1

$X_{-1}=0$, $X_0=1$; $X_1=0$, $X_2=0$, $X_3=0$, $X_4=1$, $X_5=0$, $X_6=1$, $X_7=1$, $X_8=1$

$X_9=0$, $X_{10}=1$

a)

$$P = \begin{array}{c|cc} & 0 & 1 \\ 0 & 0.7 & 0.3 \\ 1 & 0.5 & 0.5 \end{array}$$

$2 = N_n(00)$
$4 = N_n(01)$
$3 = N_n(10)$
$2 = N_n(11)$

$$\mathcal{L} = 0.7^{2} \times 0.3^{4} \times 0.5^{3} \times 0.5^{2}$$

b)

a \ $X_{n-1}X_{n-2}$	$P(a\|00)$	$P(a\|01)$	$P(a\|10)$	$P(a\|11)$
0	0.7	0.5	0.4	0.8
1	0.3	0.5	0.6	0.2

$N_n(000)=1$, $N_n(001)=1$
$N_n(100)=1$, $N_n(101)=2$
$N_n(010)=2$, $N_n(011)=1$
$N_n(110)=1$, $N_n(111)=1$

$$\mathcal{L} = 0.7^{1} \times 0.3^{1} \times 0.5^{1} \times 0.5^{2} \times 0.4^{2} \times 0.6^{1} \times 0.8^{1} \times 0.2^{1}$$

c) $\hat{P} = \begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix}$

$$\mathcal{L}(\hat{P}) = \alpha^2 \times (1-\alpha)^4 \times \beta^3 \times (1-\beta)^2$$

$$\log \mathcal{L}(\hat{P}) = 2\log(\alpha) + 4\log(1-\alpha) + 3\log\beta + 2\log(1-\beta)$$

$$\frac{\partial \log \mathcal{L}(\hat{P})}{\partial \alpha} = \frac{2}{\alpha} - \frac{4}{1-\alpha} = 0 \Rightarrow \frac{2}{\alpha} = \frac{4}{1-\alpha} \Rightarrow 2-2\alpha = 4\alpha$$

$$\alpha = 1/3$$

$$\frac{\partial \log \mathcal{L}(\hat{P})}{\partial \beta} = \frac{3}{\beta} - \frac{2}{1-\beta} = 0 \Rightarrow \frac{3}{\beta} = \frac{2}{1-\beta} \Rightarrow 3-3\beta = 2\beta$$

$$3/5 = \beta$$

Outra via de solução utilizando fórmula stimador máximo verosimil das propabilidades:

$$\alpha = \frac{N_n(00)}{\sum_{z\in A} N_n(0z)} = \frac{N_n(00)}{N_n(00)+N_n(01)} = \frac{2}{2+4} = \frac{1}{3}$$

$$\beta = \frac{Nu(10)}{\sum_{z \in A} N(1z)} = \frac{Nu(10)}{Nu(10) + Nu(11)} = \frac{3}{3+2} = \frac{3}{5}$$

Para caso inciso b:

$$\hat{p} = \begin{array}{c|cccc} a & p(a|00) & p(a|01) & p(a|10) & p(a|11) \\ \hline 0 & \alpha & \beta & \gamma & \delta \\ 1 & 1-\alpha & 1-\beta & 1-\gamma & 1-\delta \end{array}$$

$$\mathcal{L}(\hat{p}) = \alpha \times (1-\alpha) \times \beta \times (1-\beta)^2 \times \gamma^2 \times (1-\gamma) \times \delta \times (1-\delta)$$

$$\log \mathcal{L}(\hat{p}) = \log \alpha + \log(1-\alpha) + \log \beta + 2\log(1-\beta) + 2\log \gamma + \log(1-\gamma) + \log \delta + \log(1-\delta)$$

$$\frac{\partial \mathcal{L}(\hat{p})}{\partial \alpha} = \frac{1}{\alpha} - \frac{1}{1-\alpha} = 0 \Rightarrow \frac{1}{\alpha} = \frac{1}{1-\alpha} \Rightarrow \alpha = 1/2$$

$$\frac{\partial \mathcal{L}(\hat{p})}{\partial \beta} = \frac{1}{\beta} - \frac{2}{1-\beta} = 0 \Rightarrow \frac{1}{\beta} = \frac{2}{1-\beta} \Rightarrow 1-\beta = 2\beta \Rightarrow \beta = 1/3$$

$$\frac{\partial \mathcal{L}(\hat{p})}{\partial \gamma} = \frac{2}{\gamma} - \frac{1}{1-\gamma} = 0 \Rightarrow \frac{2}{\gamma} = \frac{1}{1-\gamma} \Rightarrow 2-2\gamma = \gamma \Rightarrow \gamma = 2/3$$

$$\frac{\partial \mathcal{L}(\hat{p})}{\partial \delta} = \frac{1}{\delta} - \frac{1}{1-\delta} = 0 \Rightarrow \frac{1}{\delta} = \frac{1}{1-\delta} \Rightarrow \delta = 1/2.$$

