On the classification of positions and complex structures in Banach spaces

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A R T I C L E   I N F O

A B S T R A C T

A topological setting is defined to study the complexities of the relation of equivalence of embeddings (or "position") of a Banach space into another and of the relation of isomorphism of complex structures on a real Banach space. The following results are obtained: a) if X is not uniformly finitely extensible, then there exists a space Y for which the relation of position of Y inside X reduces the relation E_0 and therefore is not smooth; b) the relation of position of ℓ_p inside ℓ_p, or inside L_p, $p \neq 2$, reduces the relation E_1 and therefore is not reducible to an orbit relation induced by the action of a Polish group; c) the relation of position of a space inside another can attain the maximum complexity $E_{\max}$; d) there exists a subspace of L_p, $1 \leq p < 2$, on which isomorphism between complex structures reduces E_1

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the action of a Polish group.

1. Introduction

In this paper we are interested in defining a topological setting to compute the complexity of certain natural equivalence relations appearing in the theory of positions and/or complex structures. Our objective is to provide examples towards the idea that these relations are not "well classifiable", or in other words, to obtain high lower bounds of complexity for natural instance of these relations. Our starting points are the previous results in which a continuum of equivalence classes were already obtained, without information on the complexity of the associated equivalence relation: examples of spaces with a continuum of mutually non-isomorphic complex structures [1], or examples of classical spaces with continuum many different positions inside another, see [6] and [18].

In this introduction we recall some basics of the theories of positions of Banach spaces, of complex structures, as well as of classification of analytic equivalence relations on Polish spaces. In section 2, after defining the appropriate topological setting, we obtain lower bounds for the complexity of position of a space inside another, in different cases. We prove that if X is not uniformly finitely extensible, then there exists a space Y for which the relation of position of Y inside X reduces the relation E_0 and therefore is not smooth (Theorem 2.7). Through a result about complexity of positions inside ℓ_p-sums of non-uniformly extensible spaces (Proposition 2.10), we extend this and prove that the relation of position of ℓ_p inside ℓ_p, or inside L_p, $p \neq 2$, reduces the relation E_1 and therefore is not reducible to an orbit relation induced by the action of a Polish group, Theorem 2.12. Then through the study of complemented positions we use the main result of [10] to show that the complexity of positions may be $E_{\max}$, the maximum complexity of analytic equivalence relations, Proposition 2.15. We end the section by providing the appropriate topological setting to study complex structures. In section 3, we describe an example to prove that there exists a subspace of L_p, $1 \leq p < 2$, on which isomorphism between complex structures reduces E_1 and therefore is not reducible to an orbit relation induced by the action of a Polish group.

1.1. Positions of Banach spaces

The notion of relative positions of Banach spaces arose in [5] where the definition of *automorphic* space was first introduced in connection with a classical result of Lindenstrauss and Rosenthal [15]: *c_0 has the property that every isomorphism between two of its infinite codimensional subspaces can be extended to an automorphism of the whole space.*

A separable space with such a property is said to be automorphic, or in other words, all its subspaces are in the same "position". The following problem remains open.

Question 1.1. *Are c_0 and ℓ_2 the only separable Banach spaces with that property?*

The papers [5,18,6,4] were devoted to the study of different aspects of the automorphic problem. In [18,6] in particular, it is provided a general theory of positions for subspaces of a Banach space, by defining equivalent embeddings. Namely, given two infinite codimensional embeddings $T, U : Y \to X$ between separable Banach spaces, we let $\sim$ be the equivalence relation: $T \sim U$ if and only if there exists an automorphism A of X such that $T = AU$. A *position* of Y in X is an $\sim$-equivalence class on the set of infinite codimensional embeddings from Y to X.

The notion of automorphy index $a(Y, X)$ was introduced in [18] and it measures how many different positions a space Y admits in another Banach space X. The automorphy index of X is defined as $a(X) = \sup_Y a(Y, X)$ and, of course, a Banach space is said to be automorphic if $a(X) = 1$. In [6] it is estimated the automorphy indices $a(Y, X)$ for classical Banach spaces. The authors obtain, among other results: $a(c_0, X) = \{0, 1, 2, \aleph_0\}$ for every separable Banach space X; $a(Y, \ell_p) = c$ for all subspaces of ℓ_p, $p \neq 2$, and $a(Y, L_p) = c$ for all subspaces of L_p, $p > 2$ not isomorphic to ℓ_2; while $a(\ell_2, L_p) = 1$; for $1 < p < 2$ one has $a(Y, L_p) = c$ for all nonstrongly embedded subspaces of L_p; $a(Y, L_1) = c$ for all nonreflexive subspaces of L_1, while $a(\ell_2, L_1) = c$; $a(Y, C[0, 1]) \in \{1, c\}$ for every separable Banach space Y.

So once we have defined a topological setting for embeddings of a space Y into another space X and for the relation of being in the same position, we shall prove that the complexity of this relation is high for some of the above examples. Here, this can be interpreted as measuring the difficulty, once two embeddings T, U of Y into X are given, of determining whether there exists an automorphism A proving that these embeddings correspond to the same position.

We shall need the notion of *uniformly finitely extensible* space considered in [6]. A space is uniformly finitely extensible (or UFO) if there exists $\lambda \geq 1$ such that for every finite dimensional subspace $E \subset X$, each linear operator $t : E \to X$ may be extended to a linear operator $T : X \to X$ with $\|T\| \leq \lambda \|t\|$. In [4] it was proved that the UFO property is equivalent to being *compactly extensible*, meaning that every compact operator from a subspace Y of X into X may be extended to the whole space. Note that $\mathcal{L}_\infty$-spaces satisfy this property.

According to [6] every automorphic space is UFO, and conversely, any UFO space is either an $\mathcal{L}_\infty$-space or a weak type 2 near-Hilbert space with the Maurey projection property. It remains open whether the UFO property is equivalent to being either $\mathcal{L}_\infty$-space or Hilbert.

1.2. Complex structures

A second theory that we shall revisit from the point of view of "definable" equivalence relations is the one of *complex structures* on real Banach spaces. A real Banach space

$(X, \|\cdot\|)$ admits a *complex structure* if there exists a multiplication of the elements of X by complex scalars which is compatible with the norm:

$$\|\lambda x\| = |\lambda| \|x\|, \qquad \forall x \in X, \qquad \forall \lambda \in \mathbb{C},$$

or compatible with an equivalent norm to $\|\cdot\|$.

The complex structures on a real Banach space $(X, \|\cdot\|)$ correspond (in a one-to-one manner) to the $\mathbb{R}$-linear isomorphisms T on X satisfying $T^2 = -Id$: if there is a complex structure we can take $Tx = ix$; conversely, we can define $(a + ib)x = ax + bTx$ which is compatible with the norm

$$\||x\|| = \sup_{\theta \in [0, 2\pi]} \|(\cos \theta)x + (\sin \theta)Tx\|.$$

The isomorphic theory of complex structure addresses questions of existence, uniqueness, or the possible structure of the set of complex structures up to isomorphism.

By employing probabilistic methods, S. Szarek constructed in [21] the first example of an infinite dimensional real Banach space which does not admit a complex structure. Using similar methods, J. Bourgain [3] exhibited an example of an infinite dimensional complex Banach space X not isomorphic to its *complex conjugate* $\overline{X}$: $\overline{X}$ has the same elements and norm as X, the same addition of vectors, while the multiplication by scalars is given by $\lambda \odot x = \overline{\lambda}x$, for $\lambda \in \mathbf{C}$, $x \in X$. Since it is clear that X and $\overline{X}$ are identical as real Banach spaces, Bourgain's construction provides an example of a real Banach space with at least two non-isomorphic complex structures.

The work of V. Ferenczi [9] shows that it is possible to construct, for all positive integers $n \geq 1$, explicit examples of infinite dimensional real Banach spaces which admit precisely n complex structures, up to isomorphism. W. Cuellar Carrera [7] gave an example of a separable real Banach space with exactly infinite countably many complex structures, up to isomorphism, while R. Anisca [1] constructed subspaces of L_p, for $1 \leq p < 2$, with a continuum of complex structures, up to isomorphism.

1.3. Theory of complexity of equivalence relations

We recall the theory of classification of analytic equivalence relations on Polish spaces by Borel reducibility. This area of research originated from the works of H. Friedman and L. Stanley [11] and independently from the works of L.A. Harrington, A.S. Kechris and A. Louveau [12]. It may be thought of as an extension of the notion of cardinality in terms of complexity, when one counts equivalence classes.

