The null set of a polytope, and the Pompeiu property for polytopes

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joint work with Sinai Robins - arXiv:2104.01957

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Pompeiu property

Let $P \subset \mathbb{R}^d$ be a bounded set with nonempty interior. P has the Pompeiu property if, for $f \in \mathcal{C}(\mathbb{R}^d)$,

$$\int_{\sigma(P)} f(x)\,\mathrm{d}x = 0.$$

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The group $M(d)$ of rigid motions in $\mathbb{R}^d$ is the group generated by all translations and rotations.

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Equivalently, P has the Pompeiu property if the values

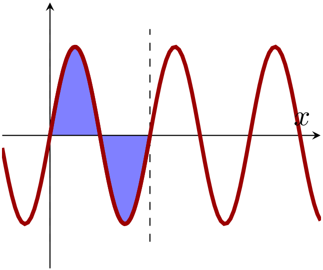
$$\int_{\sigma(P)} f(x)\,\mathrm{d}x$$

over all rigid motions $\sigma \in M(d)$ uniquely determine $f \in \mathcal{C}(\mathbb{R}^d)$.

$d = 1$

An interval does not have the Pompeiu property:

$$\int_c^{c+L} \sin\left(\frac{2\pi}{L}x\right) \mathrm{d}x = 0$$



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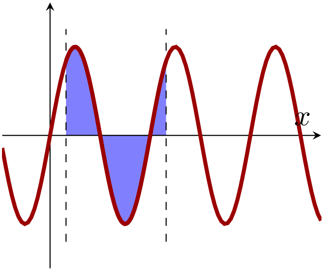

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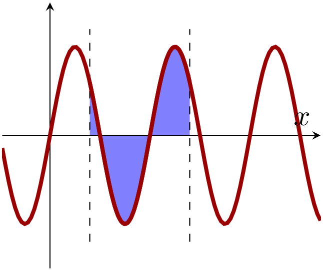

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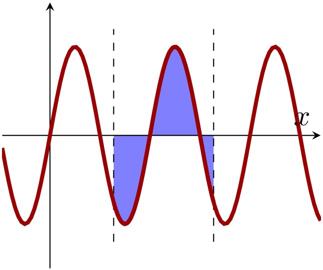

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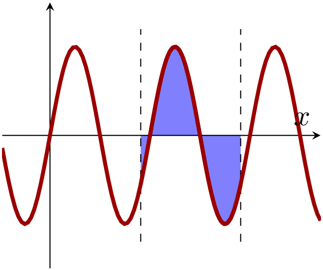

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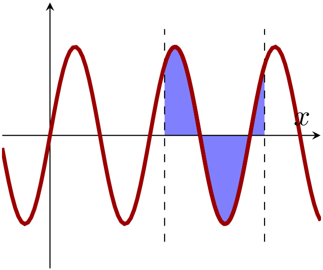

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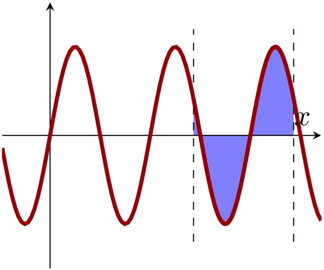
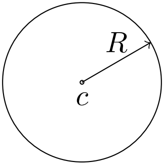


$d \geq 2$

A ball does not have the Pompeiu property. If R is the radius of the ball and a is such that $J_{d/2}(aR) = 0$, then

$$\int_{\|x-c\|\leq R} \sin(ax_1)\,\mathrm{d}x = 0.$$



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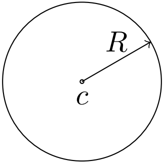
$$\int_{\|x\| \leq R} e^{2\pi i \langle \xi, x \rangle}\,\mathrm{d}x = \left(\frac{R}{\|\xi\|}\right)^{d/2} J_{d/2}(2\pi R \|\xi\|).$$



Let $P \subset \mathbb{R}^d$ be a convex body. P has the Pompeiu property if, for $f \in \mathcal{C}(\mathbb{R}^d)$,

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Pompeiu problem

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Many conditions are known that imply the Pompeiu property:

- Brown, Schreiber, Taylor (1973): In the planar case, if the boundary of P has a "corner", then it has the Pompeiu property.
- Williams (1976): If P does *not* have the Pompeiu property and has a portion of a real analytic surface on its boundary, then any analytic extension of this surface also lies in the boundary.

