A GENTLE INTRODUCTION TO FOURIER ANALYSIS ON POLYTOPES - EXERCISES FROM CHAPTER 3

FABRÍCIO C. MACHADO

Exercises from the book: Robins, S., "A friendly introduction to Fourier analysis on polytopes", available at `https://arxiv.org/abs/2104.06407`. This solution sheet is available at `https://www.ime.usp.br/~fabcm/33coloquio-impa/`.

3.1. Recalling that the L^2-norm is defined by $\|x\|_2 := \sqrt{x_1^2 + \cdots + x_d^2}$, and the L^1-norm is defined by $\|x\|_1 := |x_1| + \cdots + |x_d|$, we have the following elementary norm relations.

(a) Show that $\|x\|_2 \leq \|x\|_1$, for all $x \in \mathbb{R}^d$.

(b) On the other hand, show that we have $\|x\|_1 \leq \sqrt{d}\, \|x\|_2$, for all $x \in \mathbb{R}^d$.

3.3. We know that the functions $u(t) := \cos t = \frac{e^{it}+e^{-it}}{2}$ and $v(t) := \sin t = \frac{e^{it}-e^{-it}}{2i}$ are natural, partly because they parametrize the unit circle: $u^2 + v^2 = 1$. Here we see that there are other similarly natural functions, parametrizing the hyperbola.

(a) Show that the following functions parametrize the hyperbola $u^2 - v^2 = 1$:

$$u(t) := \frac{e^t + e^{-t}}{2}, \quad v(t) := \frac{e^t - e^{-t}}{2}.$$

(This is the reason that the function $\cosh t := \frac{e^t+e^{-t}}{2}$ is called the hyperbolic cosine, and the function $\sinh t := \frac{e^t-e^{-t}}{2}$ is called the hyperbolic sine)

(b) The hyperbolic cotangent is defined as $\coth t := \frac{\cosh t}{\sinh t} = \frac{e^t+e^{-t}}{e^t-e^{-t}}$. Using Bernoulli numbers, show that $t \coth t$ has the Taylor series:

$$t \coth t = \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!} B_{2n} t^{2n}.$$

3.5. We continue with the same function as in the previous exercise, $f(x) := e^{-2\pi t|x|}$.

(a) Show that $\hat{f}(\xi) = \frac{t}{\pi} \frac{1}{\xi^2+t^2}$, for all $\xi \in \mathbb{R}$.

(b) Using Poisson summation, show that:

$$\frac{t}{\pi} \sum_{n\in\mathbb{Z}} \frac{1}{n^2 + t^2} = \sum_{m\in\mathbb{Z}} e^{-2\pi t|m|}.$$

3.6. Here we evaluate the Riemann zeta function at the positive even integers.

Date: July 22, 2021.

(a) Show that
$$\sum_{n\in\mathbb{Z}} e^{-2\pi t|n|} = \frac{1+e^{-2\pi t}}{1-e^{-2\pi t}} := \coth(\pi t),$$
for all $t>0$.

(b) Show that the cotangent function has the following (well-known) partial fraction expansion:
$$\pi\cot(\pi x) = \frac{1}{x} + 2x\sum_{n=1}^{\infty}\frac{1}{x^2-n^2},$$
valid for any $x\in\mathbb{R}-\mathbb{Z}$.

(c) Let $0<t<1$. Show that
$$\frac{t}{\pi}\sum_{n\in\mathbb{Z}}\frac{1}{n^2+t^2} = \frac{1}{\pi t} + \frac{2}{\pi}\sum_{m=1}^{\infty}(-1)^{m+1}\zeta(2m)\, t^{2m-1},$$
where $\zeta(s) := \sum_{n=1}^{\infty}\frac{1}{n^s}$ is the Riemann zeta function, initially defined by the latter series, which is valid for all $s\in\mathbb{C}$ with $Re(s)>1$.

(d) Here we show that we may quickly evaluate the Riemann zeta function at all even integers, as follows. We recall the definition of the Bernoulli numbers, namely:
$$\frac{z}{e^z-1} = 1 - \frac{z}{2} + \sum_{m\geq 1}\frac{B_{2m}}{2m!}z^{2m}.$$
Prove that for all $m\geq 1$,
$$\zeta(2m) = \frac{(-1)^{m+1}}{2}\frac{(2\pi)^{2m}}{(2m)!}B_{2m}.$$
Thus, for example, using the first 3 Bernoulli numbers, we have: $\zeta(2)=\frac{\pi^2}{6}$, $\zeta(4)=\frac{\pi^4}{90}$, and $\zeta(6)=\frac{\pi^6}{945}$.

