Semidefinite programming bounds for the kissing number

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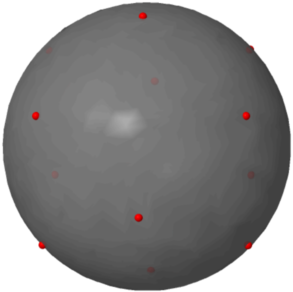
Spherical codes

$x \cdot y := \sum_{i=1}^{n} x_i y_i$

$S^{n-1} := \{x \in \mathbb{R}^n : x \cdot x = 1\}$

$d(x, y) := \arccos(x \cdot y)$

$A(n, \theta) := \max\{|C| : C \subset S^{n-1},$
$d(x, y) \geq \theta \text{ for } x, y \in C, x \neq y\}$



Find upper bounds for the size of spherical codes with minimum angular distance θ.

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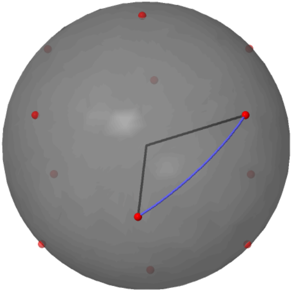

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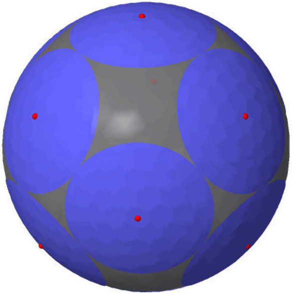
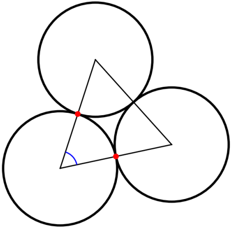
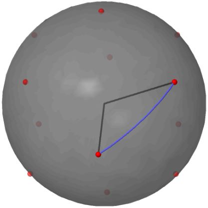
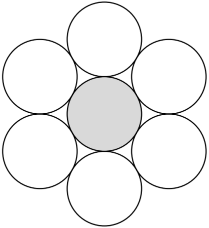


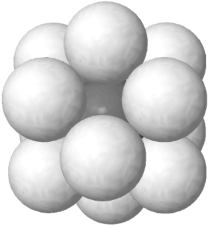
Find upper bounds for the size of spherical codes with minimum angular distance θ.

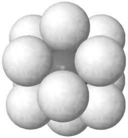
The kissing number (case $\theta = \pi/3$)











n	lower bound	upper bound
3	**12**	**12**
4	**24**	**24**
5	40	44
6	72	78
7	126	134
8	**240**	**240**
9	306	363
10	500	553
11	582	869
12	840	1356
13	1154	2066

n	lower bound	upper bound
14	1606	3177
15	2564	4866
16	4320	7355
17	5346	11014
18	7398	16469
19	10668	24575
20	17400	36402
21	27720	53878
22	49896	81376
23	93150	123328
24	**196560**	**196560**

Table 1. Lower and upper bounds for the kissing number $A(n, \pi/3)$ in dimensions $3, \ldots, 24$.

Contents

1. Preliminaries
   - Representation theory
   - Continuous, positive and invariant kernels
   - Conic programming

Basic definitions from representation theory

A *representation* of a group G on a vector space V is a homomorphism $\rho\colon G \to \mathrm{GL}(V)$.

A *G-homomorphism* between two representations $\rho\colon G \to \mathrm{GL}(V)$ and $\tau\colon G \to \mathrm{GL}(W)$ is a homomorphism that commutes with the representations. That is, $T\colon V \to W$ linear such that

$$T\rho(g) = \tau(g)T$$

for every $g \in G$. If T is invertible, the representations are said to be *equivalent*.

Basic definitions from representation theory

We say that a representation space V is *irreducible* if there is no proper and nonzero invariant subspace W of V (in the sense that $\rho(g)w \in W$ for every $w \in W$ and $g \in G$).

