the desired functions and, for n large enough,

$$\mho\left(\sum_{j=1}^{n} e_j\right) = g_n(\frac{1}{2}) = \frac{-1}{2\log\sqrt{n}} \sum^{n} e_j.$$

Similarly, the couple $(L_\infty(\mu), L_1(\mu))$ generates by complex interpolation at θ the space $L_p(\mu)$, $p = \theta^{-1}$ with associated derivation $\Omega_p(f) = pf \log \frac{|f|}{\|f\|_p}$ and derived space the "big" Kalton-Peck $\overline{Z}_p$ space. The spaces Z_p and $\overline{Z}_p$ are identical in definition but not in their properties simply because ℓ_p and $L_p(\mu)$ spaces have different properties depending on whether the measure is atomic or not. The reverse pair $(\delta_\theta, \delta'_\theta)$ generates the reverse decomposition with the Lorentz spaces $L_f, (L_f)^*$ and reverse derivation $\mho_p$.

5.4. **Weighted Köthe spaces.** Fix a Köthe function space X with the Radon-Nikodym property, let w_0 and w_1 be weight functions, and consider the interpolation couple (X_0, X_1), where $X_j = X(w_j)$, $j = 0, 1$ with their natural norms. In [12, Proposition 4.1] we showed that $X_\theta = X(w_\theta)$ for $0 < \theta < 1$, where $w_\theta = w_0^{1-\theta} w_1^\theta$. For $\Psi = \delta'_\theta$ and $\Phi = \delta_\theta$ we obtain $\Omega_{\Psi,\Phi} f = \log \frac{w_1}{w_0} \cdot f$, a linear map. Let us determine $\mathrm{Dom}(\Omega_{\Psi,\Phi})$ and $\mathrm{Ran}(\Omega_{\Psi,\Phi})$:

Claim 1: $\mathrm{Dom}(\Omega_{\Psi,\Phi}) = X(w_\theta) \cap X(w_\theta \left|\log \frac{w_1}{w_0}\right|)$ *with equivalence of norms.*

Indeed, $x \in \mathrm{Dom}(\Omega_{\Psi,\Phi})$ if and only if both x and $\Omega_{\Psi,\Phi}(x) = \log \frac{w_1}{w_0} x$ belong to $X(w_\theta)$.

Claim 2: $\mathrm{Ran}(\Omega_{\Psi,\Phi}) = X(w_\theta) + X(w_\theta \left|\log \frac{w_1}{w_0}\right|^{-1})$ *with equal norms.*

If $w \in \mathrm{Ran}(\Omega_{\Psi,\Phi})$, then we may write $w = w - \log \frac{w_1}{w_0} x + \log \frac{w_1}{w_0} x$ with $w - \log \frac{w_1}{w_0} x \in X(w_\theta)$ and $x \in X(w_\theta)$. Then $\log \frac{w_1}{w_0} x \in X(w_\theta \left|\log \frac{w_1}{w_0}\right|^{-1})$, hence $w \in X(w_\theta) + X(w_\theta \left|\log \frac{w_1}{w_0}\right|^{-1})$ and

$$\|w\|_{X(w_\theta)+X(w_\theta \left|\log \frac{w_1}{w_0}\right|^{-1})} \leq \|w\|_{\mathrm{Ran}(\Omega_{\Psi,\Phi})}.$$

If $w \in X(w_\theta) + X(w_\theta \left|\log \frac{w_1}{w_0}\right|^{-1})$, then $w = y + z$ with $y \in X(w_\theta)$ and $z \in X(w_\theta \left|\log \frac{w_1}{w_0}\right|^{-1})$. Also, there is $x \in X(w_\theta)$ such that $z = \log \frac{w_0}{w_1} x$ and $\|z\|_{X(w_\theta \left|\log \frac{w_1}{w_0}\right|^{-1})} = \|x\|_{X(w_\theta)}$. So we get the other inclusion and the other norm estimate.

To finish the description of $\mathrm{Dom}(\Omega_{\Psi,\Phi})$ and $\mathrm{Ran}(\Omega_{\Psi,\Phi})$, let us denote $\omega_\wedge = \min\{\omega_0, \omega_1\}$ and $\omega_\vee = \max\{\omega_0, \omega_1\}$.

Claim 3: $X(\omega_0) \cap X(\omega_1) = X(\omega_\vee)$ *with equivalence of norms.*

Let $x \in X(\omega_\vee)$. Then $\max\{\|\omega_0 x\|_X, \|\omega_1 x\|_X\} \leq \| = \max\{\|x\|_{X(\omega_0)}, \|x\|_{X(\omega_1)}\} \leq \|x\|_{\omega_\vee}$.

If $x \in X(\omega_0) \cap X(\omega_1)$ and A is the set where $\omega_0 \leq \omega_1$ then $\omega_\vee x = \omega_1 x \chi_A + \omega_0 x (1 - \chi_A)$, so

$$\|\omega_\vee x\|_X \leq \|\omega_1 x\|_X + \|\omega_0 x\|_X \leq 2\|x\|_{X(\omega_0)\cap X(\omega_1)}$$

and we get the other inclusion.

Claim 4: $X(\omega_0) + X(\omega_1) = X(\omega_\wedge)$ *with equivalence of norms*

Let $x = x_0 + x_1 \in X(\omega_0) + X(\omega_1)$ with $x_j \in X(\omega_j)$, $j = 0, 1$. We have

$$\|\omega_\wedge x\|_X \quad \leq \quad \|\omega_\wedge x_0\|_X + \|\omega_\wedge x_1\|_X \leq \|\omega_0 x_0\|_X + \|\omega_1 x_1\|_X = \|x_0\|_{X(\omega_0)} + \|x_1\|_{X(\omega_1)}.$$

Since x_0 and x_1 are arbitrary, $\|x\|_{X(\omega_\wedge)} \leq \|x\|_{X(\omega_0)+X(\omega_1)}$. Now let $x \in X(\omega_\wedge)$, and let A be as above. Let $x_0 = x\chi_A$ and $x_1 = x(1-\chi_A)$. Then $x_j \in X(\omega_j)$, $j = 0, 1$, and $\|x_0\|_{X(\omega_0)} + \|x_1\|_{X(\omega_1)} \leq 2\|x\|_{X(\omega_\wedge)}$, so we obtain the other inclusion.