3.8. The hyperbolic secant is defined by
$$\operatorname{sech}(\pi x) := \frac{2}{e^{\pi x}+e^{-\pi x}},\quad \text{for } x\in\mathbb{R}.$$
Show that $\operatorname{sech}(\pi x)$ is its own Fourier inverse:
$$\mathcal{F}(\operatorname{sech})(\xi) = \operatorname{sech}(\xi),$$
for all $\xi\in\mathbb{R}$. (Hard! May require complex analysis)

3.21. For all $f,g\in S(\mathbb{R}^d)$, show that $\langle f,g\rangle = \langle \hat{f},\hat{g}\rangle$.

Lecture 3

- $L^1(\mathbb{R}^d)$, $L^2(\mathbb{R}^d)$.

$$L^1(\mathbb{R}^d) := \Big\{f\colon \mathbb{R}^d \to \mathbb{C} : \int_{\mathbb{R}^d} |f(x)|dx < \infty\Big\}.$$

(see Lebesgue's Dominated Convergence Theorem)

$$L^2(\mathbb{R}^d) := \Big\{f\colon \mathbb{R}^d \to \mathbb{C} : \int_{\mathbb{R}^d} |f(x)|^2 dx < \infty\Big\}.$$

Examples:

$$f(x) = \begin{cases} \frac{1}{x}, & \text{if } x \geq 1, \\ 0, & \text{otherwise.} \end{cases} \qquad g(x) = \begin{cases} \frac{1}{\sqrt{x}}, & \text{if } 0 < x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

We have $f \in L^2(\mathbb{R})$, but $f \notin L^1(\mathbb{R})$.

$$\int_{\mathbb{R}} |f(x)|^2 dx = \int_1^\infty \frac{1}{x^2} dx = \frac{-1}{x}\Big|_{x=1}^{\infty} = 1.$$

While $g \in L^1(\mathbb{R})$, but $g \notin L^2(\mathbb{R})$.

$$\int_{\mathbb{R}} |g(x)| dx = \int_0^1 \frac{1}{\sqrt{x}} dx = 2\sqrt{x}\Big|_{x=0}^{1} = 2.$$

If we replace $\mathbb{R}^d$ by a bounded domain X, then $L^2(X) \subset L^1(X)$. If we assume that a function $f \in L^1(\mathbb{R}^d)$ is bounded, then $f \in L^2(\mathbb{R}^d)$.

- Inverse Fourier transform.

If $f \in L^1(\mathbb{R}^d)$ and $\hat{f} \in L^1(\mathbb{R}^d)$, then f is almost everywhere equal to a continuous function and, assuming that f is this continuous function,

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi$$

for all $x \in \mathbb{R}^d$.

Another inversion theorem is: If $f \in L^2(\mathbb{R}^d)$, then $\hat{f}(\xi)$ is well-defined for almost every $\xi \in \mathbb{R}^d$, $\hat{f} \in L^2(\mathbb{R}^d)$ and, letting

$$\tilde{f}(x) := \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi,$$

then

$$\int_{\mathbb{R}^d} |f(x) - \tilde{f}(x)|^2 dx = 0.$$

- The Fourier transform of a compact body is infinitely smooth.

$$\frac{\partial}{\partial x_k} \hat{f}(\xi) = \frac{\partial}{\partial \xi_k} \int_P e^{-2\pi i \langle x, \xi \rangle} dx = \int_P \frac{\partial}{\partial \xi_k} e^{-2\pi i \langle x, \xi \rangle} dx = (-2\pi i) \int_P x_k e^{-2\pi i \langle x, \xi \rangle} dx.$$

(The dominated convergence theorem justifies the exchange between the derivative and the integral. Since P is compact and $x_k e^{-2\pi i\langle x,\xi\rangle}$ is continuous, it is in $L^1(\mathbb{R}^d)$.)

- Fourier series, intuition and examples.

