A GENTLE INTRODUCTION TO FOURIER ANALYSIS ON POLYTOPES - EXERCISES FROM CHAPTER 4

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Exercises from the book: Robins, S., "A friendly introduction to Fourier analysis on polytopes", available at `https://arxiv.org/abs/2104.06407`. This solution sheet is available at `https://www.ime.usp.br/~fabcm/33coloquio-impa/`.

4.2. Given a convex d-dimensional body $K \subset \mathbb{R}^d$, prove that $K - K$ is convex, and that $K - K$ is centrally symmetric.

4.3. The support of a function f is defined here as

$$\mathrm{supp(f) := \{x \in \mathbb{R}^d \mid f(x) \neq 0\}.}$$

Suppose that we are given two convex bodies $A, B \subset \mathbb{R}^d$. Show that

$$\mathrm{supp(1_A * 1_B) = A + B,}$$

where the addition is the Minkowski addition of sets.

4.4. Suppose we have a triangle Δ whose vertices v_1, v_2, v_3 are integer points. Prove that the following properties are equivalent:

(a) Δ has no other integer points inside or on its boundary.

(b) $\mathrm{Area}(\Delta) = \frac{1}{2}$.

(c) Δ is a unimodular triangle - i.e. $v_3 - v_1$ and $v_2 - v_1$ form a basis for $\mathbb{Z}^2$.

(Hint: You might begin by "doubling" the triangle to form a parallelogram.)

4.6. Show that in $\mathbb{R}^d$, an integer simplex Δ is unimodular if and only if $\mathrm{vol}(\Delta) = \frac{1}{d!}$.

4.7. Find in $\mathbb{R}^3$, an integer simplex Δ that has no other integer points inside or on its boundary (other than its vertices of course), but such that Δ is not a unimodular simplex.

4.8. Prove that for any polytope P, $\hat{1}_P$ is not a Schwartz function.

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4.11. Here we use Siegel's Theorem 4.4 to give an extension of Minkowski's classical Theorem 4.2 for bodies K that are not necessarily symmetric.

Namely, let K be any bounded, measurable subset of $\mathbb{R}^d$ (so K is not necessarily symmetric), with a positive d-dimensional measure. Let $B := \frac{1}{2}K - \frac{1}{2}K$ be the symmetrized body of K (hence B is a convex symmetric body). Let $\mathcal{L}$ be a (full rank) lattice in $\mathbb{R}^d$. Prove the following statement:

$$\text{If } \operatorname{vol} K > 2^d(\det \mathcal{L}), \text{ then } B \text{ must contain a nonzero point of } \mathcal{L} \text{ in its interior.}$$

Notes. We note that the positive conclusion of the existence of a nonzero integer point holds only for the symmetrized body B, with no guarantees for any integer points in K.

Lecture 4

- Application of Poisson summation to the theta function

$$\theta(t) := \sum_{n\in\mathbb{Z}} e^{-\pi t n^2}, \qquad \theta\Big(\frac{1}{t}\Big) = \sqrt{t}\theta(t).$$

$$G_t(x) := \frac{1}{\sqrt{t}} e^{-\frac{\pi}{t}x^2}, \qquad \hat{G}_t(\xi) = e^{-\pi t \xi^2}.$$

- Lattices

$$\mathcal{L} := \{n_1 v_1 + \cdots + n_d v_d \in \mathbb{R}^d : n_j \in \mathbb{Z}, \forall j\}, \quad M := \begin{pmatrix} | & | & \dots & | \\ v_1 & v_2 & \dots & v_d \\ | & | & \dots & | \end{pmatrix}, \quad \mathcal{L} = M(\mathbb{Z}^d).$$

$$\mathcal{L}^* := \{m \in \mathbb{R}^d : \langle n, m\rangle \in \mathbb{Z},\ \forall n \in \mathcal{L}\} = M^{-\mathsf{T}}(\mathbb{Z}^d).$$

- Poisson summation for lattices

$$\sum_{n\in\mathcal{L}} f(n+x) = \frac{1}{\det \mathcal{L}} \sum_{\xi\in\mathcal{L}^*} \hat{f}(\xi) e^{2\pi i\langle \xi, x\rangle}.$$

Recall $(\widehat{f\circ M})(\xi) = \frac{1}{|\det M|}\hat{f}\Big(M^{-\mathsf{T}}\xi\Big)$.