In order to calculate $\Omega_{\Phi,\Psi}$, recall that $X_\Psi = \mathrm{Ran}(\Omega_{\Psi,\Phi})$. If we let $\omega = \min\{1, \left|\log \frac{\omega_1}{\omega_0}\right|^{-1}\} w_\theta$ then by Claims 2 and 4 we have $X_\Psi = X(\omega)$ with equivalence of norms. Let $x \in X_\Psi$ and let us suppose that $\min\{1, \left|\log\frac{\omega_1}{\omega_0}\right|^{-1}\} = (\log\frac{\omega_1}{\omega_0})^{-1}$. Then $(\log\frac{\omega_1}{\omega_0})^{-1}x \in X(\omega_\theta) = X_\Phi$, and the function $B_\theta(x)$ given by $B_\theta(x)(z) = \left(\frac{\omega_1}{\omega_0}\right)^{z-\theta}(\log\frac{\omega_1}{\omega_0})^{-1}x$ is in $\mathcal{H}$, its norm is $\|(\log\frac{\omega_1}{\omega_0})^{-1}x\|_{X(\omega_\theta)} = \|x\|_{X(\omega)}$ and $\Psi(B_\theta(x)) = x$. So $\Omega_{\Phi,\Psi} = \Phi(B_\theta(x)(z)) = (\log\frac{\omega_1}{\omega_0})^{-1}x$.

5.5. **Orlicz spaces.** Recall that an *N-function* is a map $\varphi : [0,\infty) \to [0,\infty)$ which is strictly increasing, continuous, $\varphi(0) = 0$, $\varphi(t)/t \to 0$ as $t \to 0$, and $\varphi(t)/t \to \infty$ as $t \to \infty$. An N-function φ satisfies the Δ_2*-property* if there exists a number $C > 0$ such that $\varphi(2t) \le C\varphi(t)$ for all $t \ge 0$. When an N-function φ satisfies the Δ_2-property, the *Orlicz space* $L_\varphi(\mu)$ is $L_\varphi(\mu) = \{f \in L_0(\mu) : \varphi(|f|) \in L_1(\mu)\}$ endowed with the norm $\|f\| = \inf\{r > 0 : \int \varphi(|f|/r)d\mu \le 1\}$.

Given φ_0 and φ_1 two N-functions satisfying the Δ_2-property and $0 < \theta < 1$ then a combination of [23] and [13, 18] yields $\left(L_{\varphi_0}(\mu), L_{\varphi_1}(\mu)\right)_\theta = L_\varphi(\mu)$ where $\varphi^{-1} = \left(\varphi_0^{-1}\right)^{1-\theta}\left(\varphi_1^{-1}\right)^\theta$ is an N-function satisfying the Δ_2-property. In particular, when $t = \varphi_0^{-1}(t)\varphi_1^{-1}(t)$ we have $\left(L_{\varphi_0}(\mu), L_{\varphi_1}(\mu)\right)_{1/2} = L_2(\mu)$ the associated derivation is, for $\|f\|_2 = 1$,

$$\Omega_{1/2}(f) = f\,\log\frac{\varphi_1^{-1}(f^2)}{\varphi_0^{-1}(f^2)}.$$

The determination of the spaces $d\Omega_{1/2}$, $\mathrm{Dom}(\Omega_{1/2})$ and $\mathrm{Ran}(\Omega_{1/2})$ requires a somewhat contorted digression into the theory of Fenchel-Orlicz spaces that will appear in [8]; we combine [13, 18] and [8] to get:

Proposition 5.3. *Let* $(\ell_{\phi_0}, \ell_{\phi_1})$ *be a couple of Orlicz sequence spaces. Fix* $0 < \theta < 1$. *Then*

(1) $\left(\ell_{\phi_0}, \ell_{\phi_1}\right)_\theta = \ell_{\phi_\theta}$ *with* $\phi_\theta^{-1} = \left(\phi_0^{-1}\right)^{1-\theta}\left(\phi_1^{-1}\right)^\theta$.
(2) $\Omega_\theta(x) = x\phi_\theta\left(x\log\frac{\phi_1^{-1}(x_1)}{\phi_0^{-1}(x_0)}\right)$ *where* $x = x_0x_1$ *is a Lozanovskii factorization of* x *(see* [13]*).*
(3) $d\Omega_\theta$ *is a Fenchel-Orlicz space (see* [1]*).*
(4) $\mathrm{Dom}(\Omega_\theta) = \ell_{\Phi_\theta}$ *where* $\Phi_\theta(x) = \phi_\theta\left(x\log\frac{\phi_1^{-1}(x)}{\phi_0^{-1}(x)}\right)$

There is an especially interesting case we can mention: pick ℓ_φ an Orlicz sequence space with φ an $N-$function for which there are $p > 1$ and $M > 0$ such that for every $\lambda \in (0,1]$ and for every $s > 0$ we have $\frac{\varphi(\lambda s)}{\lambda^p\varphi(s)} \le M$. For instance, if ℓ_φ has nontrivial type then we may suppose that φ has the previous property [1]. Now, Ω_θ is a multiple of the quasilinear map defined in [1], so that $d\Omega_\theta$ turns out to be isomorphic to the Fenchel-Orlicz space ℓ_ψ, with Young function $\psi : \mathbb{C}^2 \to [0,\infty)$ given by $\psi(x,y) = \varphi(y) + \varphi(x - y\log|x|)$. The isomorphism $T : d\Omega_\theta \to \ell_\psi$ is given by $T(x,y) = (x, py)$. If $\xi(t) = \psi(0,t)$ then $\mathrm{Dom}(\Omega_\theta) = \ell_\xi$. Since $\varphi(t) \le \varphi(t\log|t|)$ on a neighborhood of 0, actually $\mathrm{Dom}(\Omega_\theta) = \ell_{\varphi(\mathrm{t}\log|\mathrm{t}|)}$.

6. Compatibility-like conditions

Some special pairs of interpolators were studied in [9, Definition 3.1 and Remark 3.2]:

Definition 6.1. *A pair of interpolators* (Ψ, Φ) *on the same space* $\mathcal{H}$ *is called* almost compatible *when* $\Psi(\ker\Phi) \subset X_\Phi$*, and it is called* compatible *when* $\Psi(\ker\Phi) = X_\Phi$.

In the case of a compatible pair of interpolators, the identity $X_{\Psi,\Phi} = d\Omega_{\Psi,\Phi}$ appears in [9, Prop. 7.2], answering question 6 of [19]. Observe that compatibility assumptions always refer to an ordered pair, which means that (Ψ, Φ) can be compatible but (Φ, Ψ) not. In fact, one has

Lemma 6.2. *If the two pairs (Ψ, Φ) and (Φ, Ψ) are compatible then $X_\Phi = X_\Psi$ and both $\Omega_{\Psi,\Phi}$ and $\Omega_{\Phi,\Psi}$ are bounded.*

Proof. If the pair (Ψ, Φ) is compatible then $X_\Phi \subset X_\Psi$. If both pairs are compatible then $X_\Phi = X_\Psi = \Psi(\ker \Phi)$ and Theorem 4.1 concludes. □

The assertion in the Lemma fails for almost-compatibility since when the pair (Ψ, Φ) is compatible then the pair (Φ, Ψ) is almost-compatible. In the case of compatible pairs (Ψ, Φ), the short exact sequence (4) becomes

$$0 \longrightarrow X_\Phi \longrightarrow d\Omega_{\Psi,\Phi} \longrightarrow X_\Phi \longrightarrow 0 \tag{11}$$

So $d\Omega_{\Psi,\Phi}$ is a twisted sum of X_Φ with itself. When (Ψ, Φ) is almost compatible, since the inclusion map $X_\Phi \to \Sigma$ is continuous, the map $\Psi : \ker \Phi \to X_\Phi$ is continuous by the closed graph theorem. Similarly, when (Ψ, Φ) is compatible, the inclusion $X_\Phi \to X_\Psi$ is continuous.