  Intuition: $f\colon [0,1]^d \to \mathbb{C}$, (+ assumptions)

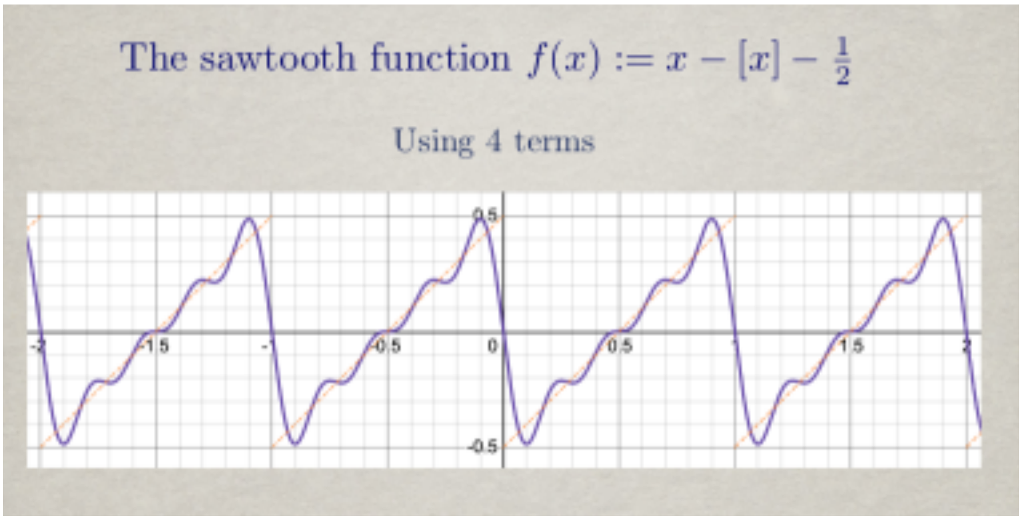
$$f(x) = \sum_{k\in\mathbb{Z}^d} f_k e^{2\pi i\langle x,k\rangle}, \quad f_k = \int_{[0,1]^d} f(x)e^{-2\pi i\langle x,k\rangle}dx.$$

  If $f \in L^2([0,1]^d)$, then

$$\int_{[0,1]^d} \left| f(x) - \sum_{k\in\mathbb{Z}^d} f_k e^{2\pi i\langle x,k\rangle} \right|^2 dx = 0.$$

  If $f\colon [0,1] \to \mathbb{C}$ ($d = 1$) is piecewise smooth, then

$$\sum_{k\in\mathbb{Z}^d} f_k e^{2\pi i x k} = \frac{\lim_{\epsilon\to 0^+} f(x-\epsilon) + \lim_{\epsilon\to 0^+} f(x+\epsilon)}{2} = \frac{f(x^-)+f(x^+)}{2}$$



- The Schwartz space.

  Functions $f \in \mathbb{R}^d \to \mathbb{C}$ infinitely differentiable and such that for all $a, k \in \mathbb{N}^d_{\geq 0}$, $|x^a D_k f(x)|$ is bounded for all $x \in \mathbb{R}^d$.

- Poisson summation

  Intuition:

$$\sum_{n\in\mathbb{Z}^d} f(x+n) = \sum_{m\in\mathbb{Z}^d} \hat{f}(m)e^{2\pi i\langle x,m\rangle}.$$

From: Rudin, W., "Real and complex analysis", McGraw-Hill

That (3) holds if $\alpha \geq 0$ follows from Proposition 1.24(*c*). It is easy to verify that (3) holds if $\alpha = -1$, using relations like $(-u)^+ = u^-$. The case $\alpha = i$ is also easy: If $f = u + iv$, then

$$\int (if) = \int (iu - v) = \int (-v) + i \int u = -\int v + i \int u = i\left(\int u + i \int v\right)$$
$$= i \int f.$$

Combining these cases with (2), we obtain (3) for any complex α. ////

1.33 Theorem *If $f \in L^1(\mu)$, then*

$$\left| \int_X f\, d\mu \right| \leq \int_X |f|\, d\mu.$$

PROOF Put $z = \int_X f\, d\mu$. Since z is a complex number, there is a complex number α, with $|\alpha| = 1$, such that $\alpha z = |z|$. Let u be the real part of αf. Then $u \leq |\alpha f| = |f|$. Hence

$$\left| \int_X f\, d\mu \right| = \alpha \int_X f\, d\mu = \int_X \alpha f\, d\mu = \int_X u\, d\mu \leq \int_X |f|\, d\mu.$$

The third of the above equalities holds since the preceding ones show that $\int \alpha f\, d\mu$ is real. ////

We conclude this section with another important convergence theorem.