Every finite dimensional representation space V can be written as a direct sum of invariant and irreducible subspaces.

$$V = V_1 \oplus \cdots \oplus V_m.$$

Basic definitions from representation theory

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The main result of this section is that a G-homomorphism $T\colon V \to V$ can be block-diagonalized with respect to the way that V decomposes as a direct sum of invariant and irreducible subspaces.

Continuous and positive kernels

Consider a compact Hausdorff space X with a positive Radon measure ω. The vector space $L^2(X)$ of square-integrable functions on X is a Hilbert space with inner product

$$(f, g) := \int_X f(x)\overline{g(x)}\, \mathrm{d}\omega(x).$$

A *(Hilbert-Schmidt) kernel* is a function $K \in L^2(X \times X)$. To a kernel is associated the integral operator $T_K \colon L^2(X) \to L^2(X)$:

$$(T_K f)(x) := \int_X K(x, y) f(y)\, \mathrm{d}\omega(y).$$

Continuous and positive kernels

A hermitian kernel (i.e. such that $K(x,y) = \overline{K(y,x)}$ for every $x, y \in X$) is *positive* if for every $f \in L^2(X)$ we have

$$(T_K f, f) \geq 0.$$

When a kernel is continuous, the following result is very useful:

A continuous and hermitian kernel K is positive if and only if for every finite subset $\{x_1, \ldots, x_N\}$ of X, the matrix $\big(K(x_i, x_j)\big)_{i,j=1}^{N}$ is positive semidefinite.

Invariant kernels

Consider that there is a compact group G which acts continuously on X and that ω is G-invariant (i.e., $\omega(gA) = \omega(A)$ for every $g \in G$ and $A \subset X$ measurable).

A continuous kernel K is *invariant* if $K(gx, gy) = K(x, y)$ for any $x, y \in X$ and $g \in G$. It can be verified that this condition is equivalent to that the associated operator T_k is a G-homomorphism with respect the representation of G in $L^2(X)$ defined by

$$(\rho(g)f)(x) = f(g^{-1}x).$$

Invariant kernels

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It is possible to represent continuous, positive and invariant kernels using positive semidefinite matrices and the block diagonalization of G-homomorphisms given by the representation theory.

Conic programming

A *duality* between two real vector spaces E and F is a non-degenerate bilinear form $\langle , \rangle \colon E \times F \to \mathbb{R}$ (non-degenerate means that $\langle e, f \rangle = 0$ for every $e \in E$ implies $f = 0$ and $\langle e, f \rangle = 0$ for every $f \in F$ implies $e = 0$).

Let $\langle , \rangle_1 \colon E_1 \times F_1 \to \mathbb{R}$ and $\langle , \rangle_2 \colon E_2 \times F_2 \to \mathbb{R}$ be two dualities. The *adjoint* of a linear transformation $A \colon E_1 \to E_2$ is a linear transformation $A^* \colon F_2 \to F_1$ such that

$$\langle Ae, f \rangle_2 = \langle e, A^* f \rangle_1 \quad \text{for all } e \in E_1,\ f \in F_2.$$

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A subset K of a vector space E is a *cone* if $\alpha x + \beta y \in K$ for every $\alpha, \beta \geq 0$ and $x, y \in K$. If $\langle , \rangle \colon E \times F \to \mathbb{R}$ is a duality, then the *dual cone* $K^* \subset F$ is defined as

$$K^* := \{y \in F : \langle x, y \rangle \geq 0 \text{ for all } x \in K\}.$$

Pair of problems primal and dual

Let $\langle , \rangle_1 \colon E_1 \times F_1 \to \mathbb{R}$ and $\langle , \rangle_2 \colon E_2 \times F_2 \to \mathbb{R}$ be two dualities, $K_1 \subset E_1$, $K_2 \subset E_2$ two convex cones and $A \colon E_1 \to E_2$ a linear transformation. For $c \in F_1$ and $b \in E_2$, we have:

Primal problem

$$\begin{aligned} \max \quad & \langle x, c \rangle_1 \\ \text{subject to} \quad & b - Ax \in K_2, \\ & x \in K_1. \end{aligned}$$

Dual problem

$$\begin{aligned} \min \quad & \langle b, y \rangle_2 \\ \text{subject to} \quad & A^* y - c \in K_1^*, \\ & y \in K_2^*. \end{aligned}$$

2. Bounds for the kissing number
   - The linear programming bound
   - The semidefinite programming bound
   - Polynomial optimization with sum of squares and symmetries
   - Application to the semidefinite programming bound

The linear programming bound

- The linear programming bound was proposed by Delsarte, Goethals and Seidel (1977) and used by Odlyzko and Sloane (1979) and, independently, Levenshtein (1979) to derive upper bounds for the kissing number in $n \leq 24$.
- It solves the cases $A(8, \pi/3) = 240$ and $A(24, \pi/3) = 196560$.
- It relies on the following property satisfied by the Gegenbauer polynomials P_k^n:

$$\sum_{x,y \in C} P_k^n(x \cdot y) \geq 0, \quad \text{for every } C \subset S^{n-1} \text{ finite.}$$

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Continuous, positive and $\mathrm{O}(\mathbb{R}^n)$-invariant kernels on S^{n-1}

The space $\mathrm{Pol}_{\leq d}(S^{n-1})$ decomposes as a direct sum of invariant and irreducible subspaces:

$$\mathrm{Pol}_{\leq d}(S^{n-1}) = H_0^n \perp H_1^n \perp \cdots \perp H_d^n,$$

where

$$H_k^n := \left\{ f \in \mathrm{Pol}_{\leq d}(S^{n-1}) : f \text{ homogeneous}, \deg f = k,\ \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} f = 0 \right\}$$

is the space of *spherical harmonic polynomials of degree* k.

Gegenbauer polynomials

Addition formula

If $\{R_1, \ldots, R_{h_k^n}\}$ is an orthonormal basis of H_k^n, then for any $x, y \in S^{n-1}$:

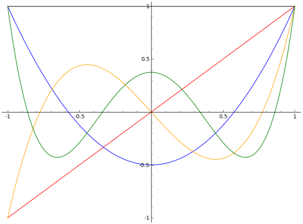
$$P_k^n(x \cdot y) = \frac{1}{h_k^n} \sum_{i=1}^{h_k^n} \overline{R_i(x)} R_i(y).$$

P_k^n is a univariate polynomial of degree k such that $P_k^n(1) = 1$.

Orthogonality relation:

$$\int_{-1}^{1} P_k^n(t) P_l^n(t)(1-t^2)^{(n-3)/2} dt = 0,$$

if $k \neq l$.



$P_0^3(x)$	1
$P_1^3(x)$	x
$P_2^3(x)$	$3/2x^2 - 1/2$
$P_3^3(x)$	$5/2x^3 - 3/2x$
$P_4^3(x)$	$35/8x^4 - 15/4x^2 + 3/8$

Table 2. Gegenbauer polynomials, $n = 3$

The linear programming bound

If $a_1, \ldots, a_d$ is a feasible solution for the following program:

$$\begin{array}{lll} \text{min} & 1 + \sum_{k=1}^{d} a_k & \\ \text{subject to} & \sum_{k=1}^{d} a_k P_k^n(t) \leq -1 & \text{for } t \in [-1, \cos\theta], \\ & a_k \geq 0 & \text{for } k = 1, 2, \ldots, d. \end{array}$$

Then $A(n,\theta) \leq 1 + \sum_{k=1}^{d} a_k$.

$$\begin{aligned} \text{min} \quad & 1 + \sum_{k=1}^{d} a_k \\ \text{s. t.} \quad & \sum_{k=1}^{d} a_k P_k^n(t) \leq -1 && \text{for } t \in [-1, \cos\theta], \\ & a_k \geq 0 && \text{for } k = 1, 2, \ldots, d. \end{aligned}$$