Let us now consider what occurs when the same derivation is used to generate twisted sums with larger spaces. This is interesting to cover the case of almost compatible interpolators.

Definition 6.3. *Let (Ψ, Φ) be a pair of interpolators on $\mathcal{H}$. We say that a subspace Z of Σ is* suitable *for (Ψ, Φ) if $\Psi(\ker \Phi) \subset Z$ and there is a norm $\|\cdot\|_Z$ on Z such that $(Z, \|\cdot\|_Z)$ is a Banach space and the inclusion $(Z, \|\cdot\|_Z) \to \Sigma$ is continuous.*

The derivation $\Omega_{\Psi,\Phi}$ and a suitable space Z for (Ψ, Φ) generate a derived space

$$d\Omega_{\Psi,\Phi}(Z) = \{(w, x) \in \Sigma \times X_\Phi : w - \Omega_{\Psi,\Phi} x \in Z\},$$

endowed with $\|(w, x)\|_{\Omega^Z_{\Psi,\Phi}} = \|w - \Omega_{\Psi,\Phi} x\|_Z + \|x\|_\Phi$, which can be showed to be a quasi-norm arguing as in Remark 1. We also obtain a short exact sequence

$$0 \longrightarrow Z \longrightarrow d\Omega_{\Psi,\Phi}(Z) \longrightarrow X_\Phi \longrightarrow 0 \tag{12}$$

with inclusion $w \to (w, 0)$ and quotient map $(w, x) \to x$.

The case (Ψ, Φ) almost compatible and $Z = X_\Phi$ was studied in [9].

Definition 6.4. *Let (Ψ, Φ) be a pair of interpolators on $\mathcal{H}$, and let Z be a suitable space for (Ψ, Φ). We define the* domain *and the* range *of $\Omega_{\Psi,\Phi}$ with respect to the short exact sequence (12) as follows:*

$$\mathrm{Dom}(\Omega^Z_{\Psi,\Phi}) = \{x \in X_\Phi : \Omega_{\Psi,\Phi}(x) \in Z\}$$

endowed with $\|x\|_{\mathrm{Dom}(\Omega^Z_{\Psi,\Phi})} = \|\Omega_{\Psi,\Phi} x\|_Z + \|x\|_\Phi$, and

$$\mathrm{Ran}(\Omega^Z_{\Psi,\Phi}) = \{w \in \Sigma : \exists x \in X_\Phi,\ w - \Omega_{\Psi,\Phi}(x) \in Z\}$$

endowed with $\|w\|_{\mathrm{Ran}(\Omega^Z_{\Psi,\Phi})} = \inf\{\|w - \Omega_{\Psi,\Phi}(x)\|_Z + \|x\|_\Phi : x \in X_\Phi,\ w - \Omega_{\Psi,\Phi}(x) \in Z\}$.

The arguments in the proof of Proposition 3.4 give the following result:

Proposition 6.5. *Let (Ψ, Φ) be a pair of interpolators on $\mathcal{H}$, and let Z be a suitable space for (Ψ, Φ). Then the maps $Jx = (0, x)$ and $Q(w, y) = w$ define a short exact sequence*

$$0 \longrightarrow \mathrm{Dom}(\Omega^Z_{\Psi,\Phi}) \xrightarrow{J} d\Omega_{\Psi,\Phi}(Z) \xrightarrow{Q} \mathrm{Ran}(\Omega^Z_{\Psi,\Phi}) \longrightarrow 0. \tag{13}$$

with $\|Jx\|_{\Omega^Z_{\Psi,\Phi}} = \|x\|_{\mathrm{Dom}(\Omega^Z_{\Psi,\Phi})}$ *and* $\|w\|_{\mathrm{Ran}(\Omega^Z_{\Psi,\Phi})} = \inf\{\|(w, x)\|_{\Omega_{\Psi,\Phi}} : (w, x) \in d\Omega_{\Psi,\Phi}(Z)\}$.

Corollary 6.6. *The spaces* $\mathrm{Dom}(\Omega^Z_{\Psi,\Phi})$ *and* $\mathrm{Ran}(\Omega^Z_{\Psi,\Phi})$*, endowed with their respective quasi-norms, are complete.*

The following result of [9, Theorem 3.8] gives a description of the domain in the almost-compatible case.

Proposition 6.7. *Let (Ψ, Φ) be an almost compatible pair of interpolators on $\mathcal{H}$. Then* $\mathrm{Dom}(\Omega^{X_\Phi}_{\Psi,\Phi}) = \Phi\left(\Psi^{-1} X_\Phi\right)$.

It would be interesting to determine $\mathrm{Ran}(\Omega^{X_\Phi}_{\Psi,\Phi})$ for (Ψ, Φ) almost compatible.

7. The general form of a commutator theorem

Recall that given two maps A, B in the suitable conditions, their commutator is defined as the map $[A, B] = AB - BA$. A number of papers [9, 10, 15, 19, 20, 21, 29, 33, 34] contain statements about the boundedness of commutators in interpolation scales, known as commutator theorems- The purpose of this section is to show that there is just one commutator theorem from which all the existing versions can be derived. Let τ be a linear operator acting on the couple and let

$$\begin{aligned} C(\tau) &= \max\{\|\tau : \mathrm{Dom}(\Omega_{\Phi,\Psi}) \to \mathrm{Dom}(\Omega_{\Phi,\Psi})\|, \|\tau : \mathrm{Ran}(\Omega_{\Phi,\Psi}) \to \mathrm{Ran}(\Omega_{\Phi,\Psi})\}\| \\ H(\tau) &= \inf\{\|T : \mathcal{H} \to \mathcal{H}\| : \Psi T = \tau\Psi, \quad \Phi T = \tau\Phi\}. \end{aligned}$$

One has:

Theorem 7.1 (Abstract commutator theorem I)**.** *Let (X_0, X_1) be an interpolation couple of Banach spaces and let (Ψ, Φ) be a pair of interpolators on $\mathcal{H}$. If τ is a linear operator acting on the couple then there is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Dom}(\Omega_{\Phi,\Psi}) & \longrightarrow & X_{\Phi,\Psi} & \longrightarrow & \mathrm{Ran}(\Omega_{\Phi,\Psi}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle(\tau,\tau)} & & \downarrow{\scriptstyle\tau} & & \\ 0 & \longrightarrow & \mathrm{Dom}(\Omega_{\Phi,\Psi}) & \longrightarrow & X_{\Phi,\Psi} & \longrightarrow & \mathrm{Ran}(\Omega_{\Phi,\Psi}) & \longrightarrow & 0 \end{array}$$

where (τ, τ), which represents the operator $(\tau, \tau)(w, x) = (\tau w, \tau x)$, is bounded. The commutator map $[\tau, \Omega_{\Psi,\Phi}] : \mathrm{Ran}(\Omega_{\Phi,\Psi}) \to \mathrm{Dom}(\Omega_{\Phi,\Psi})$ *is bounded and satisfies the estimate*

$$\|[\tau, \Omega_{\Psi,\Phi}]\| \leq 2H(\tau)\|B_\Phi\|. \tag{14}$$

and thus

$$\|(\tau, \tau)\| \leq C(\tau) + \|[\tau, \Omega_{\Psi,\Phi}]\| \tag{15}$$

Proof. If the reader has been surprised to see $\Omega_{\Phi,\Psi}$ instead of $\Omega_{\Psi,\Phi}$, recall that $\mathrm{Dom}(\Omega_{\Phi,\Psi}) = \Psi(\ker\Phi)$ and $\mathrm{Ran}(\Omega_{\Phi,\Psi}) = X_\Phi$ so that the commutative diagram above is exactly

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Psi(\ker\Phi) & \longrightarrow & X_{\Psi,\Phi} & \longrightarrow & X_\Phi & \longrightarrow & 0 \\ & & \tau\Big\downarrow & & \Big\downarrow(\tau,\tau) & & \Big\downarrow\tau & & \\ 0 & \longrightarrow & \Psi(\ker\Phi) & \longrightarrow & X_{\Psi,\Phi} & \longrightarrow & X_\Phi & \longrightarrow & 0 \end{array}$$

and the derivation associated to the exact sequence is $\Omega_{\Psi,\Phi}$. Now, observe that the statement makes sense since $x \in X_\Phi = \mathrm{Ran}(\Omega_{\Phi,\Psi})$ implies that $(\tau\Omega_{\Phi,\Psi} - \Omega_{\Phi,\Psi}\tau)x \in \Psi(\ker\Phi)$. Moreover, the operator $\tau : \Psi(\ker\Phi) \to \Psi(\ker\Phi)$ is well-defined and continuous: if $x \in \Psi(\ker\Phi)$, i.e., $x = \Psi(g)$ with $\Phi(g) = 0$ then $\Phi(Tg) = \tau(\Phi g) = 0$ and thus

$$\tau(x) = \tau(\Psi(g)) = \Psi(Tg)) \in \Psi(\ker\Phi).$$

The operator $(\tau,\tau) : X_{\Psi,\Phi} \to X_{\Psi,\Phi}$ is well-defined: if $(w,x) \in X_{\Psi,\Phi}$, namely $w - \Psi B_\Phi x \in \Psi(\ker\Phi)$ then $w - \Psi B_\Phi x = \Psi g$ for some $g \in \ker\Phi$ and thus

$$\begin{aligned} \tau w - \Psi B_\Phi \tau x &= \tau w - \Psi T B_\Phi x + \Psi T B_\Phi x - \Psi B_\Phi \tau x \\ &= \tau w - \tau\Psi B_\Phi x + \Psi T B_\Phi x - \Psi B_\Phi \tau x \\ &= \tau(w - \Psi B_\Phi x) + \Psi\left(T B_\Phi x - B_\Phi \tau x\right) \\ &= \tau\Psi(g) + \Psi\left(T B_\Phi x - B_\Phi \tau x\right) \\ &= \Psi(Tg) + \Psi\left(T B_\Phi x - B_\Phi \tau x\right) \end{aligned}$$

which means that $\tau w - \Psi B_\Phi \tau x \in \Psi(\ker\Phi)$ since both Tg and $TB_\Phi x - B_\Phi\tau x$ belong to $\ker\Phi$ and thus $(\tau w, \tau x) \in X_{\Psi,\Phi}$. It is continuous since, by the previous identity,

$$\begin{aligned} \|(\tau w, \tau x)\| &= \|\tau w - \Psi B_\Phi \tau x\|_{\Psi(\ker\Phi)} + \|\tau x\|_\Phi \\ &\leq \|\Psi T g\|_{\Psi(\ker\Phi)} + \|\Psi\left(T B_\Phi x - B_\Phi \tau x\right)\|_{\Psi(\ker\Phi)} + \|\tau x\|_\Phi \\ &\leq C(\tau)\|(w,x)\| + \|[\tau, \Omega_{\Psi,\Phi}](x)\|_{\Psi(\ker\Phi)}, \end{aligned}$$

and

$$\begin{aligned} \|[\tau, \Omega_{\Psi,\Phi}](x)\|_{\Psi(\ker\Phi)} &= \|\tau\Omega_{\Psi,\Phi} x - \Omega_{\Psi,\Phi}\tau x\|_{\Psi(\ker\Phi)} \\ &= \|\tau\Psi B_\Phi(x) - \Psi B_\Phi(\tau x)\|_{\Psi(\ker\Phi)} \\ &= \|\Psi T B_\Phi(x) - \Psi B_\Phi(\tau x)\|_{\Psi(\ker\Phi)} \\ &\leq 2H(\tau)\|B_\Phi\|\|x\|. \end{aligned}$$

□

Estimate 15 is there to show that an explicit estimate for the norm of the middle operator is possible. It is perhaps worthwhile to remark that $\|\Psi\|$ "does not appear" in the estimate above because $\|\Psi : \ker\Phi \to \Psi(\ker\Phi)\| = 1$. Observe that estimates (15) and (14) are not the same, even if they are equivalent. Indeed, estimate (15) means that

$$\|\tau w - \Omega_{\Psi,\Phi}\tau x\|_{\mathrm{Dom}(\Omega_{\Phi,\Psi})} \leq C'\left(\|w - \Omega_{\Psi,\Phi} x\|_{\mathrm{Dom}(\Omega_{\Phi,\Psi})} + \|x\|_{\mathrm{Ran}(\Omega_{\Phi,\Psi})}\right)$$

while (14) means

$$\|\tau\Omega_{\Psi,\Phi}x - \Omega_{\Psi,\Phi}\tau x\|_{\mathrm{Dom}(\Omega_{\Phi,\Psi})} \leq C''\|x\|_{\mathrm{Ran}(\Omega_{\Phi,\Psi})}$$

for suitable constants C', C''.