A topological space is Polish if it is separable and its topology may be generated by a complete metric. Its Borel subsets are those belonging to the smallest σ-algebra containing the open sets. An analytic subset is the continuous image of a Polish space, or equivalently, of a Borel subset of a Polish space. If R (respectively S) is an equivalence relation on a Polish space E (respectively F), then it is said that (E, R) is *Borel reducible*

to (F, S), $(E, R) \leq_B (F, S)$, if there exists a Borel map $f : E \to F$ such that $\forall x, y \in E$, $xRy \Leftrightarrow f(x)Ff(y)$. They are *Borel bireducible*, $(E, R) \sim_B (F, S)$, if both $(E, R) \leq_B (F, S)$ and $(F, S) \leq_B (E, R)$ hold. The aim is then to compare analytic equivalence relations modulo $\sim_B$.

One may note that such a map $f : E \to F$ induces an injection of E/R into F/S and therefore there are at least as many S-classes in F as R-classes in E when $(E, R) \leq_B (F, S)$. However the requirement that f is Borel will induce much finer topological regularities; actually the theory of $\leq_B$-classification is really interesting when both relations have 2^ω classes, and there is a huge variety of such relations which are not bireducible to each other.

We now list a few important equivalence relations on the $\leq_B$-scale. After the relations with finitely or countably many classes, the simplest Borel equivalence relation is $(\mathbb{R}, =)$, equality between real numbers. Actually by a result of Silver [20], any Borel equivalence relation admits at most countably many classes, or there is a Borel reduction of $(\mathbb{R}, =)$ to it. The analytic equivalence relations which are Borel reducible to $(\mathbb{R}, =)$ are called *smooth*; these are the relations which admit the real numbers as complete invariants.

An important equivalence relation is the relation E_0 of eventual agreement between sequences of 0 and 1's: on 2^ω,

$$\alpha E_0 \beta \Leftrightarrow \exists m \in \mathbb{N} : \forall n \geq m, \alpha(n) = \beta(n).$$

The relation E_0 is a Borel equivalence relation with continuum many classes and which, furthermore, is non-smooth. So $(\mathbb{R}, =) <_B E_0$. In fact E_0 is the $\leq_B$ minimum non-smooth Borel equivalence relation [12]. Therefore, the most natural criterium to prove that an analytic relation is non-smooth is to reduce E_0 to it.

Quite natural are the orbit equivalence relations induced by the continuous action of a Polish group H on a Polish space X: the relation E_H is defined on X by

$$xE_H y \Leftrightarrow \exists h \in H : y = hx,$$

and is easily seen to be analytic. The relation E_0 is one of them. For any Polish group H, it is possible to prove that there is a relation which is maximum among all orbit relations induced by actions of H. There is also a maximum E_G for orbit relations associated to the action of Polish groups; in particular $E_0 <_B E_G$.

In 1997, Kechris and Louveau [14] discover that there are analytic equivalence relations which are not reducible to any orbit equivalence relation, or in other words, to E_G. There is actually a minimal equivalence relation, called E_1, among those which are not Borel reducible to an orbit equivalence relation. It is defined as the eventual agreement between sequences of real numbers: for $x, y \in \mathbb{R}^{\mathbb{N}}$,

$$xE_1 y \Leftrightarrow \exists n \forall m \geq n \; x_m = y_m.$$

The relation E_1 is, up to now, the only known obstruction to reducibility to an orbit equivalence relation: $E_1 \not\leq_B E_G$.

Finally, the complete analytic equivalence relation $E_{\max}$ is the most complex of all analytic equivalence relations, and is strictly above E_1 and E_G. It may be defined formally as the $\leq_B$-maximum equivalence relation, and the proof of its existence uses certain universality properties of analytic sets. There also exist explicit realizations of $E_{\max}$, the most important in our setting being the relation of linear isomorphism between separable Banach spaces [10].

2. Classification of subspaces, positions and complex structures

In what follows, the notation $\simeq$ will be used for equivalence relations associated to linear isomorphism of Banach spaces, while $\sim$ will be used for relations of equivalence of positions of a space inside another. The letters T, U will usually stand for embeddings, A, B for automorphisms, P, Q for projections, J, K for complex structures.

Since the definition of the order $\leq_B$ relies on the Borel property of the function realizing the reduction from a relation to another, the topologies of the Polish spaces considered only play a role through the Borel sets they generate. In the following we shall therefore prefer to talk about standard Borel spaces than Polish spaces: a *standard Borel space* is a set equipped with a σ-algebra which is the σ-algebra of Borel sets induced by some Polish topology on the set.

Let X be a separable infinite dimensional Banach space. There is a natural way to equip the set of infinite dimensional subspaces of X with a Borel structure (see, e.g., [13]), and the relation we are more interested in, of linear isomorphism, is analytic in this setting [2]. If we choose X to be a universal space such as $C(2^\omega)$, then we obtain a description of the *standard Borel space SB* of all separable Banach spaces. What is proved in [10] is that linear isomorphism $\simeq$ between elements of SB is an analytic relation which is Borel bireducible to $E_{\max}$, or in other terms, has the maximum complexity among all analytic equivalence relations on Polish spaces. We may also restrict the relation $\simeq$ to $SB(X)$, the standard Borel space of infinite dimensional subspaces of X.

In [10] some other relations are proved to have maximum complexity $E_{\max}$: Lipschitz isomorphism or (complemented) biembeddability on SB, uniform homeomorphism of complete separable metric spaces, for example. According to [16], isometric biembeddability also has complexity $E_{\max}$. On the other hand, linear isometry on SB [17] and homeomorphism of compact metric spaces [22], for example, are of complexity E_G.

Given Y, X separable Banach spaces, we shall also need to study analytic equivalence relations on $\mathcal{B}(Y, X)$, the set of bounded linear operators from Y into X, or on some of its subsets. To do this we note that endowed with the strong operator topology, the space $\mathcal{B}(Y, X)_{\leq 1}$ of linear operators with norm less than or equal to 1 is Polish, while $\mathcal{B}(Y, X)$ is a standard Borel space with respect to the Borel structure generated by the strong operator topology (as a countable union of standard Borel spaces). This result may be found in [13] p. 80.

It will be useful to note that since multiplication of operators is continuous in the strong operator topology when restricted to $B \times \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$, where B is a norm bounded subset of $\mathcal{B}(X)$, the multiplication of operators $\mathcal{B}(X) \times \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$ is Borel.

2.1. Complemented subspaces

In this part we aim to define a Borel standard space of *complemented* infinite dimensional subspaces of a given separable Banach space X. This is done as follows: since the multiplication of operators $\mathcal{B}(X) \times \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$ is Borel, the set of projections on X is a Borel subset of $\mathcal{B}(X)$. Combined with the easy fact that the set of compact operators on X is Borel as well, we deduce that the set of projections of infinite range is Borel in $\mathcal{B}(X)$ for the SOT topology.

Definition 2.1. Let X be a separable infinite dimensional Banach space. We denote by $\mathcal{P}(X)$ the Borel standard space of projections of infinite range in $\mathcal{B}(X)$, equipped with the SOT topology.

Definition 2.2. Let X be a separable infinite dimensional Banach space. The relation $\simeq$ defined on $\mathcal{P}(X)$ by

$$P \simeq Q \Leftrightarrow PX \simeq QX$$

is called the *relation of linear isomorphism between complemented subspaces of X*.