- The convolution operation

$$(f * g)(x) := \int_{\mathbb{R}^d} f(x-y)g(y)dy.$$

$$(\widehat{f * g})(\xi) = \hat{f}(\xi)\hat{g}(\xi).$$

Let f_1 be $1_{[-\frac{1}{2},\frac{1}{2}]}$, the indicator function of the interval $[-\frac{1}{2},\frac{1}{2}]$.

$$f_2(x) := (f_1 * f_1)(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} 1_{[-\frac{1}{2},\frac{1}{2}]}(x-t)dt = \begin{cases} -|x|+1 & \text{if } |x| \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$f_3(x) := (f_2 * f_1)(x) = \int_{-1}^{1} f_2(t) 1_{[-\frac{1}{2},\frac{1}{2}]}(x-t)dt = \begin{cases} -x^2 + \frac{3}{4} & \text{if } |x| \le \frac{1}{2}, \\ \frac{1}{8}(2|x|-3)^2 & \text{if } \frac{1}{2} \le |x| \le \frac{3}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

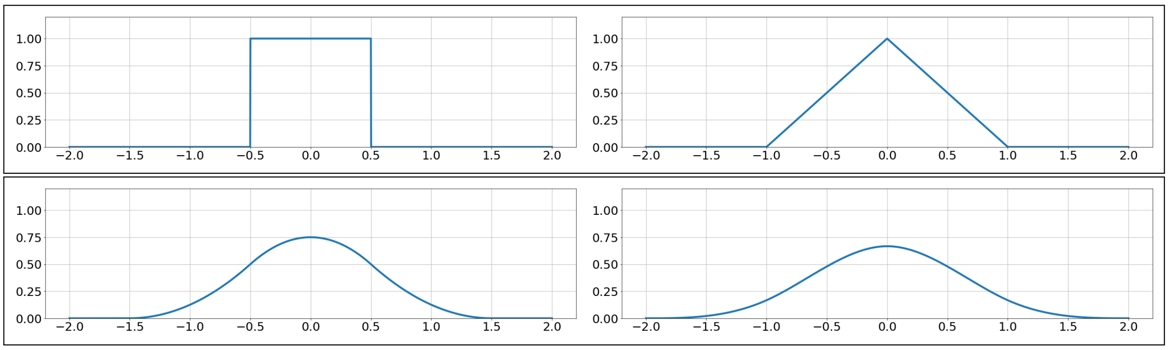

FIGURE 1. Funcions $f_1, \dots f_4$.

$$f_4(x) := (f_3 * f_1)(x) = \int_{-\frac{3}{2}}^{\frac{3}{2}} f_3(t) 1_{[-\frac{1}{2},\frac{1}{2}]}(x-t)dt = \begin{cases} \frac{1}{2}|x|^3 - x^2 + \frac{2}{3} & \text{if } |x| \le 1, \\ -\frac{1}{6}(|x|-2)^3 & \text{if } 1 \le |x| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

- The geometry of numbers: Siegel's formula. Let $B \subset \mathbb{R}^d$ be a d-dimensional convex body, symmetric about the origin. If the only integer point in its interior is the origin, then
$$2^d = \operatorname{vol} B + \frac{4^d}{\operatorname{vol} B} \sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} \left| \hat{1}_{\frac{1}{2}B}(\xi) \right|^2.$$
(Poisson summation applied to $1_{\frac{1}{2}B} * 1_{-\frac{1}{2}B}$.)

- The geometry of numbers: Minkowski's first theorem. Let $B \subset \mathbb{R}^d$ be a d-dimensional convex body, symmetric about the origin. If $\operatorname{vol} B > 2^d$, then B must contain a nonzero integer point in its interior.

Exercise 4.2. Given a convex d-dimensional body $K \subset \mathbb{R}^d$, prove that $K-K$ is convex, and that $K-K$ is centrally symmetric.

