Applications of harmonic analysis to discrete geometry

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Chapters

1. Harmonic analysis

Harmonic Analysis

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Analysis of function spaces under the action of some group.

Harmonic Analysis

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Analysis of function spaces under the action of some group.

- Let G be a topological group that acts continuously in a topological space X.

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Analysis of function spaces under the action of some group.

- Let G be a topological group that acts continuously in a topological space X.
- We also have an action on a space V of functions $X \to \mathbb{C}$:
$$(L(g)f)(x) = f(g^{-1}x),$$
for $x \in X$, $g \in G$, $f \in V$.

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- Determine how V can be decomposed as a "direct sum" of irreducible subspaces is useful to determine the linear operators $T\colon V \to V$ that commutes with the action:

$$L(g)T = TL(g)$$

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Example $\mathbb{R}/\mathbb{Z}$:

Let $X = \mathbb{R}/\mathbb{Z}$ and $G = \mathbb{R}/\mathbb{Z}$.

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Let $X = \mathbb{R}/\mathbb{Z}$ and $G = \mathbb{R}/\mathbb{Z}$. For $m \in \mathbb{Z}$, the exponential functions

$$\phi_m(x) = e^{2\pi i m x}$$

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Fourier series

$$f(x) \sim \sum_{m \in \mathbb{Z}} \hat{f}_m \phi_m(x), \quad \hat{f}_m = \int_0^1 f(x) e^{-2\pi i \langle m, x \rangle}\, \mathrm{d}x$$

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2. Harmonic Analysis

2 4. k-point semidefinite programming bounds for equiangular lines

k-point semidefinite programming bounds for equiangular lines

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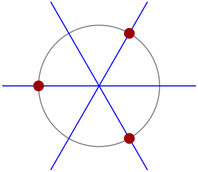
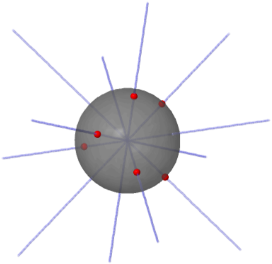
- Joint work with D. de Laat, F.M. de Oliveira Filho, and F. Vallentin.





k-point semidefinite programming bounds for equiangular lines

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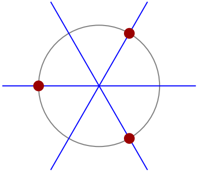
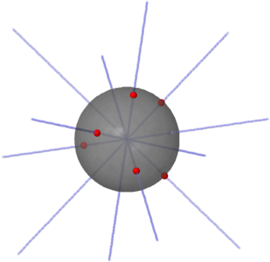
- Derive bounds for the maximum number of equiangular lines in a given dimension and with a given angle.





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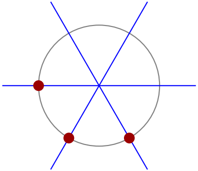

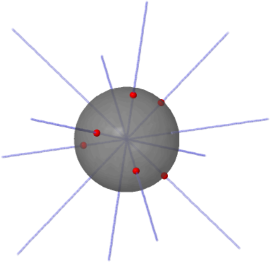

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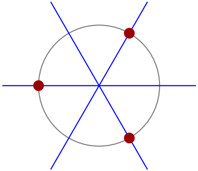

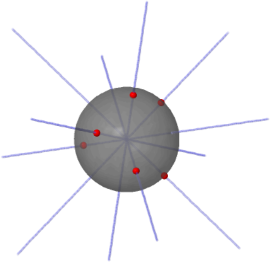

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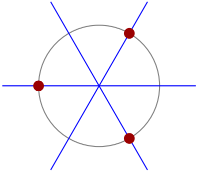
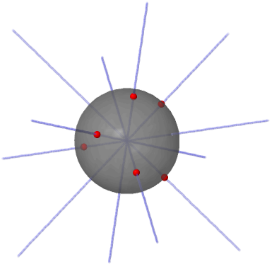
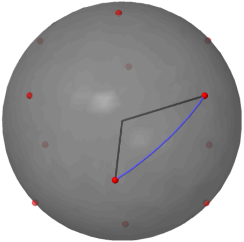
- Derive bounds for the maximum number of equiangular lines in a given dimension and with a given angle.
- We use semidefinite programming and constraints based on k-tuple of points. $(k = 2, \ldots, 6)$
- Domain: S^{d-1}, group: $\mathrm{O}(d)$.





