A GENTLE INTRODUCTION TO FOURIER ANALYSIS ON POLYTOPES - EXERCISES FROM CHAPTER 2

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Exercises from the book: Robins, S., "A friendly introduction to Fourier analysis on polytopes", available at `https://arxiv.org/abs/2104.06407`. This solution sheet is available at `https://www.ime.usp.br/~fabcm/33coloquio-impa/`.

2.2. Show that the Fourier transform of the unit cube $C := [0,1]^d \subset \mathbb{R}^d$ is:

$$\hat{1}_C(\xi) = \frac{1}{(2\pi i)^d} \prod_{k=1}^{d} \frac{1 - e^{-2\pi i \xi_k}}{\xi_k},$$

valid for all $\xi \in \mathbb{R}^d$, except for the union of hyperplanes defined by
$H := \{x \in \mathbb{R}^d \mid \xi_1 = 0 \text{ or } \xi_2 = 0, \ \ldots \text{ or } \xi_d = 0\}$.

2.4. To gain some facility with generating functions, show by a brute-force computation with Taylor series that the coefficients on the right-hand-side of equation (2.13)[1], which are called $B_n(x)$ by definition, must in fact be polynomials in x.

In fact, your direct computations will show that for all $n \geq 1$, we have

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}\, x^k,$$

where B_j is the j'th Bernoulli number.

2.6. Show that for all $n \geq 1$, we have

$$B_n(x+1) - B_n(x) = nx^{n-1}.$$

2.8. Prove that:

$$\sum_{k=1}^{n-1} k^{d-1} = \frac{B_d(n) - B_d}{d},$$

for all integers $d \geq 1$ and $n \geq 2$.

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[1]
$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}. \tag{2.13}$$

2.9. Show that the periodic Bernoulli polynomials $P_n(x) := B_n(\{x\})$, for all $n \geq 2$, have the following Fourier series:

$$P_n(x) = -\frac{n!}{(2\pi i)^n} \sum_{k \neq 0} \frac{e^{2\pi i k x}}{k^n}, \tag{1}$$

valid for all $x \in \mathbb{R}$. For $n \geq 2$, these series are absolutely convergent. We note that from the definition above, $B_n(x) = P_n(x)$ when $x \in (0, 1)$.

2.12. Here we give a different method for defining the Bernoulli polynomials, based on the following three properties that they enjoy:

(a) $B_0(x) = 1$.

(b) For all $n \geq 1$, $\frac{d}{dx} B_n(x) = n B_{n-1}(x)$.

(c) For all $n \geq 1$, we have $\int_0^1 B_n(x) dx = 0$.

Show that the latter three properties imply the original defining property of the Bernoulli polynomials (2.13).

Lecture 1

- Fourier transform of an interval
$$\hat{1}_{[a,b]}(\xi) = \int_a^b e^{-2\pi i \xi x} dx = \frac{e^{-2\pi i \xi b} - e^{-2\pi i \xi a}}{-2\pi i \xi}.$$

  If $I \subset \mathbb{R}$ is an interval and $f \in I \to \mathbb{C}$, then $f(x) = \mathrm{Re}(f)(x) + i\mathrm{Im}(f)(x)$, with $\mathrm{Re}(f), \mathrm{Im}(f) \colon I \to \mathbb{R}$ and
$$\int_I f(x)dx := \int_I \mathrm{Re}(f)(x)dx + i \int_I \mathrm{Im}(f)(x)dx.$$

  $f(x) = e^{-2\pi i \xi x} = \cos(-2\pi \xi x) + i \sin(-2\pi \xi x)$

  $f \colon \mathbb{C} \to \mathbb{C}$, $f(z) = e^{-2\pi i \xi z}$, $F(z) = \frac{e^{-2\pi i \xi z}}{(-2\pi i \xi)}$, $F'(z) = f(z)$ for all $z \in \mathbb{C}$.

- Euler formula and series definitions for the exponential, sine, and cossine
- The sawtooth function and its Fourier series
$$x - [x] - \frac{1}{2} = -\frac{1}{2\pi i} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{e^{2\pi i n x}}{n}.$$
- Fourier transform as an extension to the volume of an object
$$\hat{1}_P(0) = \mathrm{vol}(P).$$
$$\lim_{t \to 0} \frac{e^{-2\pi i t b} - e^{-2\pi i t a}}{-2\pi i t} = \lim_{t \to 0} \frac{(-2\pi i) b e^{-2\pi i t b} - (-2\pi i) a e^{-2\pi i t a}}{-2\pi i} = b - a.$$
- Tiling of nice retangles

From: Remmert, R., "Theory of Complex Functions", Springer-Verlag, New York

complex-valued continuous functions, to the extent necessary for the needs of function theory. $I = [a, b]$, with $a \le b$ will designate a compact interval in $\mathbf{R}$.