1.34 Lebesgue's Dominated Convergence Theorem *Suppose $\{f_n\}$ is a sequence of complex measurable functions on X such that*

$$f(x) = \lim_{n \to \infty} f_n(x) \tag{1}$$

exists for every $x \in X$. If there is a function $g \in L^1(\mu)$ such that

$$|f_n(x)| \leq g(x) \qquad (n = 1, 2, 3, \ldots; x \in X), \tag{2}$$

then $f \in L^1(\mu)$,

$$\lim_{n \to \infty} \int_X |f_n - f|\, d\mu = 0, \tag{3}$$

and

$$\lim_{n \to \infty} \int_X f_n\, d\mu = \int_X f\, d\mu. \tag{4}$$

From: Stromberg, K., "An Introduction to Classical Real Analysis", American Mathematical Society

(1) and (3.119, i), we have

$$\|f - h\|_u = \sup\{|P(z)| : z \in X\} = |P(z_0)| = \beta > |P(0)| = 1.$$

If $h = 0$, then $\|f - h\|_u = \|f\|_u = 1$. It follows that $\text{dist}(f, A) = 1$, and so $f \notin A^-$.

If we require A to be closed under complex conjugation, then this difficulty dissolves.

(3.121) Definition We say that a complex function algebra A is *self-adjoint* if $\bar{f} \in A$ whenever $f \in A$.

(3.122) Stone–Weierstrass Theorem [compact-complex] *Let X be a compact space and let A be a self-adjoint subalgebra of $C(X)$ that separates the points of X. Then the uniform closure A^- of A satisfies either*

(i) $A^- = C(X)$

or

(ii) *there is some $p \in X$ such that $A^- = \{f \in C(X) : f(p) = 0\}$.*

Proof Let $A^r = \{f \in A : f(X) \subset \mathbb{R}\}$. Then $f \in A$ implies

$$\text{Re} f = (f + \bar{f})/2 \in A^r, \qquad \text{Im} f = (f - \bar{f})/(2i) \in A^r.$$

Also, for $x \neq y$ in X, we can choose $f \in A$ such that $f(x) \neq f(y)$, hence

$$\text{Re}\, f(x) \neq \text{Re}\, f(y) \quad \text{or} \quad \text{Im}\, f(x) \neq \text{Im}\, f(y),$$

and so A^r separates the points of X. Obviously A^r is a subalgebra of $C^r(X)$. Therefore, either (3.117.i) or (3.117.ii) obtains for A^{r-}. Thus, given $f \in C(X)$ and $\epsilon > 0$ [if (3.117.ii) obtains for A^{r-}, we suppose $f(p) = 0$], we can choose $h_1, h_2 \in A^r$ such that

$$\|\text{Re}\, f - h_1\|_u < \epsilon/2, \qquad \|\text{Im}\, f - h_2\|_u < \epsilon/2.$$

Writing $h = h_1 + ih_2$, we have $h \in A$ and $\|f - h\|_u < \epsilon$. □

(3.123) Example This example shows that compactness is important in (3.117) and (3.122). Let X be any noncompact metric space and let $(x_n)_{n=1}^\infty$ be a sequence of distinct points of X having no convergent subsequence. Define $A = \{f \in C^r(X) : \lim_{n\to\infty} f(x_n) \text{ exists}\}$. Plainly, A is a subalgebra of $C^r(X)$. Let $a \neq b$ in X. Choose $N \in \mathbb{N}$ such that $a \neq x_n$ for all $n > N$. Write $E = \{b\} \cup \{x_n : n > N\}$. The function f defined on X by

$$f(x) = \frac{\text{dist}(x, E)}{\text{dist}(x, \{a\}) + \text{dist}(x, E)}$$

satisfies $f(a) = 1$, $f(b) = 0$, $f \in C^r(X)$, and $\lim_{n\to\infty} f(x_n) = 0$; hence, $f \in A$ and so A separates the points of X. Since A contains 1, A vanishes nowhere on X. However,

From: Stromberg, K., "An Introduction to Classical Real Analysis", American Mathematical Society

Trigonometric Series and Fourier Series

(8.1) Definitions A *trigonometric series* is any series of the form

(i) $$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(kt) + b_k \sin(kt)),$$

where $(a_k)_{k=0}^{\infty}$ and $(b_k)_{k=1}^{\infty}$ are sequences of complex numbers and $t \in \mathbb{R}$. The nth *partial sum* of (i) is the function s_n defined on $\mathbb{R}$ by