- The problem is a special case of a spherical code problem, which in turn can be classified as a geometrical packing problem.
- Let $D \subseteq [-1, 1)$ be the set of allowable inner products.



$$A(n, D) := \max \left\{ |C| \,:\, C \subseteq S^{n-1}, \quad x \cdot y \in D \text{ for all distinct } x, y \in C \right\}$$

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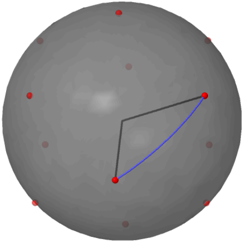

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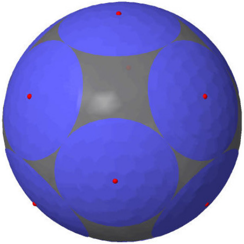

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- Chapter 3 shows how this problem relates with other geometrical packing problems and the similarities between the methods used to deal with them.

3. Packing problems

Summary of main contributions

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- Produced a new sequence $\Delta_k(G)^*$ of semidefinite programming bounds for $A(n, D)$ when D is finite. When $k = 2$ and 3, this bound reduces to known bounds in the literature.
- We computed the bound for $k = 4$, 5, and 6 and found improved bounds for the maximum number of equiangular lines in Euclidean space with a fixed common angle.
- For $k > 4$, this is the first time a k-point bound is computed for a geometric problem.

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Results $D = \{-1/7, 1/7\}$

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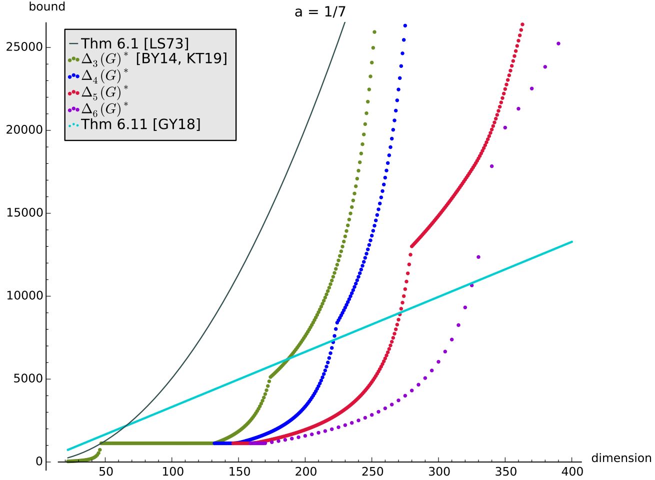

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4. k-point semidefinite programming bounds for equiangular lines

3 6. Coefficients of the solid angle and Ehrhart quasi-polynomials

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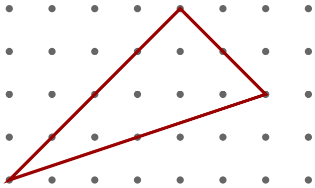

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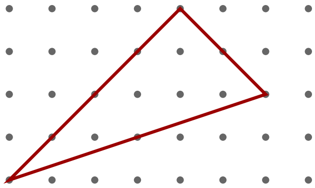

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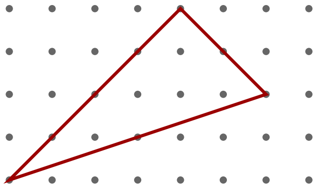
- Domain: $\mathbb{R}^d/\mathbb{Z}^d$, group: $\mathbb{R}^d/\mathbb{Z}^d$.



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The Ehrhart's function $L_P(t)$

$$L_P(t) := |tP \cap \mathbb{Z}^d|, \quad \text{for } t \in \mathbb{Z}$$

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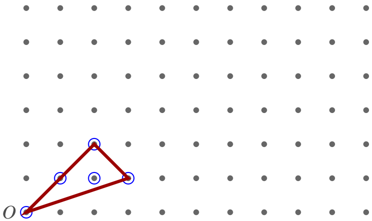

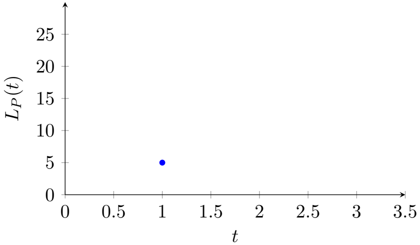

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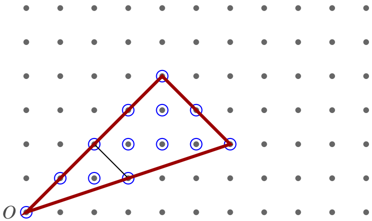

