Improving the Semidefinite Programming Bound for the Kissing Number by Exploiting Polynomial Symmetry

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Abstract

The *kissing number of* $\mathbb{R}^n$ is the maximum number of pairwise-nonoverlapping unit spheres that can simultaneously touch a central unit sphere. Mittelmann and Vallentin (2010), based on the semidefinite programming bound of Bachoc and Vallentin (2008), computed the best known upper bounds for the kissing number for several values of $n \leq 23$. In this work, we exploit the symmetry present in the semidefinite programming bound to provide improved upper bounds for $n = 9, \ldots, 23$.

Introduction

For $x, y \in \mathbb{R}^n$, denote by $x \cdot y = x_1 y_1 + \cdots + x_n y_n$ the Euclidean inner product and let $S^{n-1} = \{\, x \in \mathbb{R}^n : x \cdot x = 1 \,\}$ be the $(n-1)$-dimensional unit sphere. A *spherical code with minimum angular distance* θ is a set $C \subseteq S^{n-1}$ such that $x \cdot y \leq \cos\theta$ for all distinct $x, y \in C$. Determining the parameter

$$A(n,\theta) := \max\{\, |C| : C \subseteq S^{n-1} \text{ and } x \cdot y \leq \cos\theta \text{ for distinct } x, y \in C \,\}$$

is a problem of interest in communication theory. When $\theta = \pi/3$, the quantity $\tau_n := A(n, \pi/3)$ amounts to the maximum number of pairwise-nonoverlapping unit spheres that can simultaneously touch a central unit sphere and is called the kissing number.

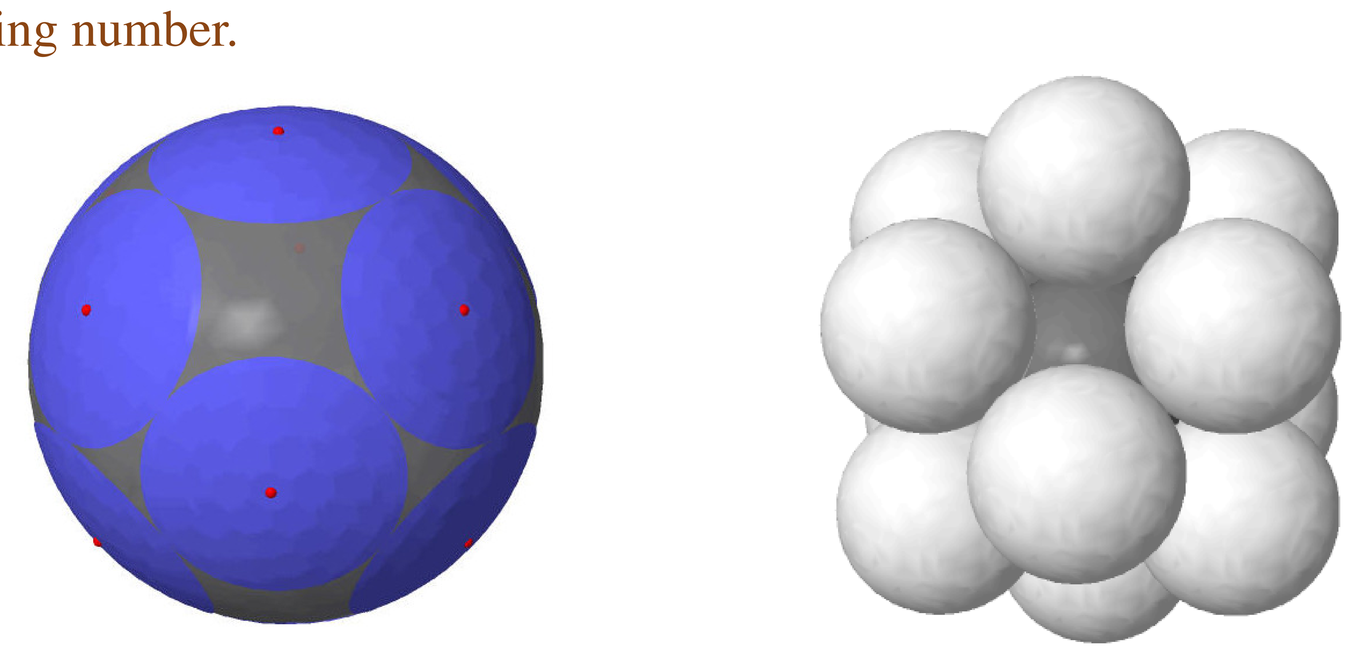

Figure 1: A spherical code with minimum angular distance $\pi/3$ in $\mathbb{R}^3$ and its corresponding kissing configuration.

A proof that $\tau_3 = 12$ appeared only in 1953, given by Schütte and van der Waerden (1953). Delsarte, Goethals, and Seidel (1977) proposed an upper bound for $A(n,\theta)$, known as the linear programming bound, that was later used by Odlyzko and Sloane (1979), and independently Levenshtein (1979), to prove $\tau_8 = 240$ and $\tau_{24} = 196560$. Musin (2008) used a stronger version of this bound to show $\tau_4 = 24$ and Bachoc and Vallentin (2008) strengthened it further via semidefinite programming.