We may relate $(\mathcal{P}(X), \simeq)$ to $(SB(X), \simeq)$ as follows:

Proposition 2.3. *Let X be a separable infinite dimensional Banach space. Then the map from $\mathcal{P}(X)$ into $SB(X)$ defined by $P \mapsto PX$ is Borel. In particular the relation $\simeq$ is analytic on $\mathcal{P}(X)$ and*

$$(\mathcal{P}(X), \simeq) \leq_B (SB(X), \simeq).$$

Proof. Let $\mathcal{O}_U$ be a typical Borel set generating the Effros Borel structure of $SB(X)$, i.e. $\mathcal{O}_U = \{Y \subset X : Y \cap U \neq \emptyset\}$, where $U \neq \emptyset$ is open. Then given $(x_n)_n$ a dense family in X, we note that

$$PX \in \mathcal{O}_U \Leftrightarrow PX \cap U \neq \emptyset \Leftrightarrow \exists n \in \mathbb{N} \, Px_n \in U.$$

This last condition is Borel in $\mathcal{P}(X)$. $\quad \square$

This means that the complexity of isomorphism between complemented subspaces of X will be $\leq_B$ below the complexity of isomorphism between subspaces of X, via the set $\{PX, P \in \mathcal{P}(X)\}$. The result of [10] about maximum complexity of isomorphism between subspaces may be extended to complemented subspaces as follows:

Proposition 2.4. *The complexity of linear isomorphism between complemented subspaces of U is $E_{\max}$, where U is Pełczyński universal unconditional space.*

Proof. It is proved in [10] that $E_{\max}$ is Borel reducible to isomorphism between subspaces generated by subsequences of the unconditional basis of a certain space, which therefore we may assume to be U [19]. Noting that every subsequence of the basis $(u_n)_{n \in A}$ of U is complemented by the natural projection P_A, it is enough to prove that the map

$$\mathbb{N}^{\mathbb{N}} \mapsto \mathcal{P}(U)$$

taking an infinite set A to P_A is Borel. This is clear since

$$P_A \in \mathcal{O}_{P,x_1,\ldots,x_n,\epsilon} \Leftrightarrow \forall i = 1, \ldots, n, \; \|Px_i - P_A x_i\| < \epsilon$$

which is an open condition in $\mathbb{N}^{\mathbb{N}}$. $\square$

2.2. Complexity of positions

Given infinite dimensional separable Banach spaces Y, X, we shall use the notation $\mathrm{Emb}(Y, X)$ for the set of linear operators which are infinite codimensional embeddings of Y into X (i.e. onto infinite codimensional subspaces of X). We also denote by $GL(X)$ the group of automorphisms on X. We let $\sim$ be the equivalence relation on $\mathrm{Emb}(Y, X)$ defined by

$$T \sim U \Leftrightarrow \exists A \in GL(X) : T = AU.$$

Definition 2.5. Let X, Y be infinite dimensional and separable. A *position* of Y in X is an $\sim$-equivalence class on $\mathrm{Emb}(Y, X)$.

By the *complexity* of the positions of Y in X, we mean the complexity of the equivalence relation $\sim$ on $\mathrm{Emb}(Y, X)$ along the $\leq_B$-scale. For this to make sense, we just need to note the following:

Proposition 2.6. *The space $\mathrm{Emb}(Y, X)$ is a Borel standard space and $\sim$ is an analytic relation on it.*

Proof. It is an easy exercise to check that the space $\mathrm{Emb}(Y, X)$ is a Borel subset of $\mathcal{B}(Y, X)$ (recalling that these sets are equipped with the SOT), and therefore a Borel standard space in its own right. Fix $(y_n)_n$ and $(x_n)_n$ dense in Y and X respectively. We let $B \subset X^\omega \times \mathrm{Emb}(Y, X)^2$ be defined by

$((z_n)_n, T, U) \in B \Leftrightarrow$

the map $x_n \mapsto z_n$ extends to an isomorphism A on X satisfying $T = AU$.

We claim that B is Borel, and therefore $\sim$ is analytic by

$$T \sim U \Leftrightarrow \exists (z_n)_n \in X^\omega \ ((z_n)_n, T, U) \in B.$$

That B is Borel follows from

$$
\begin{aligned}
((z_n)_n, T, U) \in B \Leftrightarrow \ & \forall n \in \mathbb{N}, \forall \epsilon \in \mathbb{Q}^{+*}, \exists m \in \mathbb{N} \ \|x_n - z_m\| \leq \epsilon \\
& \wedge \ \exists K \in \mathbb{N} \big(\forall (\lambda_i)_i \in c_{00}(\mathbb{Q}), K^{-1} \| \sum_i \lambda_i x_i \| \leq \| \sum_i \lambda_i z_i \| \\
& \leq K \| \sum_i \lambda_i x_i \| \\
& \wedge \ \forall m, n \in \mathbb{N}, \forall \epsilon \in \mathbb{Q}^{+*}, \|U y_m - x_n\| \leq \epsilon \Rightarrow \|T y_m - z_n\| \leq K\epsilon \big). \quad \square
\end{aligned}
$$

We now turn to the notion of uniformly finitely extensible (or UFO) space recalled in the introduction. Since every automorphic space has this property, non-UFO spaces admit subspaces in at least two positions. We shall now extend this to prove that the relation of position is not even smooth in these instances. Recall that UFO spaces are either $\mathcal{L}_\infty$-spaces or near Hilbert, meaning that non-UFO spaces include most of the classical spaces.

Given $n \in \mathbb{N}$, we write for $\alpha, \beta \in 2^\omega$, $\alpha E_0^n \beta$ to mean that $\alpha_i = \beta_i$ for all $i \geq n$. We also define for two embeddings of Y into X, $U \sim_n V$ to mean that there exists an automorphism T of X with $TU = V$, such that $\max\{\|T\|, \|T^{-1}\|\} \leq n$.

Theorem 2.7. *If X is a separable, infinite dimensional, non-uniformly finitely extensible space, then there is some subspace Y of X such that the relation E_0 is Borel reducible to $(\mathrm{Emb}(Y, X), \sim)$. In particular the positions of Y in X are not smooth.*

Proof. Since X is not UFO there exists (see [18]) a subspace $Y \subset X$ admitting a finite dimensional decomposition $Y = \sum Y_n$ and a sequence of norm-one operators $T_n : Y_n \to X$ such that every extension of T_n to X has norm not less than 2^{2n}. Let $\alpha \in 2^\omega$, we define an operator $T_\alpha : Y \to X$, $T_\alpha(\sum y_n) = \sum_n 2^{-n} T_n^{\alpha(n)} y_n$, where

$$
T^{\alpha(n)}(y_n) = \begin{cases} T_n y_n & \text{if } \alpha(n) = 1 \\ y_n & \text{if } \alpha(n) = 0. \end{cases}
$$

The operator T_α is obviously compact, $\|T_\alpha\| \leq 1$ and does not admit any extension to an operator $X \to X$. Take now $0 < \varepsilon < 1$ and consider the $(1+\varepsilon)$-isometry $A_\alpha : Y \to X$, $A_\alpha = Id + \varepsilon T_\alpha$; this map does not admit any extension to X as neither T_α does it. Let $Y_\alpha = A_\alpha(Y)$, we define the following map

$$(2^\omega, E_0) \xrightarrow{\ f\ } (\mathrm{Emb}(Y, X), \simeq), \ \ f(\alpha) = A_\alpha.$$

The map f is well defined, uniformly bounded, and Borel (actually continuous).

Let us see that $\alpha E_0 \beta$ if and only if Y_α and Y_β are in the same position. Assume first that $\alpha E_0 \beta$ and let m be such that $\alpha E_0^m \beta$, then it follows that $T_{\alpha|_{\sum_{n \geq m} Y_n}} = T_{\beta|_{\sum_{n \geq m} Y_n}}$, and we can write

$$Y_\alpha = A_\alpha \Big(\sum_{n < m} Y_n \Big) \oplus B \text{ and } Y_\beta = A_\beta \Big(\sum_{n < m} Y_n \Big) \oplus B,$$

where $B = A_\alpha(\sum_{n \geq m} Y_n) = A_\beta(\sum_{n \geq m} Y_n)$. The (canonical) projections $P_\alpha : Y_\alpha \to Y_\alpha$ and $P_\beta : Y_\beta \to Y_\beta$, with ranges $P_\alpha(Y_\alpha) = A_\alpha(\sum_{n < m} Y_n)$ and $P_\beta(Y_\beta) = A_\beta(\sum_{n < m} Y_n)$ of finite dimension, admit extensions $\widehat{P_\alpha} : X \to A_\alpha(\sum_{n < m} Y_n)$ and $\widehat{P_\beta} : X \to A_\beta(\sum_{n < m} Y_n)$, respectively, which are also projections. Let us write $X_\alpha = (1_X - \widehat{P_\alpha})(X)$ and $X_\beta = (1_X - \widehat{P_\beta})(X)$, hence $B \subset X_\alpha \cap X_\beta$, and since $X = A_\alpha(\sum_{n < m} Y_n) \oplus X_\alpha = A_\beta(\sum_{n < m} Y_n) \oplus X_\beta$ we can easily define an automorphism τ of X such that $\tau A_\alpha = A_\beta$ since the finite dimensional pieces have the same dimension and so X_α and X_β are isomorphic (as all hyperplanes in a Banach space are). Let us note for future use that, by the well-known fact that all subspaces of codimension k in a Banach space are $c(k)$-isomorphic, for some $c(k)$, we may deduce that X_α and X_β are c_m-isomorphic, where c_m only depends on m. So we can actually control the norms of τ and τ^{-1} by some constant c'_m depending only on m, i.e. $Y_\alpha \sim_{c'_m} Y_\beta$ once $\alpha E_0^m \beta$.