If $a_1, \ldots, a_d$ is a feasible solution for the program, then

$$A(n, \theta) \leq 1 + \sum_{k=1}^{d} a_k.$$

If $C \subset S^{n-1}$, $C \neq \emptyset$, is a spherical code of minimum angular distance θ,

$$\sum_{x,y \in C} \left(1 + \sum_{k=1}^{d} a_k P_k^n(x \cdot y)\right) = |C|^2 + \sum_{k=1}^{d} a_k \sum_{x,y \in C} P_k^n(x \cdot y) \geq |C|^2.$$

$$\sum_{x,y \in C} \left(1 + \sum_{k=1}^{d} a_k P_k^n(x \cdot y)\right) = \sum_{x \in C} \left(1 + \sum_{k=1}^{d} a_k P_k^n(x \cdot x)\right) + \sum_{\substack{x,y \in C, \\ x \neq y}} \left(1 + \sum_{k=1}^{d} a_k P_k^n(x \cdot y)\right)$$

$$\leq |C| \left(1 + \sum_{k=1}^{d} a_k\right).$$

The semidefinite programming bound

- The semidefinite programming bound was proposed by Bachoc and Vallentin (2008) and is responsible for all the upper bounds currently known for the kissing number.
- It improves the linear programming bound with constraints based on relations between triples of points.
- It relies on the following property satisfied by the zonal matrices S_k^n whose coefficients are symmetric polynomials in three variables:

$$\sum_{x,y,z \in C} S_k^n(x \cdot z, y \cdot z, x \cdot y) \succeq 0, \text{ for every } C \subset S^{n-1} \text{ finite.}$$

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Continuous positive and H-invariant kernels on S^{n-1}

Fix one point $e \in S^{n-1}$ and consider the subgroup H that stabilizes e under the action of $\mathrm{O}(\mathbb{R}^n)$:

$$H := \{T \in \mathrm{O}(\mathbb{R}^n) : Te = e\},$$

writing $\mathbb{R}^n = \mathbb{R}^{n-1} \perp \mathbb{R}e$, we have that $H \simeq \mathrm{O}(\mathbb{R}^{n-1})$.

The spaces H_k^n are not irreducible with respect to the action of the subgroup H. They can be decomposed as a direct sum of smaller invariant and irreducible subspaces:

$$H_k^n = H_{0,k}^{n-1} \perp H_{1,k}^{n-1} \perp \cdots \perp H_{k,k}^{n-1},$$

where the representation of H in $H_{i,k}^{n-1}$ is equivalent to the representation of $\mathrm{O}(\mathbb{R}^{n-1})$ in H_i^{n-1}.

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$$\begin{array}{rcccccccc}
Pol_{\leq d}(S^{n-1}) & = & H_0^n & \perp & H_1^n & \perp & \cdots & \perp & H_d^n \\
 & = & H_{0,0}^{n-1} & \perp & H_{0,1}^{n-1} & \perp & \cdots & \perp & H_{0,d}^{n-1} \\
 & & & \perp & H_{1,1}^{n-1} & \perp & \cdots & \perp & H_{1,d}^{n-1} \\
 & & & & & \perp & \cdots & \perp & \cdots \\
 & & & & & & & \perp & H_{d,d}^{n-1}
\end{array}$$

The representations of H in the subspaces of a same line are equivalent to each other and using an expression similar to the addition formula, one can define the matrices S_k^n of dimension $(d-k+1) \times (d-k+1)$ for each line $k = 0, \ldots, d$.

