k-point semidefinite programming bounds for equiangular lines

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(joint work with D. de Laat, F.M. de Oliveira Filho, and F. Vallentin)
arXiv:1812.06045

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SIAM Conference on Applied Algebraic Geometry
Bern, July 9-13, 2019

Acknowledgements

F.C.M was supported by grant #2017/25237-4, from the São Paulo Research Foundation (FAPESP) and was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior — Brasil (CAPES).

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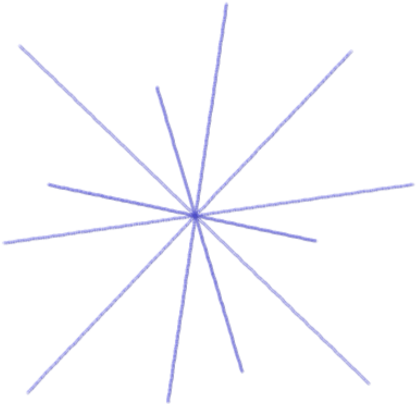
1. Problem: equiangular lines and spherical codes

Equiangular lines



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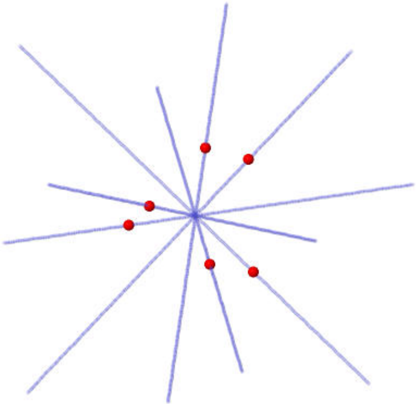

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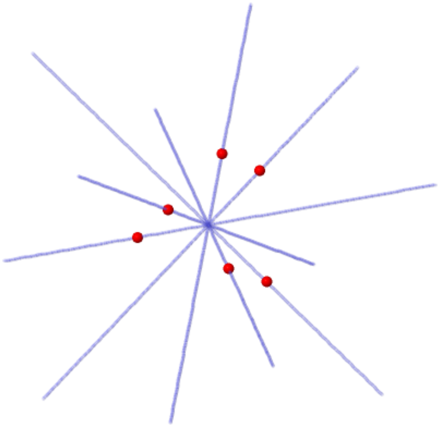

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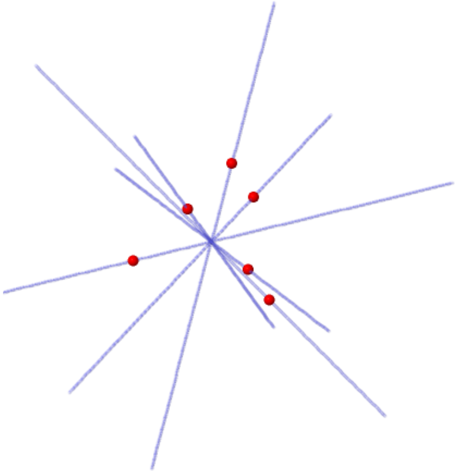

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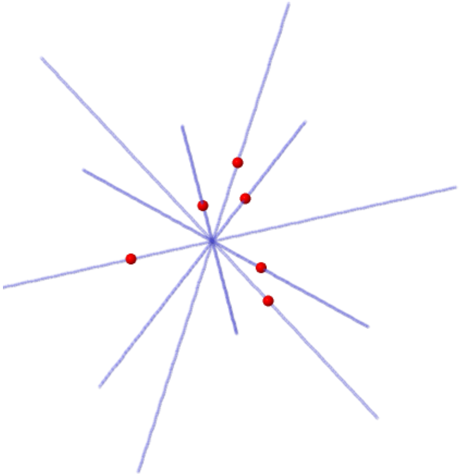

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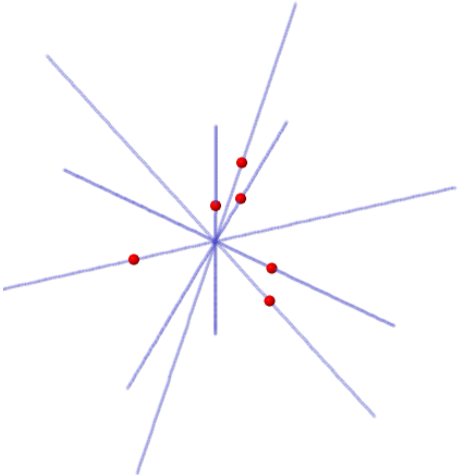