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Which in particular implies that polytopes have the Pompeiu property.

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$$\hat{1}_P(z) = \int_P e^{-2\pi i \langle x,z \rangle} \, \mathrm{d}x$$

An equivalent condition

Brown, Schreiber, Taylor (1973)

A convex body $P \subset \mathbb{R}^d$ has the Pompeiu property if and only if the Fourier-Laplace transform of P, namely $\hat{1}_P(z)$, does not vanish identically on any of the complex varieties $S_{\mathbb{C}}(\alpha)$, for any $\alpha \in \mathbb{C} \setminus \{0\}$.

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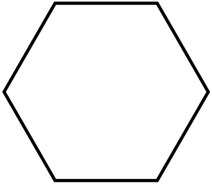
P does not have Pompeiu property $\iff \exists \alpha \neq 0 : S_{\mathbb{C}}(\alpha) \subset N(P)$

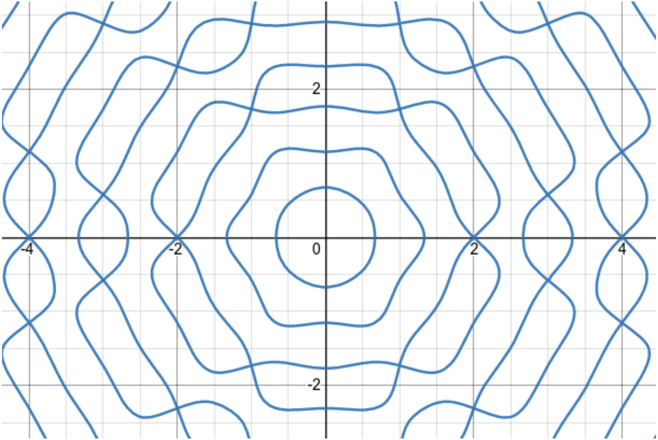
Example

Let $P \subset \mathbb{R}^2$ be an hexagon,



Example

In blue is $N(P) \cap \mathbb{R}^2$ and in red is a circle.



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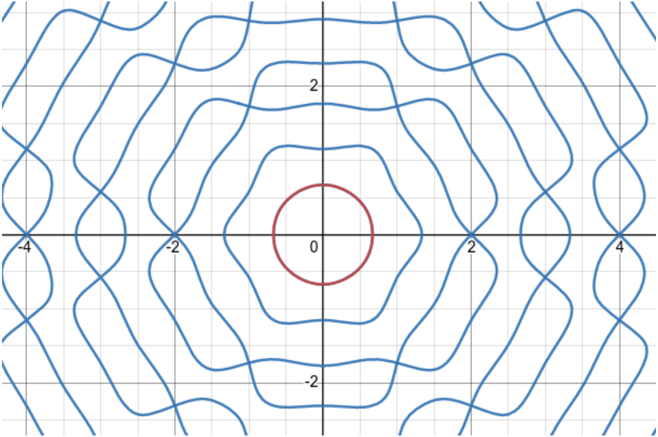