On the other direction, we shall prove that if α and β are not E_0^m-related and $n \geq m$ is such that $\alpha_n = 0$ and $\beta_n = 1$, then any map $\tau : X \to X$ such that $\tau A_\alpha = A_\beta$ has norm at least $\frac{1}{2}(\varepsilon 2^m - 1)$. This implies that if α and β are not E_0-related (and without loss of generality, $\alpha_n = 0$ and $\beta_n = 1$ for infinitely many n) then there is no automorphism $\tau : X \to X$ such that $\tau A_\alpha = A_\beta$.

So let $n \geq m$ be such that $\alpha_n = 0$ and $\beta_n = 1$, and $\tau : X \to X$ be such that $\tau A_\alpha = A_\beta$. In particular, $\tau_{|_{Y_\alpha}} = A_\beta A_\alpha^{-1}$, which means that $\tau : X \to X$ extends $A_\alpha^\beta = A_\beta A_\alpha^{-1} : Y_\alpha \to Y_\beta$. Since

$$\tau(y) = A_\beta(y) - \varepsilon \tau T_\alpha(y),$$

take $y = y_n \in Y_n$.

$$\tau(y_n) = A_\beta(y_n) - \varepsilon \tau T_\alpha(y_n)$$
$$= y_n + \varepsilon 2^{-n} T_n(y_n) - \varepsilon 2^{-n} \tau(y_n)$$

whence

$$(1 + \varepsilon 2^{-n}) \tau(y_n) - y_n = \varepsilon 2^{-n} T_n(y_n)$$

then

$$\frac{2^n}{\varepsilon} \Big((1 + \varepsilon 2^{-n}) \tau(y_n) - y_n \Big) = T_n(y_n),$$

so

$$\frac{2^n}{\varepsilon} \left((1 + \varepsilon 2^{-n})\tau - id \right)$$

extends T_n. Therefore

$$2^{2n} \leq \|\frac{2^n}{\varepsilon} \left((1 + \varepsilon 2^{-n})\tau - id \right)\| \leq \frac{2^n}{\varepsilon}(2\|\tau\| + 1)$$

and $\|\tau\| \geq \frac{1}{2}(\varepsilon 2^n - 1) \geq \frac{1}{2}(\varepsilon 2^m - 1)$.   □

Beyond the automorphic space problem, it would be interesting to investigate for which X the relation of position of Y in X is smooth, for all choices of Y. The above shows that X would have to be uniformly finitely extensible, leading to the question:

Question 2.8. *Find a non-automorphic, uniformly finitely extensible space X, such that the position of Y inside X is smooth for all subspaces Y of X.*

We recall that it is an open conjecture (see [4] and [6]) whether the UFO property is equivalent to being either $\mathcal{L}_\infty$ or isomorphic to the Hilbert space. Among $\mathcal{L}_\infty$ (and therefore UFO) spaces which are not automorphic, it remains fairly open when E_0 can be reduced to positions of subspaces. For example, remembering that in [6] it is shown that $a(Y, C(0,1)) = c$ for many choices of Y (such as $Y = \ell_p$, $p \neq 1$, or $Y = C(0,1)$ itself):

Question 2.9. *Find a space Y such that E_0 is reducible to the positions of Y inside $C(0,1)$.*

Note that the relation of equivalence of positions of Y inside X is the orbit relation of the action $(A, T) \mapsto AT$ of the group $GL(X)$ on the standard Borel space $\mathrm{Emb}(Y, X)$. So although $GL(X)$ is not a Polish group, one may be led to look for uniformity arguments to prove that equivalence of positions of space Y inside X is reducible to an orbit relation induced by action of some Polish group, and in particular is not maximum among analytic equivalence relations. One of our main results, however, is that this is not so. To prove this we turn to reductions of the relation E_1 in the case of the classical spaces ℓ_p and L_p.

Proposition 2.10. *Let $1 \leq p < \infty$. Let Y, X be separable. Assume there is a Borel reduction r of $(2^\omega, E_0)$ to $(\mathrm{Emb}(Y, X), \sim)$ with the following properties:*

(a) $r(\alpha)$ is bounded uniformly on $\alpha \in 2^\omega$;
(b) there exists a sequence $(c_k)_k$ of integers such that

$$\alpha E_0^k \beta \Rightarrow r(\alpha) \sim_{c_k} r(\beta);$$

(c) *there exists a sequence $(d_k)_k$ of integers tending to infinity such that if α and β are not E_0^k-related and we assume $\alpha(i) = 0$ and $\beta(i) = 1$ for some $i \geq k$, then there is no map T on X of norm less than d_k such that $Tr(\alpha) = r(\beta)$.*

Then the relation E_1 is Borel reducible to $(\mathrm{Emb}(\ell_p(Y), \ell_p(X)), \sim)$ and to

$$(\mathrm{Emb}(c_0(Y), c_0(X)), \sim).$$

Proof. To each $\alpha = (\alpha_n)_n \in (2^\omega)^\omega$, where for each $n \in \omega$, $\alpha_n = (\alpha_n(k))_{k \in \omega}$, associate

$$R(\alpha) = (r((\alpha_n(1))_{n \in \omega}), r((\alpha_n(2))_{n \in \omega}), \ldots, r((\alpha_n(k))_{n \in \omega}), \ldots).$$

Because r is bounded by (a), this defines an embedding of $\ell_p(Y)$ into $\ell_p(X)$ (of $c_0(Y)$ into $c_0(X)$), and R is Borel. We denote by Y_k and X_k the k-th copies of Y and X respectively.

Note that if $\alpha E_1 \beta$ then there is $m \in \mathbb{N}$ such that for $k \geq m$, $\alpha_k = \beta_k$, which implies that for each k, $\alpha_n(k)_n$ and $\beta_n(k)_n$ are E_0-related and actually only differ by at most the first m-terms. Then by the property (b), we may paste the maps $T_k : X_k \to X_k$ for which $T_k r(\alpha_n(k)_n) = r(\beta_n(k)_n)$ and $\max\{\|T_k\|, \|T_k^{-1}\|\} \leq c_m$, to define a global automorphism T witnessing that $R(\alpha)$ and $R(\beta)$ are in the same position.

On the other hand assume α and β are not E_1-related but that there is an automorphism T of $\ell_p(X)$ such that $R(\alpha) = TR(\beta)$; let K be an infinite subset of $\mathbb{N}$ such that for all $k \in K$, $\alpha_k \neq \beta_k$ and for each $k \in K$ let i_k be such that $\alpha_k(i_k) \neq \beta_k(i_k)$. Without loss of generality assume $\alpha_k(i_k) = 0$ and $\beta_k(i_k) = 1$ with K still infinite. Since $r(\alpha_n(i_k)_n)$ (resp. $r(\beta_n(i_k)_n)$) is an embedding of Y_{i_k} into X_{i_k}, we have

$$r(\alpha_n(i_k)_n) = P_k T_k r(\beta_n(i_k)_n),$$

where $T_k : X_{i_k} \to \ell_p(X)$ (or $T_k : X_{i_k} \to c_0(X)$) is the restriction of T to X_{i_k} and P_k the canonical projection onto X_{i_k}. Note that $(\alpha_n(i_k))_n$ and $(\beta_n(i_k))_n$ are not E_0^k-related, since $\alpha_k(i_k) = 0 \neq 1 = \beta_k(i_k)$. Therefore by (c) the map $P_k T_k$ has norm at least d_k. And therefore $\|T\| \geq d_k$ for all $k \in K$, which is a contradiction. $\quad\square$

We finally prove that E_1 is reducible to positions of Y inside X for classical spaces such as the ℓ_p's, and therefore this relation is not reducible to the orbit relation induced by the action of a Polish group.