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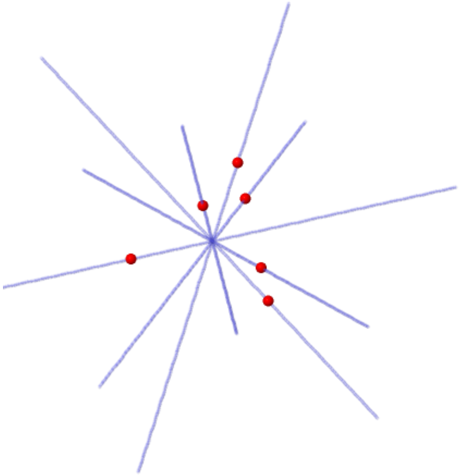

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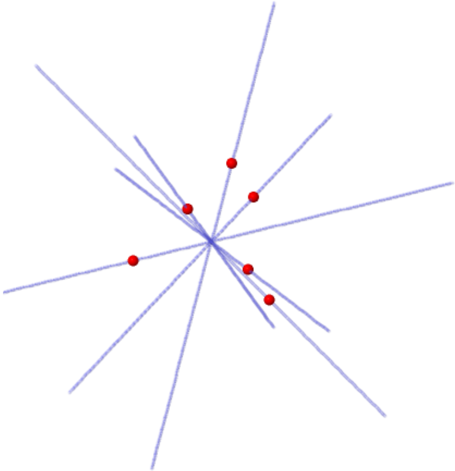

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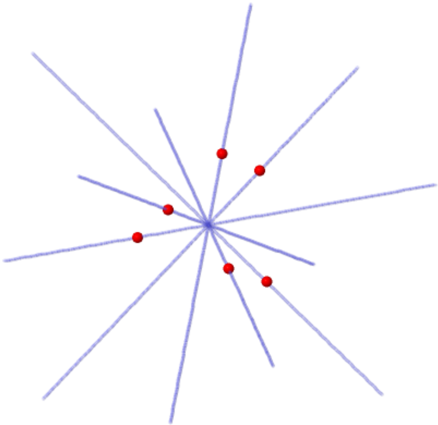

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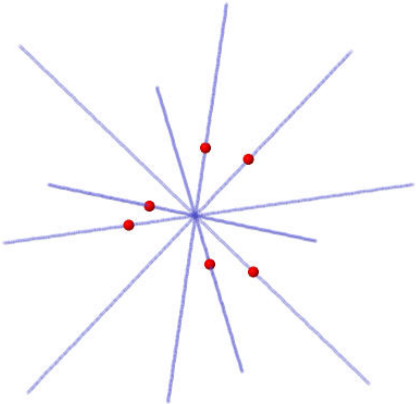

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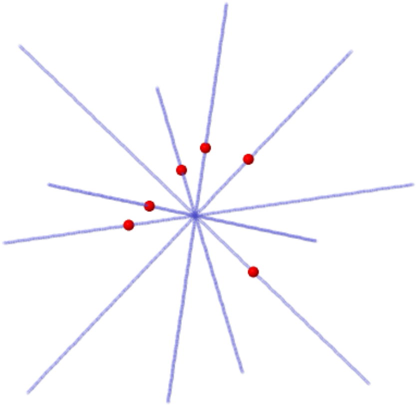

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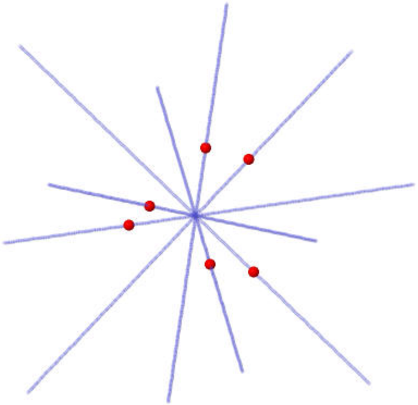