The semidefinite programming bound

For square matrices A, B of the same dimensions, we write $\langle A, B\rangle := \operatorname{tr}(B^{\mathsf T} A)$ and for a square matrix A, we write $A \succeq 0$ to mean that A is positive semidefinite. Let $O(n)$ be the orthogonal group of $\mathbb{R}^n$ and denote by H the stabilizer subgroup of a fixed point $e \in S^{n-1}$ with respect to the action of $O(n)$ in S^{n-1}. Let $\operatorname{Pol}_{\leq d}(S^{n-1})$ denote the space of polynomial functions in $\mathbb{R}^n$ of degree at most d with domain restricted to S^{n-1}.

By considering the action of H onto $\operatorname{Pol}_{\leq d}(S^{n-1})$, Bachoc and Vallentin (2008) computed matrices S_k^n of size $(d-k+1) \times (d-k+1)$ for $0 \leq k \leq d$, whose coefficients are symmetric polynomials of degree at most $2d$ in three variables and with the following property:

$$\text{For all finite } C \subset S^{n-1}, \sum_{a,b,c \in C^3} S_k^n(a\cdot b, a\cdot c, b\cdot c) \succeq 0. \qquad (1)$$

Let Δ be the set of all triples $(u, v, t) \in \mathbb{R}^3$ that are possible inner products between three points in a spherical code of minimum angular distance θ and J denote the all-ones matrix. Using (1), it is possible to prove:

Theorem 1 (Bachoc and Vallentin, 2008)**.** *If $F_0, \ldots, F_d$ are positive semidefinite matrices such that*

$$3\sum_{k=0}^{d} \langle F_k, S_k^n(u,u,1)\rangle \leq -1, \quad \textit{for } u \in [-1, \cos\theta], \textit{ and}$$

$$\sum_{k=0}^{d} \langle F_k, S_k^n(u,v,t)\rangle \leq 0, \quad \textit{for } (u,v,t) \in \Delta,$$

then $A(n,\theta) \leq 1 + \langle F_0, J\rangle$.

Exploiting symmetry

The constraints in Theorem 1 can be read as polynomials that are required to be nonnegative on certain domains. These constraints can be rewritten with sum-of-squares polynomials and semidefinite programming, resulting in a finite semidefinite program that can be solved by a computer. The problem is that this approach leads to matrices indexed by the set of all monomials of degree at most d, which with polynomials in three variables leads to a program with matrices of size $\binom{d+3}{3}$. This limits the value of d that can be used in practice. Mittelmann and Vallentin (2010) used this approach with degree $d = 14$ and computed bounds with a computation time of several weeks.

Since the polynomials occurring in the S_k^n matrices are symmetric, the semidefinite program has many symmetries. Using a method from Gatermann and Parrilo (2004), it is possible to block-diagonalize the matrices needed to represent the sum-of-squares polynomials, leading us to smaller and more stable problems. More specifically, let $\mathbb{R}[u,v,t]_{\leq 2d}$ be the space of polynomials in three variables and of degree at most $2d$, we have:

Theorem 2 (Gatermann and Parrilo, 2004)**.** *For each integer $d > 0$, there are square matrices V_d^{trv}, V_d^{alt}, and V_d^{std}, whose entries are symmetric polynomials in $\mathbb{R}[u,v,t]_{\leq 2d}$, such that a polynomial $p \in \mathbb{R}[u,v,t]_{\leq 2d}$ is symmetric and a sum of squares if and only if there are positive semidefinite matrices Q^{trv}, Q^{alt}, and Q^{std} of appropriate sizes satisfying*

$$p = \langle Q^{\mathrm{trv}}, V_d^{\mathrm{trv}}\rangle + \langle Q^{\mathrm{alt}}, V_d^{\mathrm{alt}}\rangle + \langle Q^{\mathrm{std}}, V_d^{\mathrm{std}}\rangle.$$

If moreover the dimensions of the matrices V_d^{trv}, V_d^{alt}, and V_d^{std} are a, b, and c, respectively, then $\binom{d+3}{3} = a + b + 2c$.

This theorem comes from representation theory applied to the action of the symmetric group $\mathcal{S}_3$ on $\mathbb{R}[u,v,t]_{\leq d}$ and the fact that $\mathcal{S}_3$ has three irreducible representations: the *trivial* and *alternating*, both of dimension one, and the *standard* representation, of dimension two.

Results

The symmetry reduction leads to big improvements in practice. For instance, for $d = 11$ a computation time of 9 days was reduced to less than 12 hours. In this way, it was possible to make computations with $d = 16$ within a computing time of 6 weeks and improve previous upper bounds for the kissing number on dimensions 9 to 23.

By finding a solution with positive *definite* matrices and relatively small numerical error, one can prove that it can be turned into a feasible solution, without changing its objective value. With the help of an interval arithmetic library, this analysis produces a rigorous proof of the bounds obtained.

Main references

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