The semidefinite programming bound

$$\begin{aligned}
\text{min} \quad & 1 + \sum_{k=1}^{d} a_k + b_{11} + \langle F_0, J_{d+1} \rangle \\
\text{subject to} \quad & \text{(a)} \ \sum_{k=1}^{d} a_k P_k^n(u) + 2b_{12} + b_{22} \\
& \qquad + 3 \sum_{k=0}^{d} \langle F_k, S_k^n(u,u,1) \rangle \leq -1 \quad \text{for } u \in [-1, \cos\theta], \\
& \text{(b)} \ b_{22} + \sum_{k=0}^{d} \langle F_k, S_k^n(u,v,t) \rangle \leq 0 \qquad \text{for } (u,v,t) \in \Delta, \\
& a_k \geq 0 \qquad \text{for } k = 1, \ldots, d, \\
& B = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix} \succeq 0, \\
& F_k \in \mathbb{R}^{(d-k+1)\times(d-k+1)}, \quad F_k \succeq 0 \qquad \text{for } k = 0, \ldots, d.
\end{aligned}$$

The semidefinite programming bound

Where

$$\Delta = \{\, (u, v, t) \in \mathbb{R}^3 : g_i(u, v, t) \geq 0 \text{ for } i = 1, \ldots, 4\},$$

with

$$g(u) := (u + 1)(\cos\theta - u),$$
$$g_1(u, v, t) := g(u), \quad g_2(u, v, t) := g(v),$$
$$g_3(u, v, t) := g(t), \quad g_4(u, v, t) := 1 + 2uvt - u^2 - v^2 - t^2.$$

Polynomial optimization with sum of squares

Write a real polynomial $p(x)$ (with $x \in \mathbb{R}^n$) in the form

$$p = r + \sum_{i=1}^{s} r_i g_i$$

with $r, r_1, \ldots, r_s$ sum of squares is a sufficient condition for p to be non-negative in the semialgebraic set

$$\{x \in \mathbb{R}^n : g_1(x) \geq 0, g_2(x) \geq 0, \ldots, g_s(x) \geq 0\}.$$

A polynomial r of degree $2d$ can be expressed as a sum of squares if and only if there exists a matrix $Q \succeq 0$ such that

$$r = z^t Q z = \langle Q, zz^t \rangle,$$

where z is a vector with a basis for $\mathbb{R}[x]_{\leq d}$ in its entries.

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$$r = z^t Q z = \langle Q, zz^t \rangle,$$

where z is a vector with a basis for $\mathbb{R}[x]_{\leq d}$ in its entries.

$$\min \quad 1 + \sum_{k=1}^{d} a_k + b_{11} + \langle F_0, J_{d+1} \rangle$$

$$\text{s. t.} \quad \text{(a)} - 1 - \sum_{k=1}^{d} a_k P_k^n(u) - 2b_{12} - b_{22} - 3\sum_{k=0}^{d} \langle F_k, S_k^n(u,u,1) \rangle = q(u) + p(u) q_1(u),$$

$$\text{(b)} - b_{22} - \sum_{k=0}^{d} \langle F_k, S_k^n(u,v,t) \rangle = r(u,v,t) + \sum_{i=1}^{4} r_i(u,v,t) g_i(u,v,t),$$

$$a_k \geq 0 \qquad \text{for } k = 1, \ldots, d,$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix} \succeq 0,$$

$$F_k \in \mathbb{R}^{(d-k+1)\times(d-k+1)}, \quad F_k \succeq 0 \qquad \text{for } k = 0, 1, \ldots, d,$$

$q, q_1, r, r_1, \ldots, r_4$ sum of squares.

Sum of squares with invariant polynomials

(b) $-b_{22} - \sum_{k=0}^{d} \left\langle F_k, S_k^n(u,v,t) \right\rangle = r(u,v,t) + \sum_{i=1}^{4} r_i(u,v,t) g_i(u,v,t)$

$$r = \left\langle Q, zz^t \right\rangle, \qquad \dim \mathbb{R}[u,v,t]_{\leq d} = \binom{d+3}{3}$$