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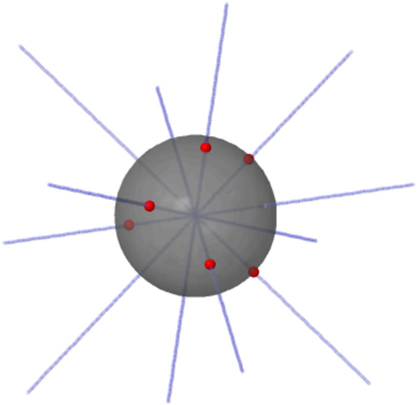


Spherical codes

$x \cdot y := \sum_{i=1}^{n} x_i y_i$

$S^{n-1} := \{x \in \mathbb{R}^n : x \cdot x = 1\}$

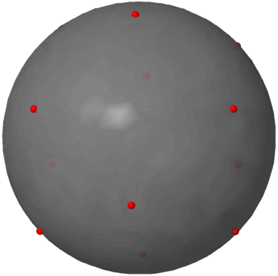
$D \subseteq [-1, 1)$ - set of allowable inner products

Examples:

$D = \{-a, a\}$ equiangular lines with common angle $\arccos a$

$|D| = s$ spherical s-distance sets

$D = [-1, a]$ packing of spherical caps



$A(n, D) := \max \{|C| : C \subseteq S^{n-1},\ x \cdot y \in D$ for all distinct $x, y \in C\}$

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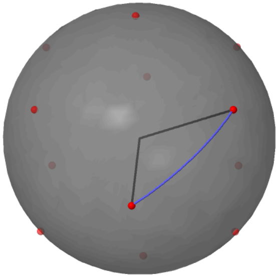

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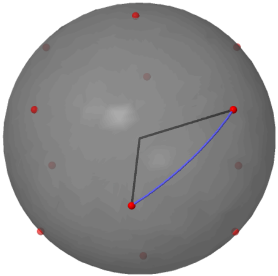

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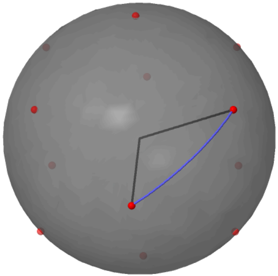

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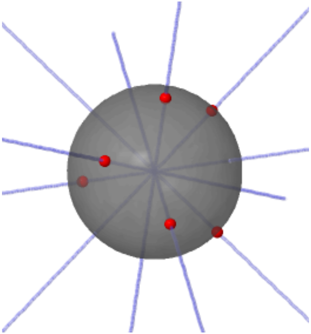

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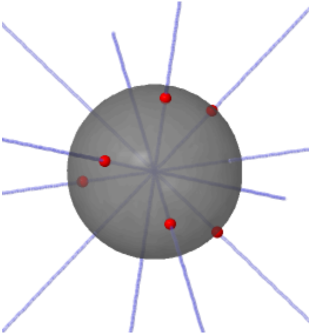

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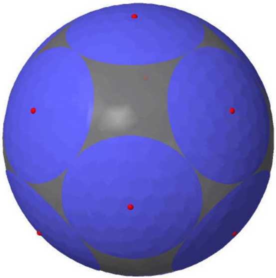

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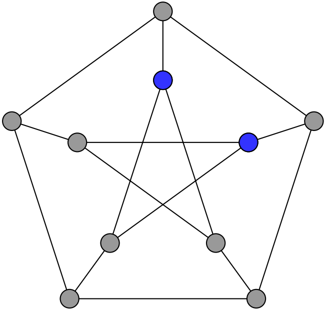
2 A hierarchy of k-point bounds for packing problems

- Independent sets
- The bound $\Delta_k(G)^*$

Independent sets

Let $G = (V, E)$ be a graph.



- $\alpha(G) := \max\{|S| : S$ is a independent set$\}$
- Let I_k be the set of independent sets in G of size at most k.
  ($I_{=k}$ for size exactly k.)