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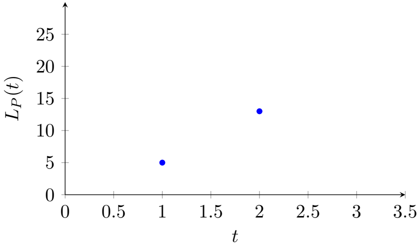

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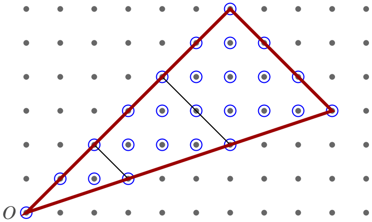

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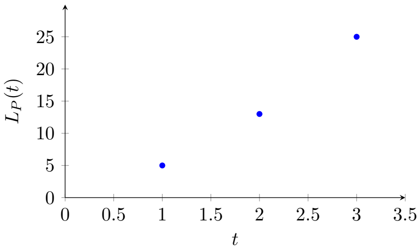

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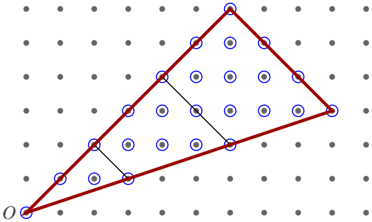

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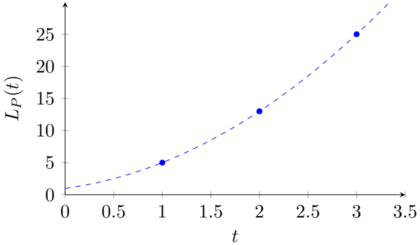

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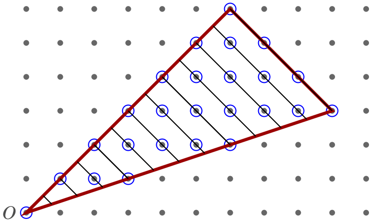

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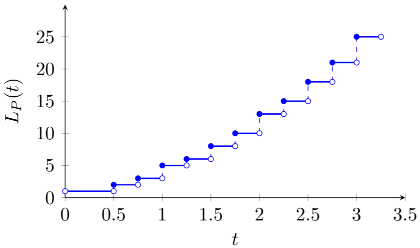

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Ehrhart's theorem

If P is a rational d-dimensional polytope, then $L_P(t)$ is a quasi-polynomial in t of degree d. The period of its quasi-coefficients divides the denominator m of P.

$$L_P(t) = e_d(t)t^d + e_{d-1}(t)t^{d-1} + \cdots + e_0(t),$$

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- $e_k(t+m) = e_k(t)$ for all $0 \leq k \leq d$ and $t \in \mathbb{R}$.

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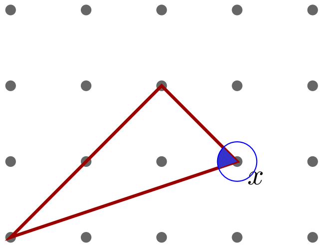
- $e_d(t) = \mathrm{vol}(P)$.
- $e_0(t) = 1$ (for P integer polytope and $t \in \mathbb{Z}$).
- $e_k(t)$ has information from the k-dimensional faces of P.

Solid angle sum $A_P(t)$

Solid angle

$$\omega_P(x) := \lim_{\epsilon \to 0^+} \frac{\mathrm{vol}(S^{d-1}(x,\epsilon) \cap P)}{\mathrm{vol}(S^{d-1}(x,\epsilon))}$$



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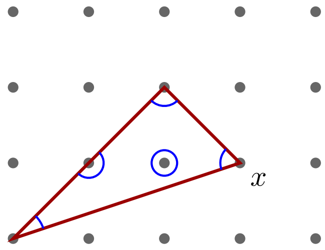

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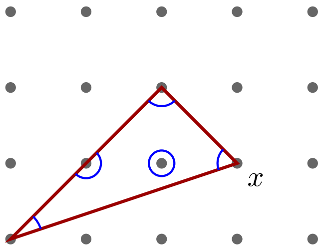

Solid angle sum $A_P(t)$

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Solid angle sum

$$A_P(t) := \sum_{x \in \mathbb{Z}^d} \omega_{tP}(x) = a_d(t)t^d + a_{d-1}(t)t^{d-1} + \cdots + a_0(t).$$

- $a_k(t+m) = a_k(t)$ for all $0 \leq k \leq d$ and $t \in \mathbb{R}$.

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- $a_{d-1}(t) = a_{d-3}(t) = \cdots = 0$ (for P integer polytope and $t \in \mathbb{Z}$).