(ii) $$s_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(kt) + b_k \sin(kt)).$$

Because of Euler's Formulas (5.5) we also have

(ii′) $$s_n(t) = \sum_{k=-n}^{n} c_k e^{ikt}$$

where, if we write $b_0 = 0$,

(iii) $$c_k = (a_k - ib_k)/2, \qquad c_{-k} = (a_k + ib_k)/2,$$ $$a_k = c_k + c_{-k}, \qquad b_k = i(c_k - c_{-k}) \quad \text{for } k \geqq 0.$$

We also write $s_0(t) = c_0 = a_0/2$. For this reason, we also write (i) in the form

(i′) $$\sum_{k=-\infty}^{\infty} c_k e^{ikt}$$

and we call s_n the nth partial sum of this series. To say that (i) or (i′) converges in some sense [pointwise, a.e., uniformly, etc.] means that the sequence $(s_n)_{n=0}^{\infty}$ of functions converges in that sense. Any function of the form (ii) or (ii′) is called a *trigonometric polynomial.*

If a function $f: \mathbb{R} \to \mathbb{C}$ is to be the pointwise sum of a trigonometric series or the pointwise limit of a sequence $(s_n)_{n=0}^{\infty}$ of trigonometric polynomials, then it must be 2π-*periodic*. That is, $f(t + 2\pi) = f(t)$ for all $t \in \mathbb{R}$. This is because every trigonometric polynomial is 2π-periodic. Thus, we isolate for study some classes of 2π-periodic functions.

(8.2) Definitions Let $f: \mathbb{R} \to \mathbb{C}$ be 2π-periodic. For a positive real number p, we write $f \in L_p(\mathbb{T})$ if f is Lebesgue measurable and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^p \, dt < \infty.$$

In this case, we define the *$L_p(\mathbb{T})$-norm* of f to be the number

$$\|f\|_p = \|f\|_{L_p(\mathbb{T})} = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^p \, dt \right)^{1/p}.$$

From: Stromberg, K., "An Introduction to Classical Real Analysis", American Mathematical Society

If f is continuous on $\mathbb{R}$ (and 2π-periodic), we write $f \in C(\mathbb{T})$ and we define the *uniform norm* of f to be the number

$$\|f\|_u = \|f\|_{C(\mathbb{T})} = \sup_{t\in\mathbb{R}} |f(t)| = \sup_{-\pi \leqq t \leqq \pi} |f(t)|.$$

We denote the set of all trigonometric polynomials by $TP(\mathbb{T})$.

Obviously, $TP(\mathbb{T}) \subset C(\mathbb{T}) \subset L_p(\mathbb{T})$ for all $p > 0$. Moreover, $TP(\mathbb{T})$ is dense in these spaces in the following sense.

(8.3) Theorem (i) *If $f \in C(\mathbb{T})$ and $\epsilon > 0$, then there exists some $P \in TP(\mathbb{T})$ such that $\|f - P\|_u < \epsilon$.*

(ii) *If $1 \leqq p < \infty$, $f \in L_p(\mathbb{T})$, and $\epsilon > 0$, then there exists some $P \in TP(\mathbb{T})$ such that $\|f - P\|_p < \epsilon$.*

Proof (i) Let $f \in C(\mathbb{T})$ and $\epsilon > 0$ be given. Write $X = \{z \in \mathbb{C} : |z| = 1\}$ and define F on X by $F(z) = f(t)$, where $z = e^{it}$ [by the periodicity of f and (5.11), the definition of F is independent of the choice of $t \in \mathbb{R}$ such that $e^{it} = z$; we could take $t = \operatorname{Arg} z$]. It is easy to see that F is continuous at every $z \in X$. In fact, $F(z) = f(\operatorname{Arg} z)$, so continuity follows from (5.15) for $z \neq -1$, but F is also continuous at -1 because $f(\pi) = f(-\pi)$. Now apply (3.129) to obtain $(c_n)_{n=-N}^{N} \subset \mathbb{C}$ such that

$$\left| F(z) - \sum_{n=-N}^{N} c_n z^n \right| < \epsilon$$

for all $z \in X$. Then (i) follows by taking $P(t) = \sum_{n=-N}^{N} c_n e^{int}$.*