Outra via di solução:

$$\alpha = \frac{Nu(000)}{\sum_{z \in A} Nu(00z)} = \frac{Nu(000)}{Nu(000) + Nu(001)} = \frac{1}{1+1} = \frac{1}{2}$$

$$\beta = \frac{Nu(100)}{\sum_{z \in A} Nu(10z)} = \frac{Nu(100)}{Nu(100) + Nu(101)} = \frac{1}{1+2} = \frac{1}{3}$$

$$\gamma = \frac{N_n(010)}{\sum_{z \in A} N_n(01z)} = \frac{N_n(010)}{N_n(010) + N_n(011)} = \frac{2}{2+1} = \frac{2}{3}$$

$$\delta = \frac{N_n(110)}{\sum_{z \in A} N_n(11z)} = \frac{N_n(110)}{N_n(110) + N_n(111)} = \frac{1}{1+1} = \frac{1}{2}$$

2

$$p = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1/4 & 3/4 \\ 1 & 3/5 & 2/5 \end{array}$$

a) $$l = \left(\frac{1}{4}\right)^{N_n(00)} \times \left(\frac{3}{4}\right)^{N_n(01)} \times \left(\frac{3}{5}\right)^{N_n(10)} \times \left(\frac{2}{5}\right)^{N_n(11)}$$

$$= \left(\frac{1}{4}\right)^{15} \times \left(\frac{3}{4}\right)^{48} \times \left(\frac{3}{5}\right)^{21} \times \left(\frac{2}{5}\right)^{16}$$

b) $$\hat{p} = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & \alpha & 1-\alpha \\ 1 & \beta & 1-\beta \end{array}$$

$$\alpha = \frac{N_n(00)}{\sum_{z \in A} N_n(0z)} = \frac{N_n(00)}{N_n(00) + N_n(01)} = \frac{15}{15+48} = \frac{15}{63}$$

$$\beta = \frac{N_n(10)}{\sum_{z \in A} N_n(1z)} = \frac{N_n(10)}{N_n(10) + N_n(11)} = \frac{21}{21+16} = \frac{21}{37}$$

c) $$l(\hat{p}) = \underbrace{\left(\frac{15}{63}\right)^{15}}_{(\alpha)} \times \underbrace{\left(\frac{48}{63}\right)^{48}}_{(1-\alpha)} \times \underbrace{\left(\frac{21}{37}\right)^{21}}_{(\beta)} \times \underbrace{\left(\frac{16}{37}\right)^{16}}_{(1-\beta)}$$

3 fila 1 e fila 3 da matriz são iguais, por tanto nos pasados $\underline{0}0$ e $\underline{1}0$ o primeiro "0" e "1" são informações não relevantes, sendo "0" contexto

$$p(0|\underset{\uparrow}{0}0) = p(0|\underline{0}1) = p(0|0)$$

irrelevante

As filas 2 e 4 são diferentes, por tanto 01 e 11 são contextos.

4 a) $X_{-1}^{10} = x_{-1}^{10} = (1,0,0,1,1,0,1,1,0,0,1,0)$

$$\mathbb{P}(X_{-1}^{10} = x_{-1}^{10}) = \prod_{a\in A}\prod_{w\in \tau} p(a|w)^{N_n(wa)}$$

w	a	$N_n(wa)$
1	1	2
1	0	4
10	0	2
10	1	1
00	0	0
00	1	2

$$= p(1|1)^2 \times p(0|1)^4 \times p(0|01)^2 \times p(1|01)^1 \times p(0|00)^0 \times p(1|00)^2$$

$$= \alpha^2 \times (1-\alpha)^4 \times (1-\beta)^2 \times \beta \times \gamma^2$$

b) Definição:

$$N_n(wa) = \sum_{t=0}^{n} \mathbb{1}\{X_{t-|w|}^{t-1} = w, X_t = a\}$$

$$N_n(w) = \sum_{t=0}^{n+1} \mathbb{1}\{X_{t-|w|}^{t-1} = w\}$$

$$\sum_{a\in A} N_n(wa) = \sum_{a\in A}\sum_{t=0}^{n} \mathbb{1}\{X_{t-|w|}^{t-1} = w, X_t = a\}$$

$$= \sum_{t=0}^{n}\sum_{a\in A} \mathbb{1}\{X_{t-|w|}^{t-1} = w, X_t = a\}$$

$$= \sum_{t=0}^{n} \mathbb{1}\{X_{t-|w|}^{t-1} = w\} = N_{n-1}(w)$$

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c) $$\sum_{w\in\tau}\sum_{a\in A} N_n(wa) = \sum_{w\in\tau} N_{n-1}(w) = n+1$$