Now, our setting so far has been to consider only one single couple (X_0, X_1) of Banach spaces and the "interpolation space" X_Φ was so with respect to linear operators $(X_0, X_1) \to (X_0, X_1)$. A standard interpolation method is often understood to mean a procedure which, given two (suitable) Banach couples (X_0, X_1) and (Y_0, Y_1) yields two spaces X, Y so that every bounded linear operator $\tau : (X_0, X_1) \to (Y_0, Y_1)$ is also a bounded map $X \to Y$. Such modification can be easily implemented: given two suitable couples $(X_0, X_1), (Y_0, Y_1)$ of Banach spaces we need

1. Two Banach spaces $\mathcal{H}^X, \mathcal{H}^Y$ of functions.
2. Two interpolators $\Phi^X : \mathcal{H}^X \to X_0 + X_1$ and $\Phi^Y : \mathcal{H}^Y \to Y_0 + Y_1$ in such a way that, for every linear operator t acting from the couple (X_0, X_1) to the couple (Y_0, Y_1) (i.e., such that t acts continuously $t : X_0 + X_1 \to Y_0 + Y_1$ as well as $X_0 \to Y_0$ and $X_1 \to Y_1$) there is a linear continuous operator $T : \mathcal{H}^X \to \mathcal{H}^Y$ such that $t \circ \Phi^X = \Phi^Y \circ T$.

Everything of what has been exposed so far translates verbatim to this two-couples context; except, perhaps, that a linear operator $t : \Sigma^X \to \Sigma^Y$ is said to *act between the couples* $X = (X_0, X_1)$ and $Y = (Y_0, Y_1)$ if $t : X_i \to Y_i$ is continuous for $i = 0, 1$. In this general situation the commutator theorem becomes:

Theorem 7.2 (Abstract commutator theorem II). *Let $X = (X_0, X_1), Y = (Y_0, Y_1)$ be couples of Banach spaces and let (Ψ^X, Φ^X) (resp. (Ψ^Y, Φ^Y)) be a pair of interpolators on $\mathcal{H}^X$ (resp. $\mathcal{H}^Y$). If τ is a linear operator acting between the couples $X \to Y$ then there is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Dom}(\Omega_{\Phi^X,\Psi^X}) & \longrightarrow & X_{\Phi^X,\Psi^X} & \longrightarrow & \mathrm{Ran}(\Omega_{\Phi^X,\Psi^X}) & \longrightarrow & 0 \\ & & {\scriptstyle\tau}\big\downarrow & & \big\downarrow{\scriptstyle(\tau,\tau)} & & \big\downarrow{\scriptstyle\tau} & & \\ 0 & \longrightarrow & \mathrm{Dom}(\Omega_{\Phi^Y,\Psi^Y}) & \longrightarrow & Y_{\Phi^Y,\Psi^Y} & \longrightarrow & \mathrm{Ran}(\Omega_{\Phi^Y,\Psi^Y}) & \longrightarrow & 0 \end{array} \tag{16}$$

where (τ, τ), which represents the operator $(\tau, \tau)(w, x) = (\tau w, \tau x)$, is bounded. Equivalently, let $[\tau, \Omega]$ denote the generalized commutator map $\tau\Omega_{\Psi^X,\Phi^X} - \Omega_{\Phi^Y,\Psi^Y}\tau$. Then $[\tau, \Omega] : \mathrm{Ran}(\Omega_{\Phi^X,\Psi^X}) \to \mathrm{Dom}(\Omega_{\Phi^Y,\Psi^Y})$ is bounded. One has similar estimates as in Theorem 7.2.

When the pairs (Ψ^X, Φ^X) and (Ψ^Y, Φ^Y) are compatible then the diagram (16) adopts the more standard form:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_{\Phi^X} & \longrightarrow & X_{\Psi^X,\Phi^X} & \longrightarrow & X_{\Phi^X} & \longrightarrow & 0 \\ & & {\scriptstyle\tau}\big\downarrow & & \big\downarrow{\scriptstyle(\tau,\tau)} & & \big\downarrow{\scriptstyle\tau} & & \\ 0 & \longrightarrow & Y_{\Phi^Y} & \longrightarrow & Y_{\Psi^Y,\Phi^Y} & \longrightarrow & Y_{\Phi^Y} & \longrightarrow & 0 \end{array}$$

and yields a recognizable estimate: the generalized commutator map $[\tau, \Omega] : X_{\Phi^X} \to Y_{\Phi^Y}$ is bounded. The reverse version of Theorem 7.2 has exactly the same form just interchanging Ψ and Φ. However, even if from the abstract point of view both theorems are "the same" they may lead to quite different concrete estimates, of which a few examples follow. For the sake of clarity and to simplify notation we will formulate the results assuming $X = Y$, but simple modifications yield the general case.

7.1. **Commutator theorem for weighted spaces.** We refer to Example 5.4. Let $X_0 = X(w_0)$ and $X_1 = X(w_1)$ be weighted versions of the same base space X. In this case we claim that the commutator theorem for $\Omega = \Omega_{\delta',\delta}$ and for its reverse $\mho = \Omega_{\delta,\delta'}$ are the same: call $a = \log \frac{w_1}{w_0}$ so that $\Omega(f) = af$ and let $min = \min\{1, |a|^{-1}\}w_\theta$ and $max = \max\{1, |a|\}w_\theta$. The continuity of the commutator $[\tau, \Omega] : X(w_\theta) \to X(w_\theta)$ means

$$\|w_\theta \left(\tau(af) - a\tau f\right)\|_X \leq \|w_\theta f\|_X$$

while when $\mho g = a^{-1}g$, the continuity of $[\tau, \mho] : X(min) \to X(max)$ means

$$\left\|\tau(a^{-1}g) - a^{-1}\tau g\right\|_{max} \leq \|g\|_{min}$$

Thus, assuming $min = |a|^{-1} w_\theta$ and $max = w_\theta |a|$ then $\|w_\theta a \left(\tau(a^{-1}g) - a^{-1}\tau g\right)\|_X \leq \|a^{-1}w_\theta g\|_X$ which, by simple change of variable $g = af$, becomes, as we knew,

$$\left\|w_\theta a \left(\tau(f) - a^{-1}\tau(af)\right)\right\|_X = \|w_\theta \left(a\tau(f) - \tau(af)\right)\|_X \leq \|w_\theta f\|_X.$$

7.2. **Commutator theorems for Kalton-Peck maps.** Picking the couple (L_{p_0}, L_{p_1}) the derivation at $\frac{1}{p} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1}$ is the Kalton-Peck map $\mathtt{K}_p(x) = p(\frac{1}{p_0} - \frac{1}{p_1})x \log \frac{|x|}{\|x\|_p}$, and the standard commutator theorem means the estimate

$$\left\|\tau\left(x \log \frac{|x|}{\|x\|_p}\right) - \tau(x) \log \frac{|\tau(x)|}{\|\tau(x)\|_p}\right\|_p \leq C\|x\|_p$$

for an operator τ acting on the couple (L_{p_0}, L_{p_1}). The reverse pair $(L_{f_p}, (L_{f_p})^*)$ (see section 5.3) with reverse derivation $\mho_p$ yields the estimate

$$\|\tau(\mho_p x) - \mho_p \left(\tau(x)\right)\|_{L_{f_p}} \leq C\|x\|_{(L_{f_p})^*}$$