(ii) Let $1 \leqq p < \infty$, $f \in L_p(\mathbb{T})$, and $\epsilon > 0$ be given. Use (6.111) to obtain $g \in C([-\pi, \pi])$ such that

$$\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f - g|^p \right)^{1/p} < \frac{\epsilon}{3}. \tag{1}$$

Choose $0 < \beta < \infty$ such that $|g(t)| \leqq \beta$ for $t \in [-\pi, \pi]$. Write $\delta = 2\pi(\epsilon/(6\beta))^p$. We may suppose $\delta < 2\pi$. Now alter g on $[\pi - \delta, \pi]$ to obtain $h \in C([-\pi, \pi])$ such that $|h| \leqq \beta$, $h = g$ on $[-\pi, \pi - \delta]$, and $h(\pi) = h(-\pi)$. For instance, one can define h on $[\pi - \delta, \pi]$ by the rule

$$h(t) = \delta^{-1}\big[(\pi - t)g(\pi - \delta) + (t - \pi + \delta)g(-\pi)\big].$$

Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |g - h|^p = \frac{1}{2\pi} \int_{\pi-\delta}^{\pi} |g - h|^p \leqq \frac{(2\beta)^p \delta}{2\pi} = \left(\frac{\epsilon}{3} \right)^p. \tag{2}$$

*A proof of (8.3.i) that does not depend on the Stone–Weierstrass Theorem (or on (8.3.i)) is given in (8.30).

From: Stromberg, K., "An Introduction to Classical Real Analysis", American Mathematical Society

Next, extend h to be 2π-periodic on $\mathbb{R}$ so that $h \in C(\mathbb{T})$. We can apply part (i) to obtain $P \in TP(\mathbb{T})$ such that $\|h - P\|_u < \epsilon/3$. Then

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|h-P|^p\right)^{1/p} < \frac{\epsilon}{3}. \tag{3}$$

Finally, (1), (2), (3), and Minkowski's Inequality (6.107) complete the proof of (ii). □

(8.4) Remark Restated, (8.3.i) says that if $f \in C(\mathbb{T})$, then there exists a sequence $(P_n)_{n=1}^{\infty} \subset TP(\mathbb{T})$ such that $P_n \to f$ uniformly on $\mathbb{R}$. However, it is not clear (or even true in general) that these P_n's can be taken to be the partial sums s_n of some fixed trigonometric series. If such a series could be found, then it is unique, as the following theorem shows.

(8.5) Theorem *Consider any trigonometric series* $\sum_{k=-\infty}^{\infty} c_k e^{ikt}$ *with partial sums* s_n *as in* (8.1.ii′). *Suppose there exists some subsequence* $(s_{n_j})_{j=1}^{\infty}$ *and some function* f *such that either* (i) $f \in C(\mathbb{T})$ *and* $\|f - s_{n_j}\|_u \to 0$ *as* $j \to \infty$ *or* (ii) $f \in L_p(\mathbb{T})$ *for some* $1 \leqq p < \infty$ *and* $\|f - s_{n_j}\|_p \to 0$ *as* $j \to \infty$. *Then, for every integer* n, *we have*

(iii)
$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt.$$

Proof If (i) obtains, then $f \in L_p(\mathbb{T})$ for all p and $\|f - s_{n_j}\|_p \leqq \|f - s_{n_j}\|_u$ for all j, and so (ii) also obtains. Now suppose that (ii) does obtain. Let n be any fixed integer. Choose $j_0 \in \mathbb{N}$ such that $n_j > |n|$ for $j > j_0$. The crucial, but obvious, fact here is that, for integers k,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikt}e^{-int}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k-n)t}\,dt = 0 \text{ or } 1$$

according as $k \neq n$ or $k = n$. Thus, for all $j > j_0$ we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} s_{n_j}(t)e^{-int}\,dt = \sum_{k=-n_j}^{n_j} c_k \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k-n)t}\,dt = c_n.$$

It follows that

$$\begin{aligned}\left|c_n - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt\right| &= \lim_{j\to\infty}\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(t) - s_{n_j}(t)]e^{-int}\,dt\right| \\ &\leqq \lim_{j\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t) - s_{n_j}(t)|\,dt \\ &\leqq \lim_{j\to\infty}\|f - s_{n_j}\|_p = 0,\end{aligned}$$

where the second inequality follows from Hölder's Inequality (6.106). □

From: Stromberg, K., "An Introduction to Classical Real Analysis", American Mathematical Society

The problem of recognizing whether or not a trigonometric series is a Fourier series by inspecting its coefficients is unsolved and seems to be extremely difficult. Equivalently, the problem is to determine the range of the mapping $f \to \hat{f}$ of $L_1(\mathbb{T})$ *into* $c_0(\mathbb{Z})$. The preceding theorem provides a necessary condition. Theorem (8.5) and the following two theorems furnish sufficient conditions.