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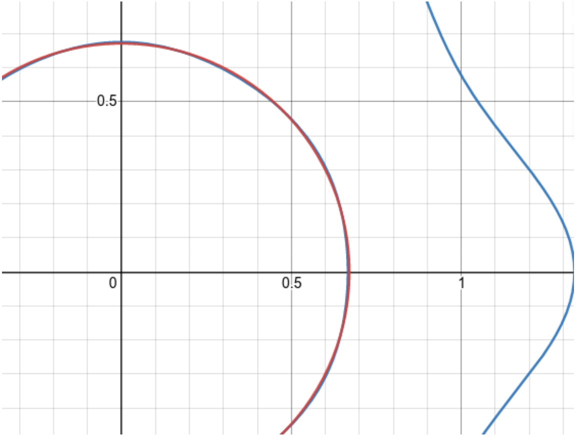

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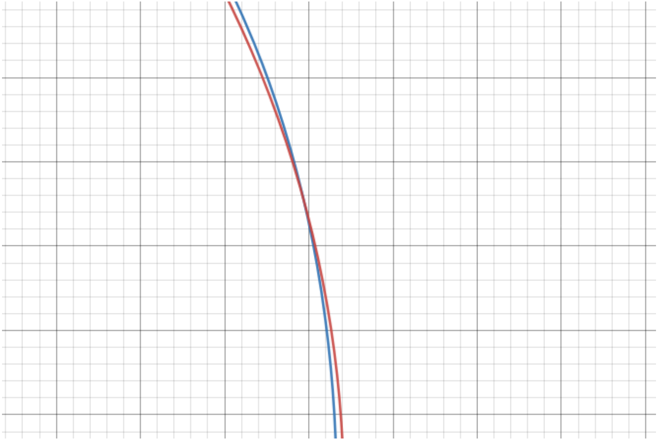


Our main contribution

Using an explicit form for the Fourier-Laplace transform of a polytope (Brion's theorem), we give a simple proof that polytopes have the Pompeiu property.

Theorem (M., Robins)

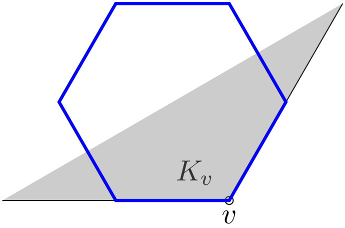
Let $P \subset \mathbb{R}^d$ be a d-dimensional polytope, $H \subset \mathbb{R}^d$ be a 2-dimensional real subspace that is not orthogonal to any edge from P, and fix an orthonormal basis $\{e, f\} \subset \mathbb{R}^d$ for H. Then

$$\left\{\alpha(\cos t)e + \alpha(\sin t)f \in \mathbb{C}^d \ : \ t \in [-\pi, \pi]\right\} \not\subset N(P)$$

for any $\alpha \in \mathbb{C} \setminus \{0\}$.

2 The Fourier transform of a polytope

Let $P \subset \mathbb{R}^d$ be a d-dimensional polytope. For each $v \in V(P)$, let K_v be the tangent cone of P at v and $K_{v,1}, \ldots, K_{v,M_v}$ be a triangulation of K_v into simplicial cones with no new edges. For each $1 \leq j \leq M_v$, let $w_{j,1}^v, \ldots, w_{j,d}^v$ be the edges of $K_{v,j}$.



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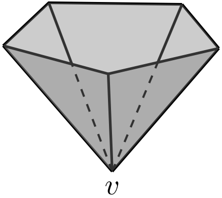

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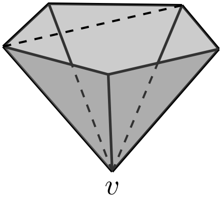

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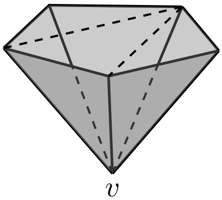

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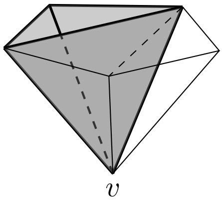

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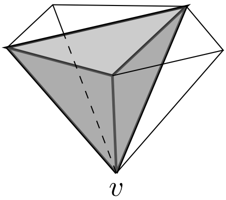