The Fourier transform of a polytope

- For any compact set $\Omega \subset \mathbb{R}^d$, its *Fourier transform* is the function

$$\hat{\mathbb{1}}_\Omega(\xi) = \int_\Omega e^{-2\pi i \langle x, \xi \rangle} \, \mathrm{d}x.$$

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- We consider $\hat{\mathbb{1}}_P$ for polytopes since we have special tools to evaluate it (the divergence theorem and Brion's theorem).
- One application of $\hat{\mathbb{1}}_P$ is in the determination of $L_P(t)$ and $A_P(t)$.

- $$L_P(t) = |tP \cap \mathbb{Z}^d| = \sum_{n \in \mathbb{Z}^d} \mathbb{1}_{tP}(n)$$

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Theorem (M., Robins 2019^+)

Let $P \subset \mathbb{R}^d$ be a rational polytope. Then the codimension two quasi-coefficient of the solid angle sum $A_P(t)$ has the following closed form for any positive real t:

$$a_{d-2}(t) = \sum_{\substack{G \subset P, \\ \dim G = d-2}} \mathrm{vol}^*(G) \left[\frac{c_G}{2k} \left(\frac{\|v_{F_2}\|}{\|v_{F_1}\|} \overline{B}_2(\langle v_{F_1}, \bar{x}_G \rangle t) + \frac{\|v_{F_1}\|}{\|v_{F_2}\|} \overline{B}_2(\langle v_{F_2}, \bar{x}_G \rangle t) \right) + \left(\omega_P(G) - \frac{1}{4} \right) \mathbb{1}_{\Lambda_G^*}(t\bar{x}_G) - s\big(h, k; (x_1 + hx_2)t, -kx_2 t\big) \right].$$

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$\omega_P(G), \Lambda_G^*, c_G, v_{F_1}, v_{F_2}, h, k, \bar{x}_G, x_1, x_2$ are all "local geometric information" of the codimension two face G of P.

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Theorem (M., Robins 2019^{+})

Let $P \subset \mathbb{R}^d$ be an integer polytope. Then for integer dilations, the codimension two coefficient of the of the solid angle sum $A_P(t)$ has the following local formula:

$$a_{d-2} = \sum_{\substack{G \subset P, \\ \dim G = d-2}} \mathrm{vol}^*(G) \left[\frac{c_G}{12k} \left(\frac{\|v_{F_1}\|}{\|v_{F_2}\|} + \frac{\|v_{F_2}\|}{\|v_{F_1}\|} \right) + \omega_P(G) - \frac{1}{4} - s(h,k) \right].$$

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Therefore, if P is an integer polytope in $\mathbb{R}^d$ with $d = 3$ or 4, its solid angle sum for integer dilations is:

$$A_P(t) = \mathrm{vol}(P) t^d + a_{d-2} t^{d-2}.$$

Obtaining the Ehrhart quasi-coefficients

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Lemma (M., Robins 2019^+)

Let $P \subset \mathbb{R}^d$ be a d-dimensional rational polytope and $a \in \operatorname{int}(P)$ be a rational vector. Then for any positive real t,

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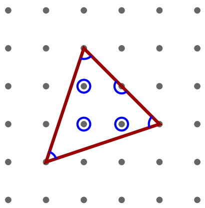

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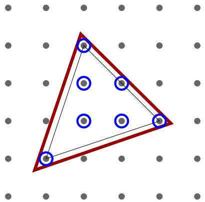

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- $a_{d-1}(t) \to e_{d-1}(t)$,
- $a_{d-2}(t) \to e_{d-2}(t)$.

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6. Coefficients of the solid angle and Ehrhart quasi-polynomials

4 7. The null set of a polytope and the Pompeiu property for polytopes

The null set of a polytope and the Pompeiu property for polytopes

- Joint work with S. Robins.

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- Let $\Omega \subset \mathbb{R}^d$ be a compact set. The Fourier transform of Ω is the function $\hat{\mathbb{1}}_\Omega \colon \mathbb{C}^d \to \mathbb{C}$:

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- The null set of Ω is $N(\Omega) = \{\xi \in \mathbb{C}^d : \hat{\mathbb{1}}_\Omega(\xi) = 0\}$.
- Using an explicit form for the Fourier transform of a polytope (Brion's theorem), we give a simple proof that polytopes have the Pompeiu property.

Pompeiu property

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Let $\Omega \subset \mathbb{R}^d$ be a bounded set with nonempty interior. Ω has the Pompeiu property if, for $f \in \mathcal{C}(\mathbb{R}^d)$,

$$\int_{\sigma(\Omega)} f(x)\,\mathrm{d}x = 0.$$

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The group $M(d)$ of rigid motions in $\mathbb{R}^d$ is the group generated by all translations and rotations.