(8.17) Riesz-Fischer Theorem *Suppose that* $(c_n)_{n=-\infty}^{\infty} \subset \mathbb{C}$ *satisfies*

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \lim_{p\to\infty} \sum_{n=-p}^{p} |c_n|^2 < \infty.$$

Then there exists a function $f \in L_2(\mathbb{T})$ *such that* $\hat{f}(n) = c_n$ *for every integer* n. *Moreover,* $\|f - s_p(f)\|_2 \to 0$ *as* $p \to \infty$.

Proof Write $s_p(t) = \sum_{n=-p}^{p} c_n e^{int}$ for $p \geqq 0$. Then $q > p \geqq 0$ implies

$$\begin{aligned}\|s_q - s_p\|_2^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |s_q(t) - s_p(t)|^2\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\sum_{p<|n|\leqq q} c_n e^{int}\Big)\Big(\sum_{p<|k|\leqq q} \overline{c_k} e^{-ikt}\Big)dt \\ &= \sum_{\substack{p<|n|\leqq q \\ p<|k|\leqq q}}\sum c_n \overline{c_k}\,\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-k)t}\,dt = \sum_{p<|n|\leqq q} |c_n|^2.\end{aligned}$$

By hypothesis, the last sum has limit 0 as $p, q \to \infty$. It follows from (6.110) that there exists $f \in L_2(\mathbb{T})$ such that $\|f - s_p\|_2 \to 0$ as $p \to \infty$. Now (8.5.ii) shows that $\hat{f}(n) = c_n$ for all n. Finally, $s_p(t) = \sum_{n=-p}^{p} \hat{f}(n) e^{int} = s_p(f, t)$. □

The next theorem has very special hypotheses, but it is often useful for producing Fourier series. Also, its proof is instructive.

(8.18) Theorem *Let* $(a_k)_{k=0}^{\infty}$ *be a nonincreasing sequence of nonnegative real numbers having limit* 0. *Suppose also that this sequence is convex*: $2a_{k+1} \leqq a_k + a_{k+2}$ *for all* $k \geqq 0$. *Then the function* f *defined on* $\mathbb{R}$ *by*

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kt) \tag{i}$$

is in $L_1^+(\mathbb{T})$ $[f \geqq 0]$ *and the series* (i) *is the Fourier series of* f. *Moreover, this series converges uniformly on* $[\delta, 2\pi - \delta]$ *whenever* $0 < \delta < \pi$, *and so* f *is continuous except possibly at integral multiples of* 2π.

Proof The last sentence, which does not require convexity, follows from (7.38), and so (i) defines a function f which is 2π-periodic and measurable

Exercise 3.1. Recalling that the L^2-norm is defined by $\|x\|_2 := \sqrt{x_1^2 + \cdots + x_d^2}$, and the L^1-norm is defined by $\|x\|_1 := |x_1| + \cdots + |x_d|$, we have the following elementary norm relations.

(a) Show that $\|x\|_2 \le \|x\|_1$, for all $x \in \mathbb{R}^d$.
(b) On the other hand, show that we have $\|x\|_1 \le \sqrt{d}\ \|x\|_2$, for all $x \in \mathbb{R}^d$.

Solution:

$$\|x\|_2^2 = \sum_{k=1}^{d} x_k^2 \le \sum_{k=1}^{d} x_k^2 + \sum_{\substack{k,l=1 \\ k\neq l}}^{d} |x_k||x_l| = \sum_{k=1}^{d} |x_k| \sum_{l=1}^{d} |x_l| = \|x\|_1^2$$

$$\implies \|x\|_2 \le \|x\|_1$$

$$\|x\|_1 = \left|\big\langle (1,\ldots,1),\ (|x_1|,\ldots,|x_d|)\big\rangle\right| \le \sqrt{d}\|x\|_2 \quad \text{(by Cauchy-Schwarz)}$$

Exercise 3.3. We know that the functions $u(t) := \cos t = \frac{e^{it}+e^{-it}}{2}$ and $v(t) := \sin t = \frac{e^{it}-e^{-it}}{2i}$ are natural, partly because they parametrize the unit circle: $u^2 + v^2 = 1$. Here we see that there are other similarly natural functions, parametrizing the hyperbola.