  $\emptyset \in I_k, \quad I_{=1} \simeq V$

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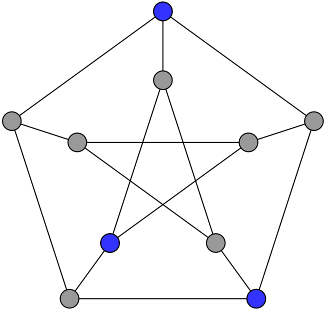

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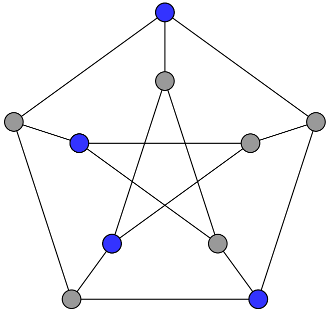

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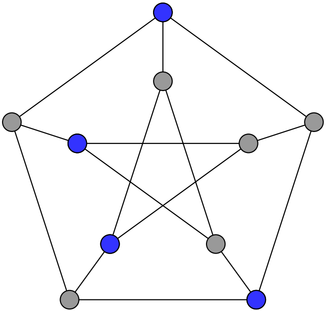

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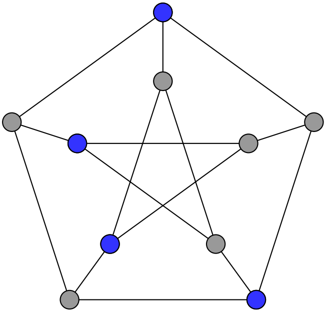

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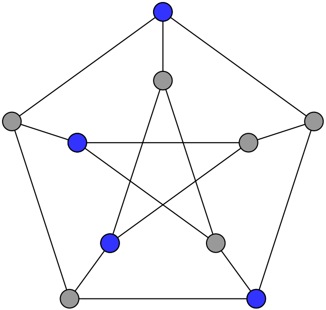

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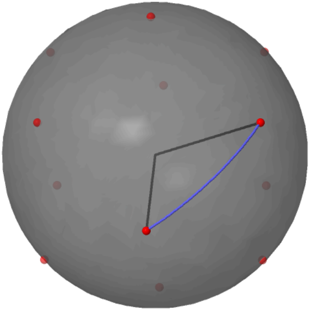
Upper bounds for $\alpha(G)$ can be computed with semidefinite programming (the Lovász theta number, Lasserre hierarchy)

A graph in the sphere



$V = S^{n-1}$

$E = \{\{x, y\} \subseteq V : x \cdot y \notin D\}$

Independent sets = spherical codes

$\alpha(G) = A(n, D)$

- Assuming D is closed, it is a "topological packing graph" and $\alpha(G)$ is finite.
- Automorphism group: $\mathrm{O}(n)$

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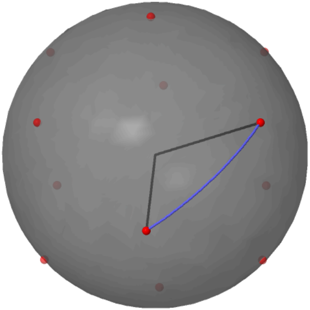

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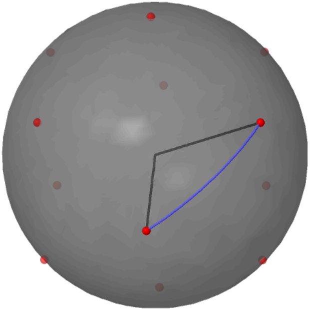

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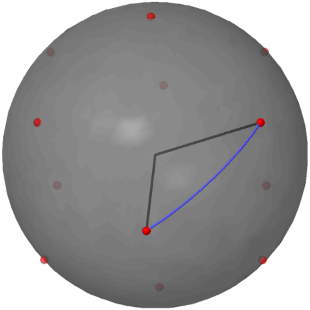

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$\Delta_k(G)^*$

$\alpha(G) \le \Delta_k(G)^*$

$$\Delta_k(G)^* := \inf\{\, 1 + \lambda : \lambda \in \mathbb{R},\ T \in \mathcal{C}(V^2 \times I_{k-2})_{\succeq 0},\ \text{and} \\ B_k T \le \lambda \chi_{I_{=1}} - 2\chi_{I_{=2}} \,\}$$

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$T \in \mathcal{C}(V^2 \times I_{k-2})_{\succeq 0}$

A function $T\colon V^2 \times I_{k-2} \to \mathbb{R}$ is in $\mathcal{C}(V^2 \times I_{k-2})_{\succeq 0}$ if it is continuous and for every $Q \in I_{k-2}$ and finite $U \subseteq V$ the matrix $\big(T(x, y, Q)\big)_{x,y \in U}$ is positive semidefinite.