When r is invariant with respect to the action of the group of permutations of three elements, Q can be block-diagonalized in three blocks of sizes m_1, m_2 and m_3 such that

$$m_1 + m_2 + 2m_3 = \binom{d+3}{3}.$$

Sum of squares with invariant polynomials

$$\text{(b)} \quad -b_{22} - \sum_{k=0}^{d} \langle F_k, S_k^n(u,v,t) \rangle = r(u,v,t) + \sum_{i=1}^{4} r_i(u,v,t) s_i(u,v,t)$$

$$r = \langle Q, zz^t \rangle, \qquad \dim \mathbb{R}[u,v,t]_{\leq d} = \binom{d+3}{3}$$

When r is invariant with respect to the action of the group of permutations of three elements, Q can be block-diagonalized in three blocks of sizes m_1, m_2 and m_3 such that

$$m_1 + m_2 + 2m_3 = \tbinom{d+3}{3}.$$

Sum of squares with invariant polynomials

$$\text{(b)} \quad -b_{22} - \sum_{k=0}^{d} \left\langle F_k, S_k^n(u,v,t) \right\rangle = r(u,v,t) + \sum_{i=1}^{4} r_i(u,v,t) s_i(u,v,t)$$

$$r = \left\langle Q, zz^t \right\rangle, \qquad \dim \mathbb{R}[u,v,t]_{\leq d} = \binom{d+3}{3}$$

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n	lower bound	previous upper bound	new upper bound
3	12	12.381810	12.368591
4	24	24.066284	24.056903
5	40	44.998997	44.981067
6	72	78.240613	78.187761
7	126	134.448817	134.270201
8	240	240.000000	240.000010
9	306	364.091929	<u>363</u>.675154
10	500	554.507549	<u>553</u>.827497
11	582	870.883116	<u>869</u>.244985
12	840	1357.889300	<u>1356</u>.603728
13	1154	2069.587585	<u>2066</u>.405173
14	1606	3183.133169	<u>3177</u>.917052
15	2564	4866.245659	<u>4858</u>.505436
16	4320	7355.809036	<u>7332</u>.776399
17	5346	11072.37543	<u>11014</u>.183845
18	7398	16572.26478	<u>16469</u>.090329
19	10668	24812.30254	<u>24575</u>.871259
20	17400	36764.40138	<u>36402</u>.675795
21	27720	54584.76757	<u>53878</u>.722941
22	49896	82340.08003	<u>81376</u>.459564
23	93150	124416.9796	<u>123328</u>.397290
24	196560	196560.0000	196560.000465

Table 3. New upper bounds for the kissing number.

Rigorous verification with interval arithmetic

The computation of the block-diagonalization and the solver use floating point arithmetic and so the obtained solution is just an approximation. It is possible to obtain a feasible solution and verify it with the following steps:

- Find an approximate solution with matrices positive definite together with lower bounds to their eigenvalues.
  - $\langle X, A\rangle = b,\ X \succeq 0 \to \langle X', A\rangle = b - \langle \lambda_{\min} I, A\rangle,\ X' \succeq 0.$
  - $X' = LL^t + \lambda_X I$
- Use interval arithmetic to find a bound for the violation in the constraints of the problem.

3. Extension to topological packing graphs
   - Topological packing graphs
   - Moment matrix and k-point bounds for finite graphs
   - 3-point bound for topological packing graphs
   - The semidefinite programming bound revisited

Topological packing graphs

Definition: a graph whose vertex set is a Hausdorff topological space is a *topological packing graph* if every finite clique is contained in an open clique.

Note that if the vertex set is compact, the topological packing condition implies that the independence number of the graph is finite.

Denote as $\mathrm{sub}_t(V)$ the family of subsets with at most t vertices, as I_t the family of independent subsets with at most t vertices, $I_{=t}$ the family of independent subsets with exactly t vertices and $I'_t = I_t \setminus \{\emptyset\}$.