Thus, if one picks the case (ℓ_∞, ℓ_1) and $p = 1/2$, the computations in Subsection 5.3 yield, recalling that $x_n = \sum^n e_j$:

$$\left\|\frac{-1}{2\log\sqrt{n}}\tau x_n - \mho_2\left(\tau x_n\right)\right\|_{\ell_{f_2}} \leq C\frac{\sqrt{n}}{W(\sqrt{n})}.$$

7.3. **Commutator theorems for convexifications.** The Kalton-Peck map admits a more general form. Fix $0 < \theta < 1$. According to [13, Prop. 3.6], if X is a Banach space with 1-unconditional basis then its p-convexification is the space $X_\theta = (\ell_\infty, X)_\theta$ for $p = \theta^{-1}$ (conversely, if X is p-convex and X^p is its p-concavification then $(\ell_\infty, X^p)_\theta = X$) with induced derivation

$$\mathtt{K}_{X_\theta}(x) = p\ x\ \log \frac{|x|}{\|x\|_{X_\theta}}$$

Accordingly, the commutator estimate is

$$\left\|\tau\left(x\log\frac{|x|}{\|x\|_{X_\theta}}\right)-\tau(x)\log\frac{|\tau(x)|}{\|\tau(x)\|_{X_\theta}}\right\|_{X_\theta}\leq C\|x\|_{X_\theta}$$

for any operator τ acting on the couple (X,ℓ_∞).

7.4. **Commutator theorems for Lorentz spaces.** Kalton-Peck maps can be defined on an arbitrary Banach space X having a 1-unconditional basis in the form

$$\mathtt{K}_X(x)=x\log\frac{|x|}{\|x\|_X}$$

Observe that, in principle, one does not know if there is a couple (X_0,X_1) so that $X=(X_0,X_1)_\theta$ for some θ or even if, even in that case, whether $\mathtt{K}_X$ is the associated derivation. In a Köthe function space $X\subset L_0$ over a measurable subset of $(0,+\infty)$, a second Kalton map [25] can be defined. Consider the rank function $r_x(t)=m\{s:|f(s)|>|f(t)| \quad \text{or} \quad |f(s)|=|f(t)| \quad \text{and} \quad s\leq t\}$, where m is the Lebesgue measure and $f\in X$. The Kalton map on X is defined as

$$\kappa_X(x)=x\ \log r_x$$

The general case of Lorentz spaces $L_{p,q}$ can be treated now following [5]: if $(L_{p_0,q_0},L_{p_1,q_1})_\theta=L_{p,q}$ with $\frac{1}{p}=(1-\theta)\frac{1}{p_0}+\theta\frac{1}{p_1}$ and $\frac{1}{q}=(1-\theta)\frac{1}{q_0}+\theta\frac{1}{q_1}$ then the associated derivation is

$$\Omega_{p,q}(x)=q\left(\frac{1}{q_1}-\frac{1}{q_0}\right)\mathtt{K}_{L_{p,q}}(x)+\left(\frac{q}{p}\left(\frac{1}{q_0}-\frac{1}{q_1}\right)-\left(\frac{1}{p_0}-\frac{1}{p_1}\right)\right)\kappa_{L_{p,q}}(x)$$

The case of L_p spaces follows from this by setting $q_0=p_0$ and $q_1=p_1$. One thus gets, for operators τ acting on the couple $(L_{p_0,q_0},L_{p_1,q_1})$, the commutator estimate

$$\|\tau\Omega_{p,q}(x)-\Omega_{p,q}(\tau(x))\|_{p,q}\leq C\|x\|_{p,q}$$

This is remarkable since neither of the estimates

$$\begin{aligned}\left\|\tau\mathtt{K}_{L_{p,q}}(x)-\mathtt{K}_{L_{p,q}}(\tau(x))\right\|_{p,q}&\leq C\|x\|_{p,q}\\ \left\|\tau\kappa_{L_{p,q}}(x)-\kappa_{L_{p,q}}(\tau(x))\right\|_{p,q}&\leq C\|x\|_{p,q}\end{aligned}$$

does, in principle, hold since neither $\mathtt{K}_{p,q}$ nor $\kappa_{L_{p,q}}$ are the right derivations. These results are new even compared with those in [11]. We cannot however leave unnoticed [11, Theorem 27] in which a remarkable connection between the complex derivation and the real derivation appears

7.5. **Commutator theorem for Orlicz spaces.** We continue form section 5.5 with the particular situation there described: when $t=\varphi_0^{-1}(t)\varphi_1^{-1}(t)$, $\big(L_{\varphi_0}(\mu),L_{\varphi_1}(\mu)\big)_{1/2}=L_2(\mu)$ and the associated derivation is $\Omega_{1/2}(f)=f\ \log\frac{\varphi_1^{-1}(f^2)}{\varphi_0^{-1}(f^2)}$ for $\|f\|_2=1$. The direct commutator estimate is thus

$$\left\|\tau\left(f\log\frac{\varphi_1^{-1}(\frac{f^2}{\|f\|_2^2})}{\varphi_0^{-1}(\frac{f^2}{\|f\|_2^2})}\right)-\tau(f)\log\frac{\varphi_1^{-1}(\frac{\tau(f)^2}{\|\tau(f)\|_2^2})}{\varphi_0^{-1}(\frac{\tau(f)^2}{\|\tau(f)\|_2^2})}\right\|_2\leq C\|f\|_2$$

for an operator τ acting on the couple $\big(L_{\varphi_0}(\mu), L_{\varphi_1}(\mu)\big)$.

7.6. **Commutator theorems for translation operators.** Cwikel, Kalton, Milman, Rochberg obtain in [20, Theorem 3.8 (ii)] a commutator theorem for translation mappings which, as they say [20, p.278]:

> *On the other hand, it is not at all clear to us at this stage how one could obtain a result like part (ii) of Theorem 3.8 in the abstract setting of* [9].

The result was integrated in the scheme of [9] by Cerdà in [15, p.1018] as Proposition 7.4 below. The idea observed by Cerdà is to consider the (non-compatible) pair of interpolators (Φ_θ, Φ_ν) associated to a differential interpolation method. In this case the derivation $\Phi_\theta B_{\Phi_\nu}$ is the translation map $\mathcal{R}_{\theta,\nu}$ as described in Subsection 5.2.