5 $$\hat{\alpha} = \frac{N_n(11)}{\sum_{a\in A} N_n(1a)} = \frac{N_n(11)}{N_n(10)+N_n(11)} = \frac{2}{4+2} = \frac{1}{3}$$

$$\hat{\beta} = \frac{N_n(101)}{\sum_{a\in A} N_n(10a)} = \frac{N_n(101)}{N_n(100)+N_n(101)} = \frac{1}{2+1} = \frac{1}{3}$$

$$\hat{\gamma} = \frac{N_n(001)}{\sum_{a\in A} N_n(00a)} = \frac{N_n(001)}{N_n(000)+N_n(001)} = \frac{2}{0+2} = 1.$$

Outra solução: Derivar expresão obtida em exercício 4:

$$f = \alpha^2 \times (1-\alpha)^4 \times (1-\beta)^2 \times \beta \times \gamma^2$$

$$\log f = 2\log\alpha + 4\log(1-\alpha) + 2\log(1-\beta) + \log\beta + 2\log\gamma$$

$$\frac{\partial \log f}{\partial\alpha} = \frac{2}{\alpha} - \frac{4}{1-\alpha} = 0 \Rightarrow 2-2\alpha = 4\alpha \Rightarrow \alpha = 1/3$$

$$\frac{\partial \log f}{\partial\beta} = -\frac{2}{1-\beta} + \frac{1}{\beta} = 0 \Rightarrow 2\beta = 1-\beta \Rightarrow \beta = 1/3$$

$$\frac{\partial \log f}{\partial\gamma} = \frac{2}{\gamma} \Rightarrow \text{Como } \gamma\in[0,1] \Rightarrow \gamma = 1$$

6 a) $\hat{p} \in \mathcal{M}_0(\{0,1\})$ ≡ Markov de orden zero (cada observação no instante t é independente do pasado).

$$\hat{p}(0) = \frac{N_n(0)}{N_n(0)+N_n(1)} = \frac{N_n(0)}{n} = \frac{6}{12} = \frac{1}{2}, \quad \hat{p}(1) = 1-\hat{p}(0) = \frac{1}{2}$$

7

a)

X_n	0	1
$p(X_n \mid X_{n-1}=0, X_{n-2}=0)$	$h(0,0) = 1/2$	$1 - h(0,0) = 1/2$
$p(X_n \mid X_{n-1}=0, X_{n-2}=1)$	$h(1,0) = 1/4$	$1 - h(1,0) = 3/4$
$p(X_n \mid X_{n-1}=1, X_{n-2}=0)$	$h(0,1) = 1/3$	$1 - h(0,1) = 2/3$
$p(X_n \mid X_{n-1}=1, X_{n-2}=1)$	$h(1,1) = 1/5$	$1 - h(1,1) = 4/5$

b) $X_{-2} = 1$, $X_{-1} = 0$ dados.

$$P(X_0 = 0 \mid X_{-1}=0, X_{-2}=1) = 1/4$$

$$\mathbb{P}(X_0 = 1 \mid X_{-1}=0, X_{-2}=1) = 3/4$$

$$\mathbb{P}(X_1=1) = \mathbb{P}(X_1=1 \mid X_0=0)\,\mathbb{P}(X_0=0) + \mathbb{P}(X_1=1 \mid X_0=1)\,\mathbb{P}(X_0=1)$$

$$= \mathbb{P}(X_1=1 \mid X_0=0, X_{-1}=0)\,P(X_0=0) +$$

$$\mathbb{P}(X_1=1 \mid X_0=1, X_{-1}=0)\,\mathbb{P}(X_0=1)$$

$$= \frac{1}{2}\cdot\frac{1}{4} + \frac{2}{3}\cdot\frac{3}{4} = \frac{1}{8} + \frac{1}{2} = \frac{1+4}{8} = \frac{5}{8}.$$

8

b)

$a \nearrow Y_{n+1} = X_{n-1}, X_n$	00	01	10	11
$p(a \mid X_{n-1}=0, X_{n-2}=0)$	1/2	1/2	0	0
$p(a \mid X_{n-1}=0, X_{n-2}=1)$	1/4	3/4	0	0
$p(a \mid X_{n-1}=1, X_{n-2}=0)$	0	0	1/3	2/3
$p(a \mid \underbrace{X_{n-1}=1, X_{n-2}=1}_{Y_n})$	0	0	1/5	4/5