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$B_k \colon \mathcal{C}(V^2 \times I_{k-2}) \to \mathcal{C}(I_k \setminus \{\emptyset\})$

$$B_k T(S) := \sum_{\substack{Q \subseteq S \\ |Q| \leq k-2}} \ \sum_{\substack{x,y \in S \\ Q \cup \{x,y\} = S}} T(x, y, Q).$$

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Example: If $S = \{a, b\}$, then

$$\begin{aligned} B_3 T(\{a,b\}) = {} & T(a, b, \emptyset) + T(b, a, \emptyset) + T(a, b, \{a\}) + T(b, a, \{a\}) \\ & + T(b, b, \{a\}) + T(a, b, \{b\}) + T(b, a, \{b\}) + T(a, a, \{b\}). \end{aligned}$$

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$$B_k T(S) \leq \begin{cases} \lambda & \text{if } |S| = 1, \\ -2 & \text{if } |S| = 2, \\ 0 & \text{if } 3 \leq |S| \leq k. \end{cases}$$

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Let $C \subseteq V$ be a nonempty independent set and let (λ, T) be a feasible solution of $\Delta_k(G)^*$.

$$\sum_{\substack{S \subseteq C \\ |S| \le k,\ S \ne \emptyset}} B_k T(S) \le \binom{|C|}{1}\lambda + \binom{|C|}{2}(-2) = |C|(1+\lambda-|C|).$$

$$\sum_{\substack{S \subseteq C \\ |S| \le k,\ S \ne \emptyset}} B_k T(S) = \sum_{\substack{S \subseteq C \\ |S| \le k,\ S \ne \emptyset}} \sum_{\substack{Q \subseteq S \\ |Q| \le k-2}} \sum_{\substack{x,y \in S \\ Q \cup \{x,y\} = S}} T(x,y,Q)$$

$$= \sum_{\substack{Q \subseteq C \\ |Q| \le k-2}} \sum_{x,y \in C} T(x,y,Q) \ge 0.$$

$T(x, y, Q)$ and $\mathrm{O}(n)$-invariance

Due the symmetries in the problem, we consider functions $T\colon S^{n-1} \times S^{n-1} \times I_{k-2} \to \mathbb{R}$ such that

$$T(\gamma x, \gamma y, \gamma Q) = T(x, y, Q) \quad \forall\ x, y \in S^{n-1}, Q \in I_{k-2}, \gamma \in \mathrm{O}(n).$$

- These functions depend only on the inner products between x, y, and Q.

$\Rightarrow$ The complexity of the computations depend only on k and not on n.

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$T(x, y, Q)$ and $\mathrm{O}(n)$-invariance

- Fix $Q \in I_{k-2}$ and define
  $K_Q \colon S^{n-1} \times S^{n-1} \to \mathbb{R}, \quad K_Q(x, y) := T(x, y, Q)$.
- $T(\gamma x, \gamma y, \gamma Q) = T(x, y, Q) \quad \forall \gamma \in \mathrm{O}(n)$

$\Rightarrow$ $K_Q(\gamma x, \gamma y) = K_Q(x, y) \quad \forall \gamma \in \mathrm{Stab}_{\mathrm{O}(n)}(Q)$

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$$\mathrm{Stab}_{\mathrm{O}(n)}(Q) := \{\gamma \in \mathrm{O}(n) : \gamma Q = Q\}$$

$T(x, y, Q)$ and $\mathrm{O}(n)$-invariance

- Fix $Q \in I_{k-2}$ and define
  $K_Q\colon S^{n-1} \times S^{n-1} \to \mathbb{R}, \quad K_Q(x,y) := T(x,y,Q)$.
- $T(\gamma x, \gamma y, \gamma Q) = T(x,y,Q) \quad \forall \gamma \in \mathrm{O}(n)$

$\Rightarrow$ $K_Q(\gamma x, \gamma y) = K_Q(x,y) \quad \forall \gamma \in \mathrm{Stab}_{\mathrm{O}(n)}(Q)$

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Gegenbauer polynomials P_l^n

The Gegenbauer polynomials P_l^n can be recursively defined as $P_0^n(t) := 1$, $P_1^n(t) := t$, and

$$P_l^n(t) := \frac{(n+2l-4)tP_{l-1}^n(t) - (l-1)P_{l-2}^n(t)}{n+l-3}.$$

Positive property

For any finite $U \subseteq S^{n-1}$, the matrix $\left(P_l^n(x \cdot y)\right)_{x,y \in U}$ is positive semidefinite.