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Equivalently, Ω has the Pompeiu property if the values

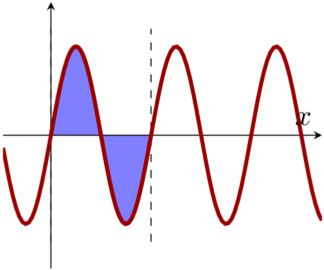
$$\int_{\sigma(\Omega)} f(x)\,\mathrm{d}x$$

over all rigid motions $\sigma \in M(d)$ uniquely determine $f \in \mathcal{C}(\mathbb{R}^d)$.

$d = 1$

An interval does not have the Pompeiu property:

$$\int_{c}^{c+L} \sin\left(\frac{2\pi}{L}x\right) \mathrm{d}x = 0$$



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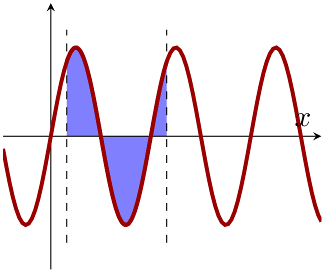

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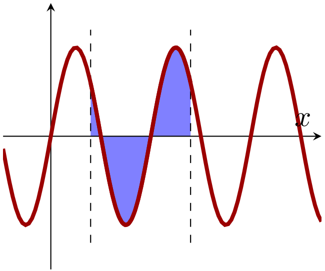

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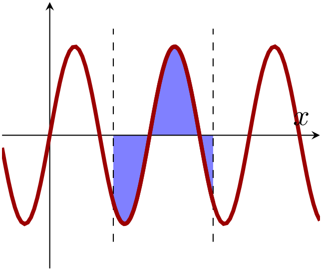

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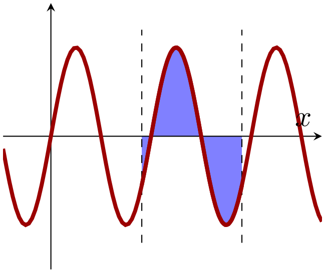

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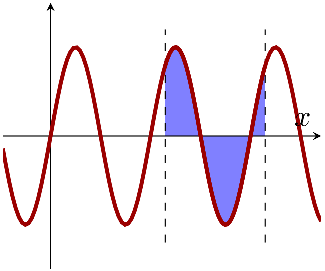

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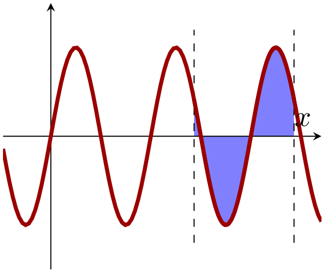
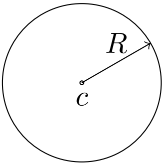


$d \geq 2$

A ball does not have the Pompeiu property. If R is the radius of the ball and a is such that $J_{d/2}(aR) = 0$, then

$$\int_{\|x-c\| \leq R} \sin(ax_1)\,\mathrm{d}x = 0.$$



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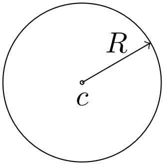
$$\int_{\|x\|\leq R} e^{2\pi i\langle \xi, x\rangle}\,\mathrm{d}x = \left(\frac{R}{\|\xi\|}\right)^{d/2} J_{d/2}(2\pi R\|\xi\|).$$



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Theorem (M., Robins)

Let $P \subset \mathbb{R}^d$ be a d-dimensional polytope, $H \subset \mathbb{R}^d$ be a 2-dimensional real subspace that is not orthogonal to any edge from P, and fix an orthonormal basis $\{e, f\} \subset \mathbb{R}^d$ for H. Then

$$\left\{\alpha(\cos t)e + \alpha(\sin t)f \in \mathbb{C}^d \;:\; t \in [-\pi, \pi]\right\} \not\subset N(P)$$

for any $\alpha \in \mathbb{C} \setminus \{0\}$.