Corollary 2.11. *Let X be a separable non-UFO space. There exists a subspace $Y \subset X$ such that E_1 is Borel reducible to $(\mathrm{Emb}(\ell_p(Y), \ell_p(X)), \sim)$ and to $(\mathrm{Emb}(c_0(Y), c_0(X)), \sim)$.*

Proof. Simply notice that the Borel map $f : (2^\omega, E_0) \to (\mathrm{Emb}(Y, X), \sim)$ in Proposition 2.7 verifies the conditions in Proposition 2.10. $\quad\square$

Theorem 2.12. *For $1 \leq p < \infty$, $p \neq 2$, the relation E_1 is Borel reducible to* $(\mathrm{Emb}(\ell_p, \ell_p), \sim)$ *and to* $(\mathrm{Emb}(\ell_p, L_p), \sim)$. *Therefore the relation of position of ℓ_p in ℓ_p (resp. in L_p) is not Borel reducible to an orbit relation induced by the action of a Polish group.*

Proof. By Proposition 3.15 in [6], there exists a subspace $Y \subset \ell_p$, admitting a FDD, which is isomorphic to ℓ_p and for which there is a Borel reduction $(2^\omega, E_0) \to$ $(\mathrm{Emb}(Y, \ell_p), \sim)$ which also verifies conditions in Proposition 2.10. The same arguments based on [6] Proposition 3.15 work for $X = L_p$. $\square$

2.3. Complexity of complemented positions

A classical method to compute equivalent positions is to look at embeddings as complemented subspaces and compare the summands. Complemented positions will also be more easily related to the relation of isomorphism of complex structures.

This motivates to define, for X, Y be infinite dimensional separable Banach spaces, the *standard Borel space of complemented embeddings of Y into X* as the Borel subspace $\mathrm{Emb}_c(Y, X)$ of $\mathrm{Emb}(Y, X) \times \mathcal{P}(X)$ given by

$$(T, P) \in \mathrm{Emb}_c(Y, X) \Leftrightarrow TY = PX.$$

We let $\sim$ be the analytic equivalence relation defined on $\mathrm{Emb}_c(Y, X)$ by

$$(T, P) \sim (U, Q) \Leftrightarrow T \sim U,$$

and we call the complexity of this relation the *complexity* of complemented positions of Y in X.

Definition 2.13. Let X, Y be infinite dimensional and separable. A *complemented position* of Y in X is an $\sim$-equivalence class on $\mathrm{Emb}_c(Y, X)$.

We note that, as is to be expected:

Proposition 2.14. *The map $\mathrm{Emb}_c(Y, X) \to \mathrm{Emb}(Y, X)$ defined by $(T, P) \mapsto T$ is Borel. In particular, the complexity of complemented positions of Y in X is a lower bound of the complexity of positions of Y in X.*

We shall now use Proposition 2.4 to show that the highest complexity $E_{\max}$ among analytic equivalence relations, can be achieved for the complexity of (complemented) positions of a space inside another. So there will be no upper bound other than $E_{\max}$ for the complexity of positions of a space inside another.

Proposition 2.15. *If U is Pełczyński universal unconditional basis, then the complexity of the relation of (complemented) positions of U in itself is $E_{\max}$.*

Proof. By Proposition 2.4 there is a Borel reduction r of $E_{\max}$ to isomorphism between subspaces of U generated by subsequences of the basis $(u_n)_n$ (identified with elements A of $\mathbb{N}^\mathbb{N}$). Since $U \simeq \ell_2(U) \simeq U \oplus \ell_2(U)$ we may use as basis of U a basis $(v_n)_n$ which is the union of infinitely countably many copies $(u_n^i)_n$ of (u_n), $i = 0, 1, \ldots$. We denote $U_i = [u_n^i]_{n \in \mathbb{N}}$, $V_0 = \oplus_{i \geq 1} U_i$ and see U as $U = \oplus_{n \in \mathbb{N}} U_n$. Note that if $A \in \mathbb{N}^\mathbb{N}$, then $Y_A = [u_n^0]_{n \notin A} \oplus V_0$ is a complemented subspace of U which is isomorphic to U by classical properties of Pelczynski's space; therefore embeddings onto subspaces Y_A and Y_B are in the same position if and only if the quotients U/Y_A and U/Y_B are isomorphic, i.e., $[u_n]_{n \in A} \simeq [u_n]_{n \in B}$. From this we deduce that there is a Borel reduction of $E_{\max}$ to the relation $\sim'$ on $\mathbb{N}^\mathbb{N}$ defined by

$$A \sim' B \Leftrightarrow \text{embeddings onto } Y_A \text{ and } Y_B \text{ have the same position in } U.$$

Let Q_A be the canonical projection onto Y_A associated to the unconditional basis $(v_n)_n$ and T_A a choice of an embedding of U into U for which $T_A(U) = Y_A$. Since

$$A \sim' B \Leftrightarrow T_A \sim T_B \Leftrightarrow (T_A, Q_A) \sim (T_B, Q_B),$$

it only remains to check that the map from $\mathbb{N}^\mathbb{N}$ to $\mathrm{Emb}_c(U, U)$ associating to A the pair (T_A, Q_A) may be chosen to be Borel. Since $A \mapsto Q_A$ is clearly Borel, let us describe a Borel choice of $A \mapsto T_A$: we extend by linearity the map for which

$$T_A(u_n^i) = u_n^{i+1}, \forall n \in A, \forall i \in \mathbb{N}$$

and

$$T_A(u_n^i) = u_n^i, \forall n \notin A, \forall i \in \mathbb{N}.$$

This is a Borel map for which T_A is an embedding of U onto Y_A, for all A. $\square$

2.4. Complex structures

The set of complex structures on a separable real space X will be identified with the set

$$\mathcal{C}(X) := \{T \in \mathcal{B}(X) \mid T^2 = -Id\}.$$

Since the multiplication of operators $\mathcal{B}(X) \times \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$ is Borel, it follows that the set $\mathcal{C}(X)$ is a Borel subset of $\mathcal{B}(X)$ and therefore a standard Borel set.

Definition 2.16. Let X be a separable real Banach space. The set

$$\mathcal{C}(X) = \{T \in \mathcal{B}(X) \mid T^2 = -Id\}$$

seen as a subspace of $\mathcal{B}(X)$ with the strong operator topology, will be called the *standard Borel space of complex structures on X*.

Definition 2.17. Given two elements J, K in $\mathcal{C}(X)$, we say that $J \simeq K$ if there exists an isomorphism $A \in GL(X)$ such that $AJA^{-1} = K$.

Note that $J \simeq K$ if and only if the associated complex structures are $\mathbb{C}$-linearly isomorphic.

Lemma 2.18. *The relation $\simeq$ is analytic on the standard Borel space $\mathcal{C}(X)$.*

Proof. We fix a countable family $(x_n)_n$ with dense linear span in X, and note that an isomorphism A on X may be coded by a family $(y_n)_n \in X^\omega$ with dense linear span and so that the map $x_n \mapsto y_n$ extends to an isomorphism onto its image. The relation $AJA^{-1} = K$ is then equivalent to $AJx_n = Ky_n$ for all n, which we may reformulate, using an upper bound M for $\|A\|$ and $\|K\|$, in terms of approximations of $K(\sum_i \lambda_i x_i)$ when $\sum_i \lambda_i x_i$ approximates y_n. We deduce the following characterization: $J \simeq K$ if and only if there exists $(y_n)_n \in X^\omega$ such that

(i) $\exists k \in \mathbb{N}, \forall (\lambda_n) \in c_{00}(\mathbb{Q}), k^{-1}\|\sum_n \lambda_n x_n\| \le \|\sum_n \lambda_n y_n\| \le k\|\sum_n \lambda_n x_n\|$,
(ii) $\forall n \in \mathbb{N}, \forall q \in \mathbb{Q}^{+*}, \exists (\lambda_i) \in c_{00}(\mathbb{Q}): \|x_n - \sum_i \lambda_i y_i\| < q$,
(iii) $\exists M \in \mathbb{N} \; \forall q \in \mathbb{Q}^{+*}, \forall (\lambda_i), (\mu_i) \in c_{00}(\mathbb{Q}), \|K(\sum_i \lambda_i x_i)\| \le M\|\sum_i \lambda_i x_i\|$ and $\forall n \in \mathbb{N}$,
$\big((\|\sum_i \lambda_i x_i - J x_n\| \le q) \wedge (\|y_n - \sum_i \mu_i x_i\| \le q)\big) \Rightarrow \|\sum_i \lambda_i y_i - K(\sum_i \mu_i x_i)\| \le 2Mq.$

Since the set of $((y_n), J, K)$ satisfying (i)–(ii)–(iii) is a Borel subset of the space $(X, \|.\|)^\omega \times \mathcal{C}(X)^2$, it follows that $\simeq$ is analytic on $\mathcal{C}(X)$. $\quad\square$