Solution: Let $x, y \in K-K$, hence $x = x_1 - x_2$ and $y = y_1 - y_2$, with $x_1, x_2, y_1, y_2 \in K$. For any $a \in [0,1]$,

$$ax + (1-a)y = a(x_1 - x_2) + (1-a)(y_1 - y_2) = ax_1 + (1-a)y_1 - \big(ax_2 + (1-a)y_2\big).$$

Since K is convex, $ax_1 + (1-a)y_1 \in K$ and $ax_2 + (1-a)y_2 \in K$. Therefore $ax + (1-a)y \in K - K$.

Tho show that $K-K$ is centrally symmetric, let $x \in K-K$, hence $x = x_1 - x_2$ with $x_1, x_2 \in K$. Since $-x = x_2 - x_1 \in K-K$, we conclude that $K-K$ is centrally symmetric.

Exercise 4.6. Show that in $\mathbb{R}^d$, an integer simplex Δ is unimodular if and only if $\text{vol}(\Delta) = \frac{1}{d!}$.

Solution: Let $S \subset \mathbb{R}^d$ be the standard simplex:

$$\begin{aligned} S &= \text{conv}(\{0, \mathrm{e}_1, \dots, \mathrm{e}_\mathrm{d}\}) \\ &= \{x \in \mathbb{R}^d : x_1 + \dots + x_d \leq 1,\ x_1, \dots, x_d \geq 0\}. \end{aligned}$$

Next we proof by induction in d that $\text{vol}(S) = \frac{1}{d!}$:

$$\begin{aligned} \text{vol}(S) &= \int_0^1 \int_0^{1-x_1} \cdots \int_0^{1-x_1-\ldots-x_{d-1}} dx_d \ldots dx_2 dx_1 \\ &= \int_0^1 \int_0^{1-x_1} \cdots \int_0^{1-x_1-\ldots-x_{d-2}} (1 - x_1 - \ldots - x_{d-1}) dx_{d-1} \ldots dx_2 dx_1 \\ &= \int_0^1 \frac{(1-x_1)^{d-1}}{(d-1)!} dx_1 = \frac{1}{d!}. \end{aligned}$$

In the second to last equality we used induction in d (verify the case $d = 2$) and, denoting the $(d-1)$-dimensional standard simplex in the coordinates $x_2, \ldots x_d$ by S', notice that $(1-x_1)S' = \{(x_2, \dots, x_d) \in \mathbb{R}^{d-1} : x_2 + \dots + x_d \leq 1 - x_1,\ x_2, \dots, x_d \geq 0\}$.

If $\Delta = \text{conv}(\mathrm{v}_0, \dots, \mathrm{v}_\mathrm{d})$, then $M = (v_1 - v_0, \dots, v_d - v_0) \in \mathbb{Z}^{d \times d}$ and

$$\begin{aligned} \text{vol}(\Delta) &= \int_\Delta dx = \int_{MS} dx = |\det M| \int_S dx \\ &= |\det M| \,\text{vol}(S) = |\det M| \frac{1}{d!}. \end{aligned}$$

We say that Δ is unimodular if $M = (v_1 - v_0, \dots, v_d - v_0) \in \mathbb{Z}^{d \times d}$ and $|\det M| = 1$. By the above equation we see that Δ is unimodular if and only if $\text{vol}(\Delta) = \frac{1}{d!}$.

Exercise 4.7. Find in $\mathbb{R}^3$, an integer simplex Δ that has no other integer points inside or on its boundary (other than its vertices of course), but such that Δ is not a unimodular simplex.

Solution: Consider Δ as the convex hull of $(0,0,0)$, $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$. Computing the determinant of the matrix with its vertices, we get $\text{vol}(\Delta) = \frac{1}{3}$. If Δ has an integer point v other than its vertices, then there is $x, y, z \geq 0$ with $x + y + z \leq 1$ such that

$$v = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{Z}^3.$$

From the first line, we get either $x+y=1$ or $x+y=0$. If $x+y=0$, then $z=0$ or $z=1$, but both cases are not possible since $v \neq (0,0,0)$ and $v \neq (0,1,1)$. So $x+y=1$ and $z=0$. From the second line, $x+z=0$ or $x+z=1$, but neither case is possible, since if $x=0$, $y=1$ we get $v=(1,0,1)$ and from $x=1$, $y=0$ we get $v=(1,1,0)$. Therefore Δ has no integer points other than its vertices.