Example

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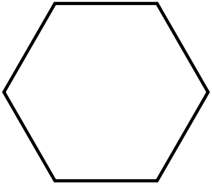
Let $P \subset \mathbb{R}^2$ be an hexagon,



Example

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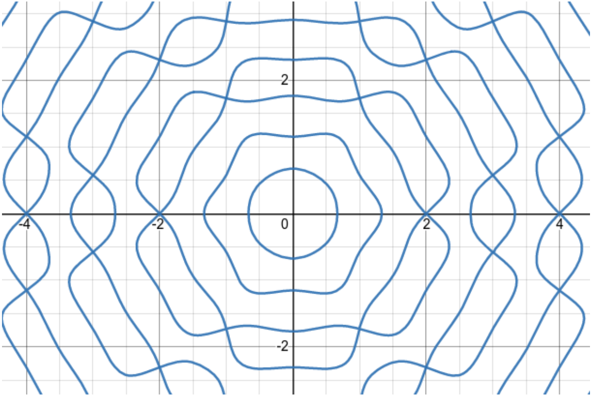
In blue is $N(P) \cap \mathbb{R}^2$



Example

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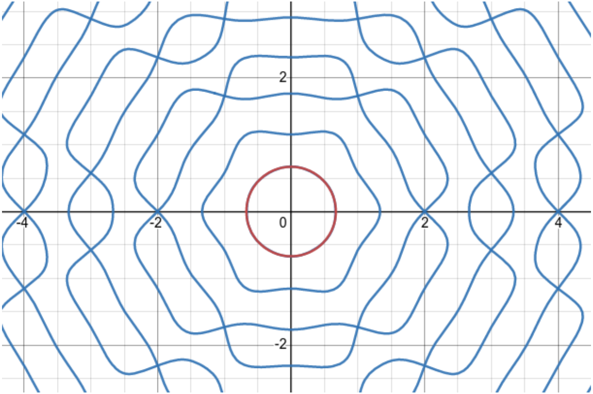
In blue is $N(P) \cap \mathbb{R}^2$ and in red is a circle.



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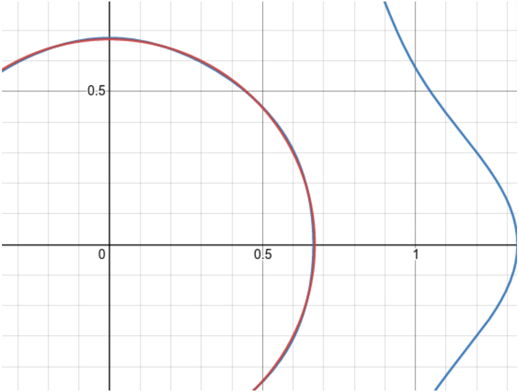

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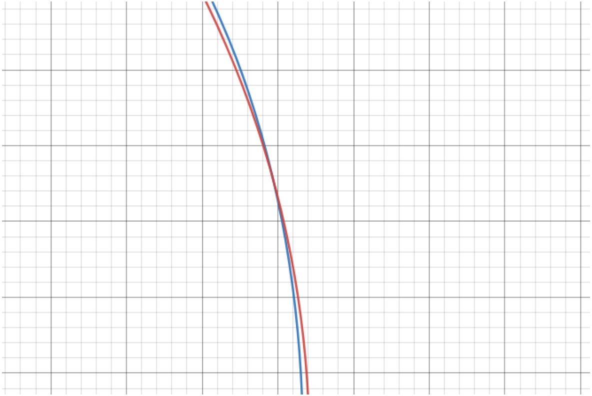

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7. The null set of a polytope and the Pompeiu property for polytopes

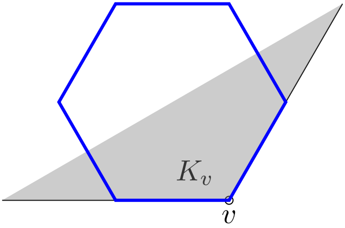
For each facet F of P, let n_F be the outer unit normal vector along F and $\mathrm{d}_F x$ denote the surface integral along F. Applying the divergence theorem to the vector field $\xi e^{-2\pi i\langle x, \xi\rangle}$, we get:

Combinatorial Stokes formula

$$\int_P e^{-2\pi i\langle x, \xi\rangle}\,\mathrm{d}x = \frac{-1}{2\pi i} \sum_{\substack{F\in\mathcal{F}(P)\\ \dim(F)=d-1}} \frac{\langle \xi, n_F\rangle}{\|\xi\|^2} \int_F e^{-2\pi i\langle x, \xi\rangle}\,\mathrm{d}_F x$$

If ξ is not orthogonal to F, we may interate the process along the lower dimensional faces of P.

Let $P \subset \mathbb{R}^d$ be a d-dimensional polytope. For each $v \in V(P)$, let K_v be the tangent cone of P at v and $K_{v,1}, \ldots, K_{v,M_v}$ be a triangulation of K_v into simplicial cones with no new edges. For each $1 \leq j \leq M_v$, let $w^v_{j,1}, \ldots, w^v_{j,d}$ be the edges of $K_{v,j}$.