Topology on the family of subsets with at most t vertices $\mathrm{sub}_t(V)$

We can associate a topology to the family of subsets with at most t vertices $\mathrm{sub}_t(V)$ using the product topology in V^t and then the quotient topology to the image of V^t under the map

$$q : (v_1, ..., v_t) \to \{v_1, ..., v_t\}$$

and finally adding $\{\emptyset\}$ with the disjoint union topology.

If V is compact, one can show with the topological packing condition that I_t is compact and $I_{=r}$ is open and closed in I_t for all $r = 1, \ldots, t$.

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Moment matrix and k-point bounds for finite graphs

For $y\colon \mathrm{sub}_{2t}(V) \to \mathbb{R}$, the *truncated moment matrix* $M_t(y)\colon \mathrm{sub}_t(V) \times \mathrm{sub}_t(V) \to \mathbb{R}$ is the matrix defined as:

$$[M_t(y)]_{S,S'} := y_{S \cup S'}.$$

The moment matrix can be used to define optimization programs that upper bounds the independence number of a graph. For instance, the *t-th step of the Lasserre's hierarchy* is:

$$\begin{aligned} \max \quad & \sum_{u \in V} y_{\{u\}} \\ \text{s. t.} \quad & y\colon \mathrm{sub}_{2t}(V) \to \mathbb{R}, \\ & y_\emptyset = 1, \\ & y_S = 0 \text{ if } S \notin I_{2t}, \\ & M_t(y) \succeq 0. \end{aligned}$$

The denomination "k-point bound" refers to the size k of the biggest subset considered. This program is a $2t$-point bound.

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The denomination "k-point bound" refers to the size k of the biggest subset considered. This program is a $2t$-point bound.

The extended Lovász theta number $\vartheta_t'(G)$

Replace $\mathrm{sub}_t(V)$ by $\mathrm{sub}_t'(V) = \mathrm{sub}_t(V) \setminus \{\emptyset\}$ and denote by $M_t'(y)$ the corresponding truncate moment matrix without the row and column associated to $\{\emptyset\}$.

$$\begin{aligned} \max \quad & \sum_{u,v \in V} y_{\{u,v\}} \\ \text{s. t.} \quad & y \colon \mathrm{sub}_{2t}'(V) \to \mathbb{R}_{\geq 0}, \\ & \sum_{u \in V} y_{\{u\}} = 1, \\ & y_S = 0 \text{ if } S \notin I_{2t}', \\ & M_t'(y) \succeq 0. \end{aligned}$$

To see that $\alpha(G) \leq \vartheta_t'(G)$, given $I \subseteq V$ independent, consider y defined as $y_S = \frac{1}{|I|}$ if $S \subseteq I$ and $y_S = 0$ otherwise.

3-point bound for finite graphs

For $u \in V$, consider the matrix $M'_u(y)\colon V \times V \to \mathbb{R}$ defined as:

$$[M'_u(y)]_{v,w} := y_{\{u,v,w\}}.$$

$$\begin{aligned}
\max \quad & \sum_{u,v \in V} y_{\{u,v\}} \\
\text{s. t.} \quad & y\colon \mathrm{sub}'_3(V) \to \mathbb{R}_{\geq 0}, \\
& \sum_{u \in V} y_{\{u\}} = 1 \\
& y_S = 0 \text{ if } S \notin I'_3, \\
& M'_1(y) \succeq 0, \\
& M'_u(y) \succeq 0 \text{ for all } u \in V.
\end{aligned}$$

3-point bound for topological packing graphs

$$\begin{aligned}
\max \quad & \sum_{u,v \in V} y_{\{u,v\}} \\
\text{s. t.} \quad & y \colon \mathrm{sub}'_3(V) \to \mathbb{R}_{\geq 0}, \\
& \sum_{u \in V} y_{\{u\}} = 1 \\
& y_S = 0 \text{ if } S \notin I'_3, \\
& M'_1(y) \succeq 0, \\
& M'_u(y) \succeq 0 \text{ for all } u \in V.
\end{aligned}$$

$$\begin{aligned}
\max \quad & \lambda(I_{=1}) + 2\lambda(I_{=2}) \\
\text{s. t.} \quad & \lambda \in \mathcal{M}(I'_3)_{\geq 0}, \\
& \lambda(I_{=1}) = 1, \\
& A_1^* \lambda \in \mathcal{M}(V \times V)_{\succeq 0}, \\
& A_{3\mathrm{PB}}^* \lambda \in \mathcal{M}(V \times V \times V)_{\succeq 0}.
\end{aligned}$$