We now relate the Borel standard space $\mathcal{C}(X)$ of complex structures to a Borel standard space of complemented subspaces as follows. Recall that if J is a complex structure on a real space X, then the space

$$X_J = \{(x, Jx), x \in X\}$$

is a complemented, $\mathbb{C}$-linear subspace of the complexification $X \oplus_{\mathbb{C}} X$ of X, which is $\mathbb{C}$-linearly isomorphic to the complex structure X^J induced by J. Actually, X_J is the image of the $\mathbb{C}$-linear projection P_J defined on $X \oplus_{\mathbb{C}} X$ by

$$P_J(x, y) = \frac{1}{2}(x - Jy, Jx + y).$$

It is clear that the map $J \mapsto P_J$ is a Borel isomorphism between $\mathcal{C}(X)$ and the Borel subspace $\{P_J, J \in \mathcal{C}(X)\}$ of $\mathcal{P}(X \oplus_{\mathbb{C}} X)$, and by definition,

$$J \simeq K \Leftrightarrow X^J \simeq X^K \Leftrightarrow P_J \simeq P_K.$$

In other words our definition of complexity of isomorphism between complex structures coincides with the natural one induced by isomorphism of complemented subspaces of $X \oplus_{\mathbb{C}} X$ on the Borel set $\{P_J, J \in \mathcal{C}(X)\}$. Therefore also the complexity of isomorphism of complex structures on X will be $\leq_B$-below the complexity of isomorphism on $SB(X \oplus_{\mathbb{C}} X)$ (resp. $\mathcal{P}(X \oplus_{\mathbb{C}} X)$), i.e., between (resp. complemented) subspaces of $X \oplus_{\mathbb{C}} X$.

This line of ideas initiated with a result of N.J. Kalton proving that if $X \oplus_{\mathbb{C}} X$ is primary then X admits unique complex structure, and is further exemplified in [8].

Let us note here that since the complexity of linear isomorphism between separable spaces is $E_{\max}$, it is natural to ask if such maximum complexity may be achieved by isomorphism between different complex structures on a given real Banach space X.

On the other hand, the relation of isomorphism between complex structures is the orbit relation of the action $(U, T) \mapsto UTU^{-1}$ of the group $GL(X)$ on the standard Borel space $\mathcal{C}(X)$. So although $GL(X)$ is not a Polish group, one might hope to prove that isomorphism between complex structures on X is reducible to an orbit relation induced by action of some Polish group. Our final result is that, similarly to what happens for positions, this is not so:

Theorem 2.19. *There exists a separable real space X such that E_1 is Borel reducible to linear isomorphism between complex structures on X. In particular, linear isomorphism between complex structures on X is not Borel reducible to the orbit equivalence relation induced by a Polish group action on a Polish space.*

The proof of the theorem is more technical than the previous ones and is given in the next section. It is striking that the level of complexity E_1 may be obtained for $\mathbb{C}$-linear isomorphism between spaces which are all $\mathbb{R}$-linearly isometric. This means that in some sense $\mathbb{R}$-linear and $\mathbb{C}$-linear structures may be quite far apart on a Banach space. Let us note the following question:

Question 2.20. *Find a separable real Banach space such that $E_{\max}$ is Borel reducible to linear isomorphism between complex structures of X.*

3. A reduction result for complex structures

Inside L_p, with $1 \leq p < 2$, we will construct a (complex) Banach space of the form:

$$X = (\oplus_{k \geq 1} X_k)_{\ell_p} \text{ with each } X_k = (\oplus_{m \geq 1} X_{k,m})_{\ell_p}, \text{ for all } k \geq 1.$$

Given $\alpha = (\alpha_k)_k \in (2^{\omega})^{\omega}$, with $\alpha_k = (\alpha_k(m))_m \in 2^{\omega}$, we define for each k

$$X_k(\alpha) = (\oplus_{m \geq 1} Y_{k,m})_{\ell_p}$$

where

$$Y_{k,m} = \begin{cases} X_{k,m}, & \text{when} \quad \alpha_k(m) = 0 \\ \overline{X}_{k,m}, & \text{when} \quad \alpha_k(m) = 1. \end{cases}$$

Then

$$X(\alpha) = (\oplus_{k \geq 1} X_k(\alpha))_{\ell_p}$$

gives a complex structure on X, treated as a real space.

Let $T_\alpha \in \mathcal{B}(X)$ be the element of the standard Borel space of complex structures on X associated to $X(\alpha)$. We may write T_α as

$$T_\alpha = (T_k(\alpha))_k,$$

where

$$T_k(\alpha) = (T_{k,m}(\alpha))_m$$

with $T_{k,m}(\alpha)$ is defined on $X_{k,m}$ by

$$T_{k,m}(\alpha)(x) = \begin{cases} ix, & \text{when} \quad \alpha_k(m) = 0 \\ -ix, & \text{when} \quad \alpha_k(m) = 1. \end{cases}$$

It is straightforward that the map $\alpha \mapsto T_\alpha$ is Borel from $(2^\omega)^\omega$ into $\mathcal{B}(X)$ and therefore into the standard Borel space of complex structures on X.

The claim is that, for a suitable X as above, we have $X(\alpha) \simeq X(\beta)$ (equivalently $T_\alpha \simeq T_\beta$) if and only if $\alpha E_1 \beta$. For such X, E_1 is therefore Borel reducible to linear isomorphism between complex structures on X. In particular

Theorem 3.1. *The equivalence relation E_1 is Borel reducible to linear isomorphism between complex structures on the subspace X of L_p.*

Let $k \geq 1$, $m \geq 1$ be arbitrarily fixed. The ingredient space $X_{k,m}$ will be constructed as a subspace of $\ell_{p_1} \oplus_p \ldots \oplus_p \ell_{p_5}$, for some suitable constants $2 > p_1 > p_2 > \ldots > p_5 > p$ depending on k and m. Furthermore, $X_{k,m}$ will admit a 1-unconditional decomposition into 2-dimensional spaces $X_{k,m} = \overline{\text{span}} \, [Z_t]_{t \geq 1}$. More specifically, if we denote by $\{f_{j,t}\}_{t \geq 1}$ the natural basis of ℓ_{p_j} $(j = 1, \ldots, 5)$, we define the vectors x_t and y_t spanning Z_t by

$$\begin{aligned} x_t &= f_{1,t} & + \gamma_1 f_{3,t} & + \gamma_2 f_{4,t} & + \gamma_3 f_{5,t} \\ y_t &= \quad\; f_{2,t} & & + \gamma_2 f_{4,t} & + i\gamma_3 f_{5,t} \end{aligned} \qquad (1)$$

for all $t \geq 1$. The constants γ_1, γ_2, γ_3 will depend on $\eta := 1/p_2 - 1/p_1 = 1/p_3 - 1/p_2 = 1/p_4 - 1/p_3 = 1/p_5 - 1/p_4$ and on a positive integer N that is chosen according to k and m. More precisely, $\gamma_1 = N^{-2\eta}$, $\gamma_2 = N^{-8\eta}$ and $\gamma_3 = N^{-24\eta}$.

Under these definitions it was proved in [1] (Corollary 2.2) that, in terms of the Banach–Mazur distance, we have

$$d(X_{k,m}, \overline{X}_{k,m}) \geq \frac{1}{100} N^\eta.$$

This was the consequence of the following fact regarding the behavior of linear operators acting from $X_{k,m}$ to $\overline{X}_{k,m}$, which will be also useful to us later in the sequence.

First, let $T : W \longrightarrow V$ be a bounded linear operator with W, V Banach spaces having finite dimensional decompositions $\{W_t\}_t$ and $\{V_t\}_t$ respectively. We say that T is *block-diagonal* with respect to $\{W_t\}_t$ and $\{V_t\}_t$ if for every t there exists a finite set $B_t \subset \{1, 2, \ldots\}$ such that

$$\begin{cases} \max B_s < \min B_t & \forall s, t \in \{1, 2, \ldots\} \text{ with } s < t, \\ \operatorname{supp} Tw \subset B_t & \forall w \in W_t, \ \forall t \in \{1, 2, \ldots\} \end{cases}$$

where $\operatorname{supp} Tw$ is taken with respect to the decomposition $\{V_t\}_t$.