For $a_0 \ldots, a_d \geq 0$, $(x, y) \mapsto \sum_{l=0}^{d} a_l P_l^n(x \cdot y)$ is $\mathrm{O}(n)$-invariant and has the positive property (Schoenberg's theorem)

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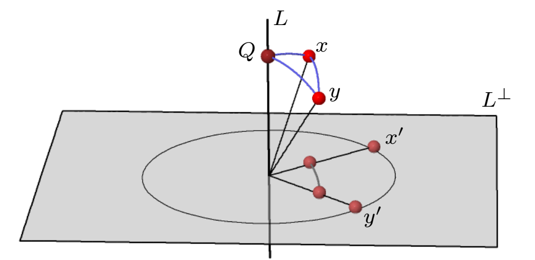
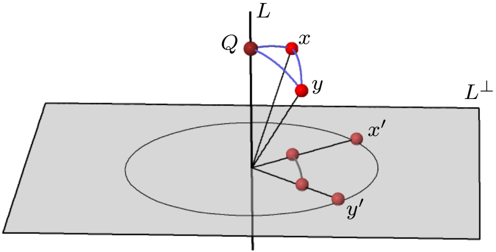
✓ $Q = \emptyset$

$Q \neq \emptyset$?

- $|Q| = 1$ — Bachoc and Vallentin (2008)
- $|Q| > 1$ — Musin (2014)



- $L := \text{span}(Q), \quad m := \dim(L)$.
- Let $C' \subseteq S^{n-m-1}$ be the projection of the spherical code C onto $L^\perp$.
- $x' \cdot y'$ can be computed in terms of the inner products between x, y, and Q.
- Positive and $\text{Stab}_{O(n)}(Q)$-invariant function can be defined with C' and the polynomials P_l^{n-m}.



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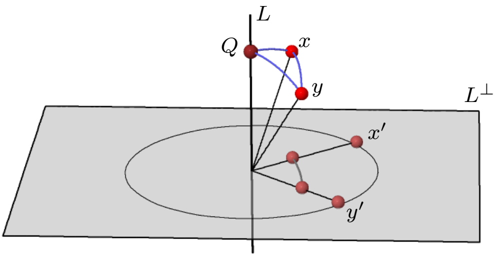

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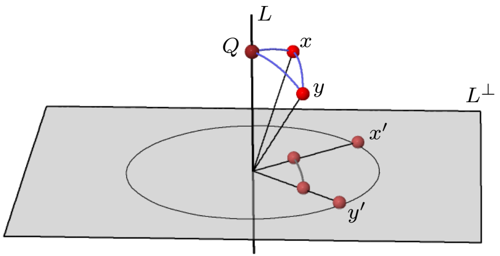