Proposition 7.3. [Commutator theorem for translation maps] *There is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Phi_\theta(\ker \Phi_\nu) & \longrightarrow & X_{\theta,\nu} & \longrightarrow & X_\nu & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle (\tau,\tau)} & & \downarrow{\scriptstyle \tau} & & \\ 0 & \longrightarrow & \Phi_\theta(\ker \Phi_\nu) & \longrightarrow & X_{\theta,\nu} & \longrightarrow & X_\nu & \longrightarrow & 0 \end{array}$$

More precisely, the commutator map $[\tau, \mathcal{R}_{\theta,\nu}] : X_\nu \to \Phi_\theta(\ker \Phi_\nu)$ *is bounded.*

It is not necessary to formulate the reverse form as we observed in Example 5.2. The statement in Proposition 7.3 is not, in principle, that in [20, Theorem 3.8 (ii)] or [15, Cor. 4.3] since those results establish that the commutator map $[\tau, \mathcal{R}_{\theta,\nu}] : X_\nu \to X_\theta$ is bounded, as it is obvious since $\mathcal{R}_{\theta,\nu} : X_\nu \to X_\theta$ is bounded, while Proposition 7.3 asserts that $[\tau, \mathcal{R}_{\theta,\nu}] : X_\nu \to \Phi_\theta(\ker \Phi_\nu)$ is bounded. However, by Corollary 5.2, $\Phi_\theta(\ker \Phi_\nu) = X_\theta$ and thus also $\mathcal{R}_{\theta,\nu} : X_\nu \to \Phi_\theta(\ker \Phi_\nu)$ is bounded. The estimate one obtains in this form [15] for the commutator is however more interesting:

Proposition 7.4. *One has*

$$\|[\tau, \mathcal{R}_{\theta,\nu}] : X_\nu \to \Phi_\theta(\ker \Phi_\nu)\| \leq g(\ker \Phi_\nu, \ker \Phi_\theta)$$

Proof. From the last estimate in the proof of Theorem 7.2 we get

$$\begin{aligned} \|[\tau, \mathcal{R}_{\theta,\nu}](x)\|_{\Phi_\theta(\mathcal{H})} &\leq \|\Phi_\theta T B_{\Phi_\nu}(x) - \Phi_\theta B_{\Phi_\nu} \tau x\|_{\Phi_\theta(\mathcal{H})} \\ &= \operatorname{dist}\left(T B_{\Phi_\nu}(x) - B_{\Phi_\nu} \tau x, \ker \Phi_\theta\right) \\ &\leq \|T B_{\Phi_\nu}(x) - B_{\Phi_\nu} \tau x\|_{\mathcal{H}} \;\; g(\ker \Phi_\theta, \ker \Phi_\theta) \\ &\leq 2\|T\|\|B_{\Phi_\nu}\|\|x\|_\nu \;\; g(\ker \Phi_\nu, \ker \Phi_\theta) \end{aligned}$$

since $T B_{\Phi_\nu}(x) - B_{\Phi_\nu} \tau x \in \ker \Phi_\nu$. □

This estimate is similar to [15, Corollary 4.2] although Cerdà uses an adaptation of the Krugljak-Milman metric [29] (see Section 8).

8. Stability issues

We now enter into stability issues; namely, what occurs when, for a fixed couple (X_0, X_1) (or two fixed couples (X_0, X_1) and (Y_0, Y_1)) of spaces we have a family $(\Phi_\alpha)_{\alpha\in E}$ of interpolators or a family $(\Psi_\alpha, \Phi_\alpha)_{\alpha\in E}$ of pairs of interpolators indexed by the elements of a metric space E. In standard applications (complex or real interpolation, differential methods, orbits method) the space E is the base space behind the construction of the generalized Calderón space $\mathcal{H}$: a domain in the complex plane, an annulus in the complex plane, the space A_0+A_1 for a fixed interpolation couple (A_0, A_1),...

Stability issues refer to the possibility of transferring properties of Φ_t (resp. (Ψ_t, Φ_t)) to properties of Φ_s (resp. (Ψ_s, Φ_s)) when s is close to t. Our basic tool is the gap, described in Section 2. Given two interpolators on the same generalized Calderón space $\mathcal{H}$ let us define

$$g(\Phi, \Psi) = g(\ker \Phi, \ker \Psi).$$

This parameter is equivalent to the Krugljak-Milman metric [29] $\rho(\Phi, \Psi) = \sup_{\|f\|\leq 1} |\Phi(f) - \Psi(f)|$ An immediate application of Proposition 2.1 yields:

Proposition 8.1. *Let (Ψ, Φ) and (Ψ_1, Φ_1) be two pairs of interpolators on the same generalized Calderón space $\mathcal{H}$, such that $\mathcal{H} = \ker \Psi + \ker \Phi$. There exists $C > 0$ such that if $\max\{g(\ker \Psi, \ker \Psi_1), g(\ker \Phi, \ker \Phi_1)\} < C$ then $\mathcal{H} = \ker \Psi_1 + \ker \Phi_1$.*

We will also need a technical lemma:

Lemma 8.2. *Let (Ψ, Φ) be two interpolators on $\mathcal{H}$. Then*

$$g(\Psi(\ker \Phi), \Phi(\ker \Psi)) = g(\ker \Phi, \ker \Psi).$$

Proof. Here, as in Proposition 3.2, we identify $\Psi(\ker \Phi)$ and $\Phi(\ker \Psi)$ with the subspaces $\ker \Phi/(\ker \Psi \cap \ker \Phi)$ and $\ker \Psi/(\ker \Psi \cap \ker \Phi)$ of $X_{\Psi,\Phi} = \mathcal{H}/(\ker \Psi \cap \ker \Phi)$. Moreover, given $f \in \mathcal{H}$, we denote $\tilde{f} = f + (\ker \Psi \cap \ker \Phi) \in X_{\Psi,\Phi}$.

For $f \in \ker \Psi$, the equality follows from

$$\begin{aligned}
\operatorname{dist}\left(\tilde{f}, \Psi(\ker \Phi)\right) &= \inf\{\|\tilde{f} - \tilde{g}\| : \tilde{g} \in \Psi(\ker \Phi)\} \\
&= \inf\{\|f - g - h\| : g \in \ker \Phi, h \in \ker \Psi \cap \ker \Phi\} \\
&= \inf\{\|f - g\| : g \in \ker \Phi\} = \operatorname{dist}(f, \ker \Phi).
\end{aligned}$$

□

8.1. **Continuous families.** Let E be a metric space and let $(\Phi_d)_{d\in E}$ and $(\Psi_d)_{d\in E}$ be two families of interpolators on the same generalized Calderón space $\mathcal{H}$.