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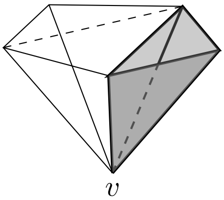

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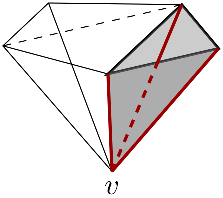

Brion's theorem

$$\hat{1}_P(z) = \sum_{v \in V(P)} \sum_{j=1}^{M_v} \frac{e^{-2\pi i \langle v, z \rangle}}{(2\pi i)^d} \frac{\det K_{v,j}}{\langle w^v_{j,1}, z \rangle \dots \langle w^v_{j,d}, z \rangle}.$$

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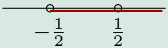

$$\hat{1}_P(z) = \frac{e^{-2\pi i(-\frac{1}{2}).z}}{(2\pi i)^1} \frac{1}{z} + \frac{e^{-2\pi i \frac{1}{2}.z}}{(2\pi i)^1} \frac{1}{(-z)} = \frac{\sin(\pi z)}{\pi z}$$

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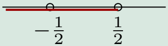

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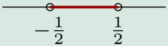

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$d = 2$

$P = \mathrm{conv}\{(0,0),(a,0),(0,b)\}$,
$w_1^0 = (1,0), \quad w_1^a = (-1,0), \quad w_1^b = (0,-1)$
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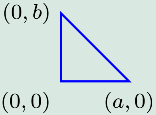

$$\hat{1}_P(z) = \left(\frac{1}{2\pi i}\right)^2 \left(\frac{1}{z_1 z_2} + \frac{b e^{-2\pi i a z_1}}{(a z_1 - b z_2) z_1} + \frac{a e^{-2\pi i b z_2}}{(-a z_1 + b z_2) z_2}\right)$$

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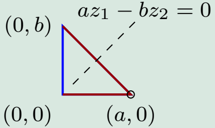

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3 Bessel functions

Bessel function of order n, $J_n(z)$:

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4 Overview of the proof

Brown, Schreiber, Taylor (1973)

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$$z(t) = (z_1, \ldots, z_d) \in \mathbb{C}^d,$$
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Assume by contradiction that

$$\hat{1}_P(z(t)) = 0 \text{ for all } t \in (-\pi, \pi].$$

$$0 = \hat{1}_P(z(t)) = \sum_{v \in V(P)} \sum_{j=1}^{M_v} \frac{e^{-2\pi i \langle v, z(t) \rangle}}{(2\pi i)^d} \frac{\det K_{v,j}}{\langle w^v_{j,1}, z(t) \rangle \dots \langle w^v_{j,d}, z(t) \rangle}$$

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Substituting $\cos t = (e^{it} + e^{-it})/2$ and $\sin t = (e^{it} - e^{-it})/(2i)$, $\langle w^v_{j,l},\, z(t)\rangle$ is a trigonometric polynomial $c_{-1}e^{-it} + c_0 + c_1 e^{it}$

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0 &= \sum_{v \in V(P)} \Big(\sum_{k=-N}^{N} c_{v,k} e^{ikt} \Big) e^{-2\pi i \langle v, z(t) \rangle} \\
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By translating P in the direction of u, we may assume that

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$$\implies c_{u,N}\, i^N\, e^{iN\phi_u} = 0 \implies c_{u,N} = 0 \quad \text{a contradiction. } \square$$

Summary

- Brion's theorem shows the Fourier-Laplace transform of a polytope as a rational-exponential function.
- The null set of a polytope P does not contain a 'circle' on any plane not orthogonal to some edge of P.
- To show this we assume otherwise and find a contradiction between the Fourier coefficients of $\hat{1}_P(z(t))$ of high order and the asymptotic behaviour of Bessel functions.
- This implies that every polytope has the Pompeiu property.

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Thank you for your attention!

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