⑨

$\mathbb{P}(X_n=1) = p \qquad 0<p<1$

$$Z = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\mathbb{E}(Z) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(X_i) = \frac{1}{n}\cdot n\cdot p = p.$$

$$\boxed{\begin{aligned}\mathbb{E}(X_i) &= 1\cdot p + 0\cdot(1-p) \\ &= p. \\ \operatorname{var}(X_i) &= \mathbb{E}(X_i^2) - \mathbb{E}(X_i)^2 \\ &= p - p^2 = p(1-p)\end{aligned}}$$

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - p\right| > 0.01\right) = \mathbb{P}\left(\left|Z - \mathbb{E}(Z)\right| > 0.01\right)$$

$$\leq \frac{\operatorname{var}(Z)}{(0.01)^2} \qquad (\text{Chebyshev.})$$

$$\boxed{\begin{aligned}\operatorname{Var}(Z) &= \operatorname{var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\ &= \frac{1}{n^2}\sum_{i=1}^{n}\operatorname{Var}(X_i) = \frac{1}{n^2}\, n\, p(1-p) = \frac{p(1-p)}{n}\end{aligned}}$$

$$= \frac{p(1-p)}{n(0.01)^2}$$

Si $0<p<1$,

$p(1-p) \leq 1/4$

$$\frac{p(1-p)}{n(0.01)^2} \leq 0.01 \Rightarrow \frac{p(1-p)}{(0.01)^3} \leq n$$

$$\frac{1/4}{10^{-6}} \leq n$$

$$\frac{10^6}{4} \leq n \Rightarrow \bar{n} = \frac{10^6}{4}.$$

(10)

a)

$$P = \begin{array}{c|cc} w \backslash a & 0 & 1 \\ \hline 0 & \alpha & 1-\alpha \\ 01 & \beta & 1-\beta \\ 11 & \gamma & 1-\gamma \end{array}$$

$Nn(00) = 2$ $\quad Nn(01) = 8$

$Nn(010) = 3$ $\quad Nn(011) = 4$

$Nn(110) = 4$ $\quad Nn(111) = 2$

$$L(p) = \prod_{a \in A} \prod_{w \in \tau} p(a|w)^{Nn(wa)}$$

$$= p(0|0)^{Nn(00)} \times p(1|0)^{Nn(01)} \times p(0|10)^{Nn(010)} \times p(1|10)^{Nn(011)}$$

$$\times p(0|11)^{Nn(110)} \times p(1|11)^{Nn(111)}$$

$$= \alpha^2 \times (1-\alpha)^8 \times \beta^3 \times (1-\beta)^4 \times \gamma^4 \times (1-\gamma)^2$$

b)

$$\alpha = \frac{Nn(00)}{Nn(00) + Nn(01)} = \frac{2}{2+8} = \frac{1}{5}$$

$$\beta = \frac{Nn(010)}{Nn(010) + Nn(011)} = \frac{3}{3+4} = \frac{3}{7}$$

$$\gamma = \frac{Nn(110)}{Nn(110) + Nn(111)} = \frac{4}{4+2} = \frac{2}{3}$$

Solução usando derivada (alternativa)

$$\log L(p) = 2\log\alpha + 8\log(1-\alpha) + 3\log\beta + 4\log(1-\beta) + 4\log\gamma + 2\log(1-\gamma)$$

$$\frac{\partial L(p)}{\partial \alpha} = \frac{2}{\alpha} - \frac{8}{1-\alpha} = 0 \Rightarrow \alpha = 1/5$$

$$\frac{\partial L(p)}{\partial \beta} = \frac{3}{\beta} - \frac{4}{1-\beta} = 0 \Rightarrow \beta = 3/7$$

$$\frac{\partial f(p)}{\partial \gamma} = \frac{4}{\gamma} - \frac{2}{1-\gamma} = 0 \Rightarrow \gamma = 2/3$$

11

$$p = \begin{array}{c|cc} & 0 & 1 \\ \hline 00 & \alpha & 1-\alpha \\ 01 & \beta & 1-\beta \\ 10 & \gamma & 1-\gamma \\ 11 & \delta & 1-\delta \end{array}$$

$$p(a|a_{-1}a_{-2}) = \frac{N_n(a_{-2}a_{-1}a)}{\sum_{b\in A} N_n(a_{-2}a_{-1}b)}$$

$$\alpha = p(0|00) = \frac{N_n(000)}{N_n(000)+N_n(001)} = \frac{150}{150+54}$$

$$\beta = p(0|10) = \frac{N_n(010)}{N_n(010)+N_n(011)} = \frac{167}{167+116}$$

$$\gamma = p(0|01) = \frac{N_n(100)}{N_n(100)+N_n(101)} = \frac{55}{55+229}.$$

$$\delta = p(0|11) = \frac{N_n(110)}{N_n(110)+N_n(111)} = \frac{116}{116+N_n(111)}$$