Where the operators $A_1 : \mathcal{C}(V \times V)_{\mathrm{sym}} \to \mathcal{C}(I'_3)$ and $A_{3\mathrm{PB}} : \mathcal{C}(V \times V \times V)_{\mathrm{sym}} \to \mathcal{C}(I'_3)$ are defined as:

$$A_1 K(S) := \sum_{\substack{u,v \in V: \\ \{u,v\} = S}} K(u,v), \qquad A_{3\mathrm{PB}} T(S) := \sum_{\substack{u,v,t \in V: \\ \{u,v,t\} = S}} T(u,v,t).$$

Primal and dual formulations

$$\begin{array}{ll} \max & \lambda(I_{=1}) + 2\lambda(I_{=2}) \\ \text{s. t.} & \lambda \in \mathcal{M}(I_3')_{\geq 0}, \\ & \lambda(I_{=1}) = 1, \\ & A_1^*\lambda \in \mathcal{M}(V \times V)_{\succeq 0}, \\ & A_{3\mathrm{PB}}^*\lambda \in \mathcal{M}(V \times V \times V)_{\succeq 0}. \end{array}$$

$$\begin{array}{lll} \min & a & \\ \text{s. t.} & a \in \mathbb{R}, \quad K \in \mathcal{C}(V \times V)_{\succeq 0}, & \\ & T \in \mathcal{C}(V \times V \times V)_{\succeq 0}, & \\ & A_1 K(S) + A_{3\mathrm{PB}} T(S) \leq a - 1 & \text{if } S \in I_{=1}, \\ & A_1 K(S) + A_{3\mathrm{PB}} T(S) \leq -2 & \text{if } S \in I_{=2}, \\ & A_1 K(S) + A_{3\mathrm{PB}} T(S) \leq 0 & \text{if } S \in I_{=3}. \end{array}$$

Symmetrization for homogeneous graphs

A graph G is *homogeneous* if there exists a compact group Γ, subgroup of the group of automorphisms of G, that acts continuously in V and such that the action of Γ in V is transitive.

Fixing $e \in V$ and letting H be the subgroup that stabilizes e under the action of Γ, there is a one-to-one correspondence Φ between $\mathcal{C}(V \times V \times V)^{\Gamma}_{\succeq 0}$ and $\mathcal{C}(V \times V)^{H}_{\succeq 0}$ given by:

$$\Phi(T)(x, y) = T(x, y, e),$$

whose inverse is

$$\Phi^{-1}(R)(x, y, z) = R(\psi_z^{-1}x, \psi_z^{-1}y),$$

where for each $z \in V$, $\psi_z \in \Gamma$ is such that $\psi_z e = z$.

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The semidefinite programming bound revisited

A kernel of $\mathcal{C}(S^{n-1} \times S^{n-1})^{H}_{\succeq 0}$ can be written in terms of the zonal matrices in such way that for $T \in \mathcal{C}(S^{n-1} \times S^{n-1} \times S^{n-1})^{\mathrm{O}(\mathbb{R}^n)}_{\succeq 0}$, $A_{3\mathrm{PB}}T$ can be approximated by expressions of the form

$$A_{3\mathrm{PB}}T(\{x,y,z\}) = \sum_{k=0}^{d} \big\langle F_k, S_k^n(x \cdot z, y \cdot z, x \cdot y)\big\rangle.$$

and, after the substitutions, the three point bound applied to the kissing number problem becomes equal to the semidefinite programming bound.

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