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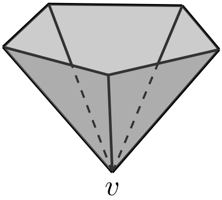

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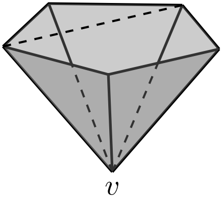

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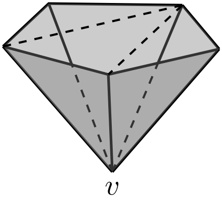

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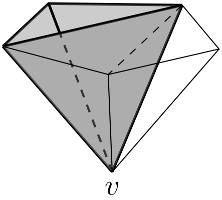

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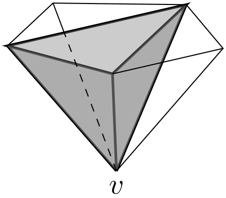

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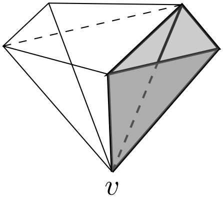

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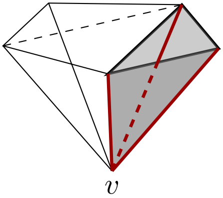
Let $P \subset \mathbb{R}^d$ be a d-dimensional polytope. For each $v \in V(P)$, let K_v be the tangent cone of P at v and $K_{v,1}, \ldots, K_{v,M_v}$ be a triangulation of K_v into simplicial cones with no new edges. For each $1 \leq j \leq M_v$, let $w_{j,1}^v, \ldots, w_{j,d}^v$ be the edges of $K_{v,j}$.



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Brion's theorem

If $P \subset \mathbb{R}^d$ is a d-dimensional polytope and for each $v \in V(P)$, $K_{v,1}, \ldots, K_{v,M_v}$ are simplicial cones with disjoint interiors such that $\mathrm{tcone}(P, v) = \bigcup_{j=1}^{M_v} K_{v,j}$ and for each $1 \leq j \leq M_v$, $w_{j,1}^v, \ldots, w_{j,d}^v$ are the generators of $K_{v,j}$. Then

$$\hat{\mathbb{1}}_P(\xi) = \sum_{v \in V(P)} \sum_{j=1}^{M_v} \frac{e^{-2\pi i \langle v, \xi \rangle}}{(2\pi i)^d} \frac{\det(w_{j,1}^v, \ldots, w_{j,d}^v)}{\langle w_{j,1}^v, \xi \rangle \ldots \langle w_{j,d}^v, \xi \rangle}.$$

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5. The Fourier transform of a polytope

5.1 Combinatorial Stokes Formula

5.2 The integral and exponential sum valuations

Thank you for your attention!

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Fourier transforms of polytopes (Diaz,Le, and Robins 2016^+)

$$L_P(t) = \sum_{n \in \mathbb{Z}^d} \mathbb{1}_{tP}(n) = \sum_{\xi \in \mathbb{Z}^d} \hat{\mathbb{1}}_{tP}(\xi) = t^d \sum_{\xi \in \mathbb{Z}^d} \int_P e^{-2\pi i \langle t\xi, x \rangle} \, \mathrm{d}x$$

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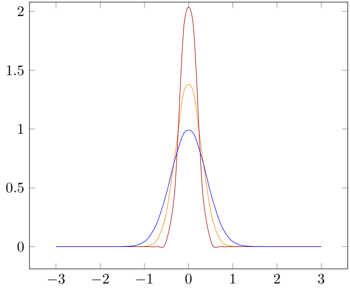
$$\phi_{d,\epsilon}(x) := \epsilon^{-d/2} e^{-\pi\|x\|^2/\epsilon}$$

$$\begin{aligned}\omega_P(n) &= \lim_{\epsilon\to 0^+} (\mathbb{1}_P * \phi_{d,\epsilon})(n)\\ &= \lim_{\epsilon\to 0^+} \int_P \phi_{d,\epsilon}(y-n)\,\mathrm{d}y.\end{aligned}$$



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Fourier transforms of polytopes (Diaz,Le, and Robins 2016^+)