Proposition 3.2. *[1] Let $I \subset \{1, 2, \ldots\}$ be an infinite set and let Y be the subspace of $X_{k,m}$ defined by $Y = \overline{\operatorname{span}}[Z_t]_{t \in I}$. Consider $T : Y \longrightarrow \overline{X}_{k,m}$ a block-diagonal operator (with respect to $\{Z_t\}_{t \in I}$ and $\{\overline{Z}_t\}_{t \geq 1}$) with $\|T\| \leq 1$. Then*

(i) *There exists a finite set $J \subset I$ such that*

$$\max\{\|Tx_t\|, \|Ty_t\|\} \leq 24N^{-\eta}, \quad \text{for all } t \in I \setminus J. \tag{2}$$

(ii) *Let $\{I_l\}_{l \geq 1}$ be a family of disjoint subsets of I with the property that $|I_l| = N$, for all $l \geq 1$. Let $\widetilde{x}_l = \sum_{t \in I_l} a_l(t) x_t$, $\widetilde{y}_l = \sum_{t \in I_l} a_l(t) y_t$ satisfy $\sum_{t \in I_l} |a_l(t)|^{p_2} = 1$, for $l = 1, 2, \ldots$. Then there exists a finite subset $J' \subset \{1, 2, \ldots\}$ such that*

$$\max\{\|T\widetilde{x}_l\|, \|T\widetilde{y}_l\|\} \leq 70N^{-\eta}, \quad \text{for all } l \in \{1, 2, \ldots\} \setminus J'. \tag{3}$$

Notice that $x_1, y_1, x_2, y_2, \ldots$ form a Schauder basis for both $X_{k,m}$ and $\overline{X}_{k,m}$. Normalized blocks of this basis satisfy an upper p_5-estimate, which in turn implies that the basis is shrinking and hence w-null. Given now any bounded linear operator $T : Y = \overline{\operatorname{span}}[Z_t]_{t \in I} \longrightarrow \overline{X}_{k,m}$, with $I \subset \{1, 2, \ldots\}$ infinite, it is easy to see that a classical gliding-hump argument allows us to approximate T on an infinite dimensional subspace $Y_0 = \overline{\operatorname{span}}[Z_t]_{t \in I_0} \subset Y$ by a block-diagonal operator $T_0 : Y_0 \longrightarrow \overline{X}_{k,m}$. We will use this fact later in the sequel.

We are now going into the specific details of choosing the indices $p_1 = p_1(k, m) > p_2 = p_2(k, m) > \ldots > p_5 = p_5(k, m)$ corresponding to each of the ingredient spaces $X_{k,m}$, for $k \geq 1$ and $m \geq 1$, as well as the positive numbers $N = N(k, m)$ which are part of the definition (1) of the basis of $X_{k,m}$. Recall that in (1) we have required

$$\frac{1}{p_2(k,m)} - \frac{1}{p_1(k,m)} = \ldots = \frac{1}{p_5(k,m)} - \frac{1}{p_4(k,m)} =: \eta(k,m). \qquad (4)$$

We start by picking a sequence $\{q(k, m)\}_{k,m}$ as follows. For a fixed $k \geq 1$ we choose $q(k, m) \longrightarrow p$ as $m \longrightarrow \infty$ such that $q(k, m) > q(k, m + 1)$ for all $m \geq 1$. Once $q(k, m)$ have been chosen for a fixed $k \geq 1$ and for all $m \geq 1$, the inductive step of picking $\{q(k + 1, m)\}_{m \geq 1}$ is done in such a way to satisfy

$$q(k, m + 1) > q(k + 1, m) > q(1, m + k + 1)$$

for all $m \geq 1$.

It is not hard to see that, once we choose $\{q(k, m)\}_{k \geq 1, m \geq 1}$ in this way, we can look at it as a decreasing sequence if on the set of double indices (k, m) we consider the order relation "$\leq$" which follows the order: $(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), \ldots, (1, i), (2, i - 1), \ldots, (i - 1, 2), (i, 1), \ldots$.

Next, for $k \geq 1$ and $m \geq 1$, we pick $p_1(k, m) > p_2(k, m) > \ldots > p_5(k, m)$ satisfying (4) together with

$$q(k, m) > p_1(k, m) > \ldots > p_5(k, m) > q(k_0, m_0) \qquad (5)$$

where (k_0, m_0) is the successor of (k, m) with respect to "$\leq$", and also

$$\frac{(m + k - 1)(m + k)}{2} \eta(k, m) < \frac{1}{p_1(k, m)} - \frac{1}{q(k, m)}. \qquad (6)$$

Lastly, we define $N(k, m)$ as

$$N(k, m) = \left[(k + 1)^{1/\eta(k,m)} \right]$$

for $k \geq 1$, $m \geq 1$, which gives the estimate

$$k < N(k, m)^{\eta(k,m)} \leq k + 1. \qquad (7)$$

Proof of Theorem 3.1. When $\alpha = (\alpha_k)_k \in (2^\omega)^\omega$, $\beta = (\beta_k)_k \in (2^\omega)^\omega$ satisfy $\alpha E_1 \beta$ we can see that the spaces $X(\alpha)$ and $X(\beta)$ are isomorphic by means of a linear operator $T : X(\alpha) \longrightarrow X(\beta)$ which is defined as follows:

- when $\alpha_k(m) = \beta_k(m)$, then $T_{|Y_{k,m}} = Id_{|Y_{k,m}}$;

- when $\alpha_k(m) = 0$ and $\beta_k(m) = 1$, then on $X_{k,m} = \overline{\mathrm{span}}\,[Z_t]_{t\geq 1}$

$$T\left(\sum_{t\geq 1}(a_t x_t + b_t y_t)\right) = \sum_{t\geq 1}(a_t \odot x_t - b_t \odot y_t) \tag{8}$$

for all scalars $\{a_t\}_{t\geq 1}$, $\{b_t\}_{t\geq 1}$;
- when $\alpha_k(m) = 1$ and $\beta_k(m) = 0$, then on $\overline{X}_{k,m} = \overline{\mathrm{span}}\,[\overline{Z}_t]_{t\geq 1}$

$$T\left(\sum_{t\geq 1}(a_t \odot x_t + b_t \odot y_t)\right) = \sum_{t\geq 1}(a_t x_t - b_t y_t) \tag{9}$$

for all scalars $\{a_t\}_{t\geq 1}$, $\{b_t\}_{t\geq 1}$.

Recall that multiplication by scalars in $\overline{X}_{k,m}$ is given by $\lambda \odot x = \overline{\lambda} x$, for $\lambda \in \mathbf{C}$, $x \in X_{k,m}$. Taking into account the definition (1) of the basis of $X_{k,m}$ and $\overline{X}_{k,m}$ we can rewrite (8) as

$$T\left(\sum_{t\geq 1} a_t f_{1,t} + b_t f_{2,t} + \gamma_1 a_t f_{3,t} + \gamma_2(a_t + b_t)f_{4,t} + \gamma_3(a_t + ib_t)f_{5,t}\right) =$$
$$= \sum_{t\geq 1} \overline{a}_t f_{1,t} - \overline{b}_t f_{2,t} + \gamma_1 \overline{a}_t f_{3,t} + \gamma_2(\overline{a}_t - \overline{b}_t)f_{4,t} + \gamma_3(\overline{a}_t - i\overline{b}_t)f_{5,t}.$$

A simple computation shows that, in this situation, we have

$$\|T_{|X_{k,m}}\| \leq 2\gamma_2\gamma_3^{-1} = 2N(k,m)^{16\eta(k.m)} \leq 2(k+1)^{16}.$$

In the case when we are dealing with (9) we also get $\|T_{|\overline{X}_{k,m}}\| \leq 2(k+1)^{16}$. Thus we can conclude that

$$\|T\| \leq 2\left(1 + \max\{k \mid \alpha_k \neq \beta_k\}\right)^{16}.$$

Now let $\alpha = (\alpha_k)_k \in (2^\omega)^\omega$, with $\alpha_k = (\alpha_k(m))_m \in 2^\omega$, and $\beta = (\beta_k)_k \in (2^\omega)^\omega$, with $\beta_k = (\beta_k(m))_m \in 2^\omega$, be elements in $(2^\omega)^\omega$ which are not E_1-equivalent. Without loss of generality assume that

$$A = \{k \mid \exists m \text{ such that } \alpha_k(m) = 0 \text{ and } \beta_k(m) = 1\}$$

is infinite.

Suppose that $T : X(\alpha) \longrightarrow X(\beta)$ is an isomorphism with $\|T\| \leq 1/4$ and $\|T^{-1}\| =: C$. For $k \geq 1$, $m \geq 1$, denote by $P_{k,m} : X(\beta) \longrightarrow Y_{k,m}$ the canonical projection of $X(\beta)$ onto

$$Y_{k,m} = \begin{cases} X_{k,m}, & \text{when } \beta_k(m) = 0 \\ \overline{X}_{k,m}, & \text{when } \beta_k(m) = 1. \end{cases}$$

Furthermore, since $Y_{k,m} \subset \ell_{p_1(k,m)} \oplus_p \ldots \oplus_p \ell_{p_5(k,m)}$, we will denote by

$$Q_j(k,m) : Y_{k,m} \longrightarrow \ell_{p_j(k,m)}$$

the canonical projection for all $j = 1, \ldots, 5$.