(a) Show that the following functions parametrize the hyperbola $u^2 - v^2 = 1$:

$$u(t) := \frac{e^t + e^{-t}}{2}, \quad v(t) := \frac{e^t - e^{-t}}{2}.$$

(This is the reason that the function $\cosh t := \frac{e^t+e^{-t}}{2}$ is called the hyperbolic cosine, and the function $\sinh t := \frac{e^t-e^{-t}}{2}$ is called the hyperbolic sine)

Solution: For any $t \in \mathbb{R}$,

$$u(t)^2 - v(t)^2 = \left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2 = \frac{e^{2t} + 2 + e^{-2t} - e^{2t} + 2 - e^{-2t}}{4} = 1.$$

Now if $x, y \in \mathbb{R}$ are such that $x^2 - y^2 = 1$, we must show that exists t such that $x = u(t)$ and $y = v(t)$. Since $u(t) \ge 0$, this will only happen if $x \ge 0$. Note that $v(t)$ is bijective from $\mathbb{R}$ to $\mathbb{R}$, hence invertible.

(b) The hyperbolic cotangent is defined as $\coth t := \frac{\cosh t}{\sinh t} = \frac{e^t+e^{-t}}{e^t-e^{-t}}$. Using Bernoulli numbers, show that $t \coth t$ has the Taylor series:

$$t \coth t = \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!} B_{2n} t^{2n}.$$

Solution: Recall

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}.$$

Using $x = 0$ and summing the expression above with t and $-t$,

$$\frac{t}{e^t - 1} + \frac{-t}{e^{-t} - 1} = \sum_{k=0}^{\infty} B_k(0)\frac{t^k}{k!} + \sum_{k=0}^{\infty} B_k(0)\frac{(-t)^k}{k!}$$
$$= \sum_{n=0}^{\infty} B_{2n}(0)\frac{2t^{2n}}{(2n)!}$$

Replacing t by $2t$,

$$\frac{2t}{e^{2t} - 1} + \frac{-2t}{e^{-2t} - 1} = \sum_{n=0}^{\infty} B_{2n}(0)\frac{2(2t)^{2n}}{(2n)!} = 2\sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!}B_{2n}t^{2n}$$
$$\frac{t}{e^{2t} - 1} + \frac{-t}{e^{-2t} - 1} = \frac{te^{-t}}{e^t - e^{-t}} + \frac{-te^t}{e^{-t} - e^t} = t\frac{e^t + e^{-t}}{e^t - e^{-t}}.$$

Exercise 3.5. We continue with the same function as in the previous exercise, $f(x) := e^{-2\pi t|x|}$.

(a) Show that $\hat{f}(\xi) = \frac{t}{\pi}\frac{1}{\xi^2+t^2}$, for all $\xi \in \mathbb{R}$.

(b) Using Poisson summation, show that:

$$\frac{t}{\pi}\sum_{n\in\mathbb{Z}} \frac{1}{n^2 + t^2} = \sum_{m\in\mathbb{Z}} e^{-2\pi t|m|}.$$

Solution:

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(t)e^{-2\pi i x\xi}dx = \int_{\mathbb{R}} e^{-2\pi t|x|}e^{-2\pi i x\xi}dx$$
$$= \int_{-\infty}^{0} e^{2\pi x(t-i\xi)}dx + \int_{0}^{\infty} e^{2\pi x(-t-i\xi)}dx$$
$$= \frac{e^{2\pi x(t-i\xi)}}{2\pi(t-i\xi)}\Big|_{x=-\infty}^{0} + \frac{e^{2\pi x(-t-i\xi)}}{2\pi(-t-i\xi)}\Big|_{x=0}^{\infty}$$
$$= \frac{1}{2\pi(t-i\xi)} + \frac{1}{2\pi(t+i\xi)} \qquad \text{(we must assume } t > 0\text{)}$$
$$= \frac{t+i\xi+t-i\xi}{2\pi(t^2+\xi^2)} = \frac{t}{\pi(t^2+\xi^2)}.$$