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\begin{aligned}
A_P(t) &= \sum_{n\in\mathbb{Z}^d} \omega_{tP}(n) = \sum_{n\in\mathbb{Z}^d} \lim_{\epsilon\to 0^+} (\mathbb{1}_{tP} * \phi_{d,\epsilon})(n) \\
&= \lim_{\epsilon\to 0^+} \sum_{n\in\mathbb{Z}^d} (\mathbb{1}_{tP} * \phi_{d,\epsilon})(n) = \lim_{\epsilon\to 0^+} \sum_{\xi\in\mathbb{Z}^d} \hat{\mathbb{1}}_{tP}(\xi)\hat{\phi}_{d,\epsilon}(\xi)
\end{aligned}
$$

Fourier transforms of polytopes (Diaz,Le, and Robins 2016^+)

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&= \lim_{\epsilon\to 0^+} \sum_{\xi\in\mathbb{Z}^d} \hat{\phi}_{d,\epsilon}(\xi) \int_{tP} e^{-2\pi i\langle \xi, x\rangle}\,\mathrm{d}x
\end{aligned}
$$

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$$
\begin{aligned}
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&= t^d \mathrm{vol}(P) + t^{d-1} \lim_{\epsilon\to 0^+} \sum_{\xi\in\mathbb{Z}^d\setminus\{0\}} \hat{\phi}_{d,\epsilon}(\xi) \sum_{\substack{F\subseteq P\\ \dim(F)=d-1}} \frac{\langle \xi, N_P(F)\rangle}{-2\pi i\|\xi\|^2} \int_F e^{-2\pi i\langle t\xi, x\rangle}\,\mathrm{d}x
\end{aligned}
$$

Theorem (Diaz, Le, and Robins 2016^+)

Let P be a d-dimensional rational polytope in $\mathbb{R}^d$, and t be a positive real number. Then we have $A_P(t) = \sum_{k=0}^{d} a_k(t)t^k$, where, for $0 \leq k \leq d$,

$$a_{d-k}(t) = \lim_{\epsilon \to 0^+} \sum_{T:\, l(T)=k} \sum_{\xi \in \mathbb{Z}^d \cap S(T)} \mathcal{R}_T(\xi) e^{-2\pi i \langle t\xi, x_{F_k} \rangle} \hat{\phi}_{d,\epsilon}(\xi),$$

where the first sum is over all chains $T = (P \to F_1 \to \cdots \to F_k)$ with F_j a facet of F_{j-1} for every j and $S(T)$ is the set of vectors ξ orthogonal to $\operatorname{lin}(F_k)$ but not to $\operatorname{lin}(F_{k-1})$.

Lattice sums — Definition

Let Λ be a k-dimensional lattice in $\mathbb{R}^d$, $w_1, \ldots, w_k$ be linearly independent vectors from

$$\Lambda^* := \{y \in \operatorname{span}(\Lambda) : \langle x,\, y\rangle \in \mathbb{Z} \text{ for all } x \in \Lambda\}$$

and $W \in \mathbb{R}^{d\times k}$ be a matrix with them as columns. For a k-tuple $e = (e_1, \ldots, e_k)$ of positive integers, let $|e| := \sum_{j=1}^k e_j$. For all $x \in \mathbb{R}^d$, we want to evaluate:

$$L_\Lambda(W, e; x) := \lim_{\epsilon\to 0^+} \frac{1}{(2\pi i)^{|e|}} \sum_{\substack{\xi\in\Lambda: \\ \langle w_j, \xi\rangle \neq 0, \forall j}} \frac{e^{-2\pi i\langle x, \xi\rangle}}{\prod_{j=1}^k \langle w_j,\, \xi\rangle^{e_j}} e^{-\pi\epsilon\|\xi\|^2}.$$

Lattice sums — Theorem

Theorem (M., Robins 2019^+)

If $W \in \mathbb{R}^{d\times k}$ is a matrix with linearly independent columns $w_1, \ldots, w_k \in \Lambda^*$, $e = (e_1, \ldots, e_k)$ is a k-tuple of positive integers and $x \in \mathbb{R}^d$, then:

$$L_\Lambda(W, e; x) = \sum_{n \in \Lambda^* \cap P_{W,x}} \frac{(-1)^k \mathcal{B}_e\big(W^+(n-x)\big)\omega_{P_{W,x}}(n)}{e_1! \cdots e_k! \det(W^\mathsf{T} W)^{1/2} \det(\Lambda)}.$$

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$B_r(x)$ are the Bernoulli polynomials with support in $[0, 1]$.
$B_1(x) = x - 1/2$, $B_2(x) = x^2 - x + 1/6$ for $x \in [0, 1]$.

$$\mathcal{B}_e(x) := B_{e_1}(x_1)\cdots B_{e_k}(x_k).$$

Theorem (M., Robins 2019^+)

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