Let $k \in A$ be arbitrarily fixed and pick $m \geq 1$ such that $\alpha_k(m) = 0$ and $\beta_k(m) = 1$. We will concentrate our attention on the action of the isomorphism T on $X_{k,m} = \overline{\text{span}} \, [Z_t]_{t \geq 1}$.

First, we notice that for $(k', m') > (k, m)$ we have the following: for every $\delta > 0$ and infinite set $L \subseteq \mathbf{N}$, there exists an infinite subset $L' \subseteq L$ such that

$$\|P_{k',m'}T_{|\overline{\text{span}}[Z_t]_{t \in L'}}\| \leq \delta.$$

Otherwise we can find $\delta_0 > 0$ and a normalized block sequence $(z_s)_s \subset X_{k,m}$ (with respect to the UFDD $\{Z_t\}_t$) satisfying

$$\delta_0 < \|P_{k',m'}Tz_s\|$$
$$\left(= (\|Q_1(k',m')P_{k',m'}Tz_s\|^p + \ldots + \|Q_5(k',m')P_{k',m'}Tz_s\|^p)^{\frac{1}{p}} \right). \tag{10}$$

By passing to a subsequence and perturbing the operator $P_{k',m'}T$ (similarly as in the remark following Proposition 3.2) we may assume that $(P_{k',m'}Tz_s)_s$ are successive in $Y_{k',m'}$, with respect to the 2-dimensional UFDD in $Y_{k',m'}$. This ensures that $(P_{k',m'}Tz_s)_s$ admit a lower $p_1(k',m')$-estimate, based on (10). On the other hand, $(z_s)_s$ admit an upper $p_5(k,m)$-estimate, and this gives a contradiction since $p_5(k,m) > q(k',m') > p_1(k',m')$.

Inductively, for every $(\delta_{k',m'})_{(k',m')>(k,m)} \searrow 0$ we can get infinite sets

$$\{L_{k',m'}\}_{(k',m')>(k,m)}$$

with the property that

$$L_{k',m'} \supseteq L_{k'',m''}, \qquad \text{whenever } (k',m') \leq (k'',m''),$$

and

$$\|P_{k',m'}T_{|\overline{\text{span}}[Z_t]_{t \in L_{k',m'}}}\| \leq \delta_{k',m'}, \quad \text{for all } (k',m') > (k,m).$$

Let $I = \{t_{k',m'}\}_{(k',m')>(k,m)}$ be the diagonal sequence of $\{L_{k',m'}\}_{(k',m')>(k,m)}$. We then obtain a subspace of $X_{k,m}$, namely $S_{k,m} := \overline{\text{span}}[Z_t]_{t \in I}$, and by a perturbation argument we can get a linear operator $T_0 : S_{k,m} \longrightarrow X(\beta)$ which satisfies

$$\frac{1}{2C}\|x\| \le \|T_0 x\| \le \frac{1}{2}\|x\|, \quad \text{for all } x \in S_{k,m},$$

and

$$P_{k',m'}T_0 z = 0,$$

for all $z \in \overline{\text{span}}[Z_t]_{t \in I, \, t \ge t_{k',m'}}$ and all $(k', m') > (k, m)$.

Denote by $R_{k,m} : X(\beta) \longrightarrow \left(\sum_{(k',m')>(k,m)} \oplus Y_{k',m'}\right)_{\ell_p}$ the canonical projection.

It is easy to see now that for all $\delta > 0$ and every infinite set $L \subseteq I$ there exists $t \in L$ satisfying

$$\|R_{k,m}T_{0 \mid Z_t}\| \le \delta.$$

Otherwise, we can find $\delta_0 > 0$, an infinite set $L_0 \subseteq I$ and, for each $t \in L_0$, normalized elements $z_t \in Z_t$ such that $\|R_{k,m}T_0 z_t\| > \delta_0$. By passing to a subsequence and perturbing the operator $R_{k,m}T_0$ we may assume that $(R_{k,m}T_0 z_t)_{t \in L_0}$ are disjoint in $\left(\sum_{(k',m')>(k,m)} \oplus Y_{k',m'}\right)_{\ell_p}$ and thus they admit a lower p-estimate. However $(z_t)_{t \in L_0}$ admit an upper $p_5(k, m)$-estimate, and this gives a contradiction since $p_5(k, m) > p$.

This allows us to obtain a subsequence $\tilde{I}$ of I and, after some perturbations, a linear operator (denoted again by) $T_0 : \overline{\text{span}}[Z_t]_{t \in \tilde{I}} \longrightarrow X(\beta)$ with the property that $R_{k,m}T_0 = 0$ and

$$\frac{1}{4C}\|x\| \le \|T_0 x\| \le \|x\|, \quad \text{for all } x \in \overline{\text{span}}[Z_t]_{t \in \tilde{I}}.$$

In addition, we can also assume that T_0 has the property that $P_{k',m'}T_0 : \overline{\text{span}}[Z_t]_{t \in \tilde{I}} \longrightarrow Y_{k',m'}$ is block-diagonal, with respect to their respective 2-dimensional decompositions, for all $(k', m') \le (k, m)$.

Looking now at $P_{k,m}T_0 : \overline{\text{span}}[Z_t]_{t \in \tilde{I}} \longrightarrow Y_{k,m} = \overline{X}_{k,m}$ we have all the conditions of Proposition 3.2 satisfied. We can then find $I_0 \subset \tilde{I}$, $|I_0| = N(k, m)$ with the property that, for $y = \sum_{k \in I_0} y_k$,

$$\|P_{k,m}T_0 y\| \le 70 N(k,m)^{-\eta(k,m)} N(k,m)^{\frac{1}{p_2(k,m)}},$$

and thus

$$\left\|\sum_{(k',m')<(k,m)} P_{k',m'}T_0 y\right\| \ge \|T_0 y\| - \|P_{k,m}T_0 y\|$$

$$\ge \left(\frac{1}{4C} - 70 N(k,m)^{-\eta(k,m)}\right) N(k,m)^{\frac{1}{p_2(k,m)}}. \qquad (11)$$

For every $(k', m') < (k, m)$ we have

$$\|P_{k',m'}T_0y\| = \|\sum_{k \in I_0} P_{k',m'}T_0y_k\|$$

$$\leq \|\sum_{k \in I_0} Q_1(k',m')P_{k',m'}T_0y_k\| + \ldots + \|\sum_{k \in I_0} Q_5(k',m')P_{k',m'}T_0y_k\|$$

$$\leq 2N(k,m)^{\frac{1}{p_1(k',m')}} + \ldots + 2N(k,m)^{\frac{1}{p_5(k',m')}} \leq 10N(k,m)^{\frac{1}{q(k,m)}}.$$

The last inequalities are consequences of (5) and the fact that $P_{k',m'}T_0$ is block-diagonal and

$$\|Q_j(k',m')P_{k',m'}T_0y_k\| \leq \|y_k\| \leq 2, \quad \forall k \in I_0, \quad \forall j = 1, \ldots, 5.$$

Since there are at most $(m+k-1)(m+k)/2$ elements $(k',m') < (k,m)$ we get, as a consequence of (6) and (7),

$$\|\sum_{(k',m')<(k,m)} P_{k',m'}T_0y\| \leq 10\frac{(m+k-1)(m+k)}{2}N(k,m)^{\frac{1}{q(k,m)}}$$

$$\leq 10k^{\frac{(m+k-1)(m+k)}{2}}N(k,m)^{\frac{1}{q(k,m)}}$$

$$< 10N(k,m)^{\frac{(m+k-1)(m+k)}{2}\eta(k,m)+\frac{1}{q(k,m)}}$$

$$\leq 10N(k,m)^{\frac{1}{p_1(k,m)}}.$$

We now conclude based on (11) that

$$10N(k,m)^{\frac{1}{p_1(k,m)}-\frac{1}{p_2(k,m)}} \geq \frac{1}{4C} - 70N(k,m)^{-\eta(k,m)}$$

which in turn gives $C \geq N(k,m)^{\eta(k,m)}/320 > k/320$ (by (4) and (7)).

As $k \in A$ was arbitrarily fixed, we obtain a contradiction and this concludes the proof. $\quad \square$

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