Temos que calcular.

$$\sum_{a_{-1}^{-2}\in A^2} \sum_{b\in A} N_n(a_{-2}a_{-1}b) = \underbrace{1000-2}.$$

↳ número de transiçoes que existem de pasados $a_{-2}a_{-1}$ para símbolo b na sequencia $X_1, \dots, X_{1000}$

Então :

$$N_{1000}(000) + N_{1000}(001) + N_{1000}(010) + N_{1000}(011) + N_{1000}(100)$$
$$+ N_{1000}(101) + N_{1000}(110) + N_{1000}(111) = 998$$

$$N_{1000}(111) = 998 - 150 - 54 - 167 - 116 - 55 - 229 - 116 = 111.$$

Observacão

Formula anterior utilizada para calular $N_n(111)$ e' a mesma que aparece no exercício 4(c), só que no exercício 4(c) a sequencia tem notação da forma

$$X_{-k}^{n} = (X_{-k}, \dots, X_n).$$

No exercício 11, a sequencia foi denotada como

$$X_1^{1000} = (X_1, \dots, X_{1000})$$

Mas se denotamos essa seqoencia do jeito

$$\bar{X}_{-2}^{997} = (X_{-2}, \dots, X_{997})$$, pode se observar

que (utilizando 4(c))

$$\sum_{u \in A^{2^2}} \sum_{b \in A} N_n(ub) = n+1 = \underbrace{997+1 = 998}_{\text{que foi o mesmo utilizado no exercício.}}$$

12 a)

$$f(\alpha,\beta,\gamma) = \prod_{w \in \tau} \prod_{a \in A} p(a|w)^{N_n(wa)}$$

$$= (1-\alpha)^3 \times (\alpha)^3 \times (1-\beta) \times \beta^2 \times (1-\gamma) \times \gamma^3$$

w\a	0	1
0	$1-\alpha$	α
01	$1-\beta$	β
11	$1-\gamma$	γ

$N_n(00) = 3$ $N_n(01) = 3$

$N_n(010) = 1$ $N_n(011) = 2$

$N_n(110) = 1$ $N_n(111) = 3$

b)

$$\alpha = \frac{N_n(01)}{N_n(01) + N_n(00)} = \frac{3}{6} = \frac{1}{2}$$

$$\beta = \frac{N_n(011)}{N_n(011) + N_n(010)} = \frac{2}{2+1} = \frac{2}{3}$$

γ =

$$\gamma = \frac{N_n(111)}{N_n(111) + N_n(110)} = \frac{3}{3+1} = \frac{3}{4}$$

Pela via da derivada:

$$\log f(\alpha,\beta,\gamma) = 3\log(1-\alpha) + 3\log\alpha + \log(1-\beta) + 2\log\beta$$
$$+ \log(1-\gamma) + 3\log\gamma$$

$$\frac{\partial f(\alpha,\beta,\gamma)}{\partial\alpha} = -\frac{3}{1-\alpha} + \frac{3}{\alpha} = 0 \Rightarrow \hat{\alpha} = 1/2$$

$$\frac{\partial f(\alpha,\beta,\gamma)}{\partial\beta} = -\frac{1}{1-\beta} + \frac{2}{\beta} = 0 \Rightarrow \hat{\beta} = 2/3$$

$$\frac{\partial f(\alpha,\beta,\gamma)}{\partial\gamma} = -\frac{1}{1-\gamma} + \frac{3}{\gamma} = 0 \Rightarrow \hat{\gamma} = 3/4$$

13 Utilizar o fato que:

$$\frac{N_n(wb)}{n} \xrightarrow[n\to\infty]{} \mu(w)\,p(b|w)$$

$$\frac{N_n(w)}{n} \xrightarrow[n\to\infty]{} \mu(w)$$

$$\frac{N_n(000)}{n} \xrightarrow[n\to\infty]{} \mu(00)\,p(0|00) = \mu(00)\,p(0|0)$$

$$\frac{N_{n-1}(00)}{n} \xrightarrow[n\to\infty]{} \mu(00)$$

$$\lim_{n\to\infty} \frac{N_n(000)}{N_n(00)} = p(0|0) = 0.4$$