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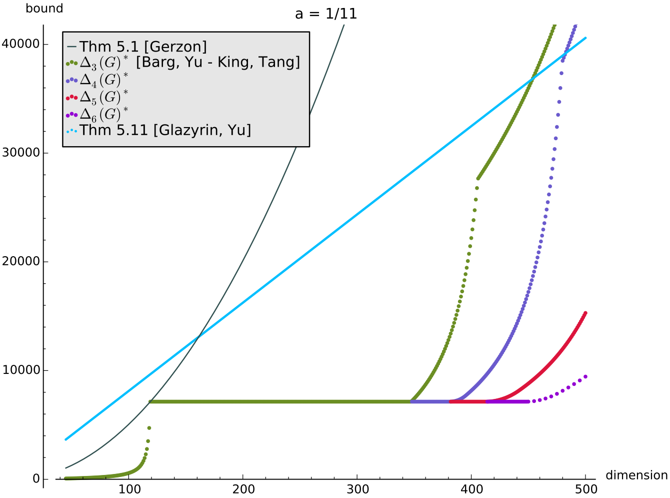
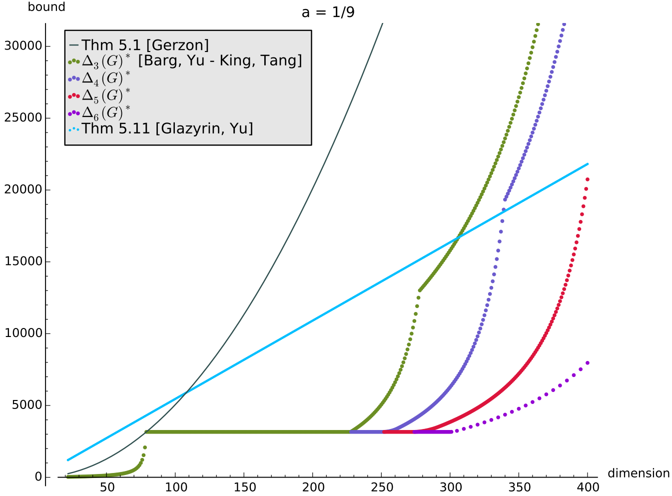
4 Results

- $a = 1/11, 1/9, 1/7, 1/5$

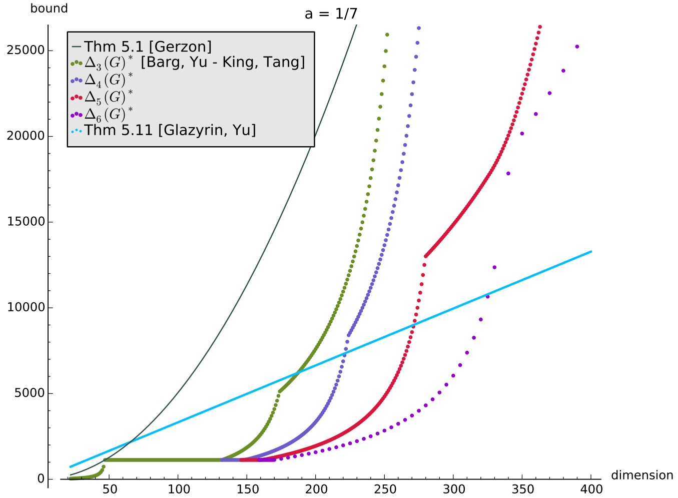
Results $D = \{-1/11, 1/11\}$



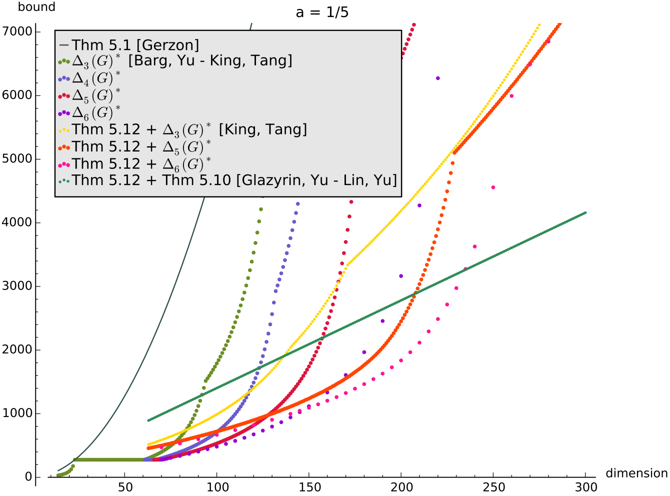
Results $D = \{-1/9, 1/9\}$



Results $D = \{-1/7, 1/7\}$



Results $D = \{-1/5, 1/5\}$



Questions

- What is the smallest n such that $M_a(n) = (1/a^2 - 2)(1/a^2 - 1)/2$? (the "stable range" in the plots)

- What is the smallest n such that $M_a(n) > (1/a^2 - 2)(1/a^2 - 1)/2$?

When $a = 1/5$, Lemmens and Seidel (1973) conjectures that $n > 185$, the SDP bound shows $n > 60$ and we show $n > 70$.

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Thank you for your attention!

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