Definition 8.3. *The family $(\Phi_d)_{d\in E}$ will be called continuous if*

$$\lim_{t\to s} g(\Phi_t, \Phi_s) = 0.$$

A family of pairs $(\Psi_d, \Phi_d)_{d\in E}$ will be called bicontinuous *if $(\Phi_d)_{d\in E}$ is continuous and*

$$\lim_{t\to s} g\big(\ker \Psi_t \cap \ker \Phi_t, \ker \Psi_s \cap \ker \Phi_s\big) = 0.$$

Continuous interpolation methods yield stability results:

Proposition 8.4. *Let $(\Phi_t)_{t\in E}$ be a continuous family of interpolators on $\mathcal{H}$ and let $\mathcal{R}_{t,s} = \Phi_t B_{\Phi_s}$ be the translation map. Each $s \in E$ has a neighborhood V such that for each $t \in V$ the derivation $\mathcal{R}_{t,s}$ is trivial if and only if $\mathcal{R}_{s,t}$ is trivial.*

Proof. Recall that "$\mathcal{R}_{t,s}$ is trivial" means that the sequence $0 \to \Phi_t(\ker \Phi_s) \to X_{t,s} \to X_s \to 0$ splits, so that $X_{t,s} = \Phi_t(\ker \Phi_s) \oplus N$ for some closed subspace N of $X_{t,s}$ and thus $\gamma(\Phi_t(\ker \Phi_s), N) > 0$. Since the family of interpolators in continuous $\lim_{t\to s} g(\ker \Phi_t, \ker \Phi_s) = \lim_{t\to s} g(\Phi_t, \Phi_s) = 0$ and thus there is some neighborhood V of s so that

$$g(\Phi_t(\ker \Phi_s), \Phi_s(\ker \Phi_t)) = g(\ker \Phi_t, \ker \Phi_s) < \gamma(\Phi_t(\ker \Phi_s), N)$$

which means that also $\Phi_s(\ker \Phi_t)$ is complemented in $X_{t,s} = X_{s,t}$ and thus the sequence $0 \to \Phi_s(\ker \Phi_t) \to X_{s,t} \to X_t \to 0$ generated by $\mathcal{R}_{s,t}$ splits. □

Also, a standard using of Proposition 2.1 yields, under the same conditions as above:

Proposition 8.5. *Fix $s \in E$.*

(1) *If $(\Phi_t)_{t\in E}$ is a continuous family of interpolators on $\mathcal{H}$ then there is $\varepsilon > 0$ such that if $d(t, s) < \varepsilon$ and the sequence $0 \to \ker \Phi_s \to \mathcal{H} \to X_s \to 0$ splits then each short exact sequence $0 \to \ker \Phi_t \to \mathcal{H} \to X_t \to 0$ splits and X_t is isomorphic to X_s*

(2) *If $(\Psi_t, \Phi_t)_{t\in E}$ is bicontinuous then there is $\varepsilon > 0$ such that if $d(t, s) < \varepsilon$ and the sequence $0 \to \ker \Psi_s \cap \ker \Phi_s \to \mathcal{H} \to X_{\Psi_s,\Phi_s} \to 0$ splits then each exact sequence $0 \to \ker \Psi_t \cap \ker \Phi_t \to \mathcal{H} \to X_{\Psi_t,\Phi_t} \to 0$ splits and X_{Ψ_t,Φ_t} is isomorphic to X_{Ψ_s,Φ_s}.*

Examples of continuous and bicontinuous families of interpolators are provided by the differential methods of [20]. With the same notation as in Section 5.1 one has:

Proposition 8.6. *The family of pairs $(\Psi_t, \Phi_t)_{t\in \mathbb{A}}$ is bicontinuous.*

Proof. We use here the identification of $b = (b_n)$ with the function on $\mathbb{A}$ given by $f_b(z) = \sum z^n b_n$. To prove the first part, pick $f \in \ker \Phi_s$ and define the function $g(z) = f(z)/(z - s)$ when $z \neq s$ and $g(s) = f'(s)$. According to [20, Lemma 3.11] the function g, identified with its Laurent expansion $g(z) = \sum z^n g_n$ is also an element of $\mathfrak{J}(X, \overline{B})$ and with a bound $\|g\| \leq C\|f\|$ for a constant $C > 0$ independent on f. Pick the function $(z - t)g(z) \in \ker \Phi_t$ to obtain

$$\|f(z) - (z - t)g(z)\| = \|(z - s)g(z) - (z - t)g(z)\| = |s - t|\|g\| \leq |s - t|C\|f\|$$

Thus $g(\ker \Phi_t, \ker \Phi_s) \leq C|t - s|$, which shows that the family (Φ_t) is continuous.

For the second part, if $f \in \ker \Psi_s \cap \ker \Phi_s$ repeat the previous argument to get the function $h \in \mathfrak{J}(X, \overline{B})$ given by $h(z) = f(z)/(z-s)$ when $z \neq s$ and $h(s) = f'(s)$ with a bound $\|h\| \leq C\|f\|$; and then the function $g \in \mathfrak{J}(X, \overline{B})$ given by $g(z) = h(z)/(z - s)$ when $z \neq s$ and $g(s) = h'(s)$ with a bound $\|g\| \leq C\|h\|$. Form the function $(z - t)^2 g(z) \in \ker \Phi_t \cap \ker \Phi_s$ to obtain

$$\begin{aligned} \|f(z) - (z - t)^2 g(z)\| &= \|(z - s)^2 g(z) - (z - t)^2 g(z)\| \\ &= |(z - s)^2 - (z - t)^2|\|g\| \\ &\leq (|s|^2 - |t|^2 + 2z|t - s|)C^2\|f\| \end{aligned}$$

Thus $\lim_{t\to s} g(\ker\Psi_t \cap \ker\Phi_t, \ker\Psi_s \cap \ker\Phi_s) = 0$. □

The bicontinuity of the pair $(\delta_t', \delta_t)_t$ associated to the complex method in the unit strip was studied in [13], obtaining the estimate

$$g(\cap_{0\le j\le n-1} \ker\delta_t^j, \cap_{0\le j\le n-1} \ker\delta_s^j) \le 2(n+1)\mathsf{h}(t,s)$$

where $\mathsf{h}(\cdot)$ be the hyperbolic distance on the strip. Observe, moreover, that $g(\delta_t, \delta_t') \in \{0,1\}$ as it can be easily shown: it follows from Lemma 8.2 that $\delta_t'(\ker\delta_t) = X_t$ by standard compatibility, while $\delta_t(\ker\delta_t') = X_t$ only when $\mathcal{H} = \ker\delta_t + \ker\delta_t'$ according to Theorem 4.1. If this is the case (the induced sequence splits) then $g(\delta_t, \delta_t') = 0$. Otherwise, $\delta_t(\ker\delta_t')$ is a proper subspace of $\delta_t'(\ker\delta_t)$.

9. Two remarks on problems derived from the research in this paper

We have attempted to present the bare bones of an interpolation method and its associated differential process. In doing so, the reader might be surprised by our "non-functorial" approach to interpolators. A categorical analysis of the material presented in this paper is perhaps possible. A second aspect is whether every interpolation method, or every interpolation method obtained via some couple $(\mathcal{H}, \Phi)$, admits an associated differential process. This question already appears implicitly formulated in [7, 6.3] and explicitly in [6] in the form: Does Calderón's "upper" method produce twisted sums? The same can be asked for the Orbits method (see the Krugljak - Milman approach via Benson spaces in [29]). This issue connects with the categorical aspects of the problem: Of course that given a method $(\mathcal{H}, \Phi)$ nothing prevents one to consider the couple of interpolators $(0, \Phi)$, but this is not what is expected. What one wants instead is an "intrinsic" way to produce Ψ out of Φ; and here it is where the categorical approach should pay off.

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