1. The integral concept. Rules of calculation and the standard estimate. For every continuous function $f : I \to \mathbf{C}$ the definition

$$\int_r^s f(t)dt := \int_r^s (\Re f)(t)dt + i\int_r^s (\Im f)(t)dt \in \mathbf{C}$$

makes sense for any $r, s \in I$ because $\Re f$ and $\Im f$ are real-valued continuous, consequently integrable, functions. We have the following simple

Rules of calculation. *For all $f, g \in C(I)$, all $r, s \in I$ and all $c \in \mathbf{C}$*

(1) $$\int_r^s (f+g)(t)dt = \int_r^s f(t)dt + \int_r^s g(t)dt\ ,\quad \int_r^s cf(t)dt = c\int_r^s f(t)dt,$$

(2) $$\int_r^x f(t)dt + \int_x^s f(t)dt = \int_r^s f(t)dt \qquad \textit{for every } x \in I,$$

(3) $$\int_s^r f(t)dt = -\int_r^s f(t)dt \qquad \textit{(reversal rule)}$$

(4) $$\Re\int_r^s f(t)dt = \int_r^s \Re f(t)dt,\quad \Im\int_r^s f(t)dt = \int_r^s \Im f(t)dt.$$

The mapping $C(I) \to \mathbf{C}$, $f \mapsto \int_a^b f(t)dt$ is thus in particular a *complex-linear form* on the $\mathbf{C}$-vector space $C(I)$. We call $\int_a^b f(t)dt$ the *integral of f along the (real) interval* $[a, b]$. For real-valued functions $f, g \in C(I)$ there is a

Monotonicity rule: $\int_a^b f(t)dt \le \int_a^b g(t)dt$ *in case* $f(t) \le g(t)$ *for all* $t \in I$.

For complex-valued functions the appropriate analog of this rule is the

Standard estimate: $\left|\int_a^b f(t)dt\right| \le \int_a^b |f(t)|dt \qquad$ *for all* $\quad f \in C(I)$.

Proof. For real-valued f this follows at once from the monotonicity rule and the inequalities $-|f(t)| \le f(t) \le |f(t)|$. The general case is reduced to this one as follows: There is a complex number c of modulus 1 such

From: Remmert, R., "Theory of Complex Functions", Springer-Verlag, New York

Exercise 2. For any polynomial $p(z)$, any $c \in \mathbf{C}$, $r \in \mathbf{R}^+$

$$\int_{\partial B_r(c)} \overline{p(\zeta)}\, d\zeta = 2\pi i r^2 \overline{p'(c)}.$$

Exercise 3. Let $\gamma : [a, b] \to \mathbf{C}$ be a continuously differentiable path with $\gamma'(t) \neq 0$ for all $t \in [a, b]$. Then there is a path $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to \mathbf{C}$ which is equivalent to γ and satisfies $|\tilde{\gamma}'(t)| = 1$ for all $t \in [\tilde{a}, \tilde{b}]$.

Exercise 4 (Sharpened standard estimate). Let γ be a path in $\mathbf{C}$ and $f \in \mathcal{C}(|\gamma|)$. If there exists $c \in |\gamma|$ such that $|f(c)| < |f|_\gamma := \max_{\zeta \in |\gamma|} |f(\zeta)|$, then

$$\left| \int_\gamma f(\zeta) d\zeta \right| < |f|_\gamma \cdot L(\gamma).$$

Exercise 5. Let $t_n(z) := 1 + z + \frac{1}{2!}z^2 + \cdots + \frac{1}{n!}z^n$ be the nth Taylor polynomial approximant to e^z. Show that $|e^z - t_n(z)| < |z|^{n+1}$ for all $n \in \mathbf{N}$ and all $z \in \mathbf{C}$ with $\Re z < 0$.

§3 Path independence of integrals. Primitives.

The path integral $\int_\gamma f d\zeta$ is, for fixed $f \in \mathcal{C}(D)$, a *function of the path* γ *in* D. Two points $z_I, z_T \in D$ can be joined, if at all, by a multitude of paths γ in D. We saw in 1.3 that, even in the case of a holomorphic function f in D, the integral $\int_\gamma f d\zeta$ in general depends not just on the initial point z_I and the terminal point z_T but on the whole course of the path γ. Here we will discuss conditions which guarantee the *path independence* of the integral $\int_\gamma f d\zeta$, in the sense that its value is determined solely by the initial and terminal points of the path.

1. Primitives. We want to generalize the concept of primitive (function) introduced in 0.2. Fundamental here is the following

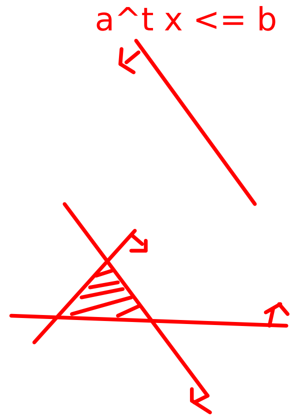
Theorem. *If f is continuous in D, the following assertions about a function $F : D \to \mathbf{C}$ are equivalent:*

i) *F is holomorphic in D and satisfies $F' = f$.*

ii) *For every pair $w, z \in D$ and every path γ in D with initial point w and terminal point z*

$$\int_\gamma f d\zeta = F(z) - F(w).$$

Lecture 2



- Polytopes (facet description, simplex)

$$P = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : a_{j1}x_1 + \cdots + a_{jd}x_d \leq b_j,\ j \in J\},$$

  with a_{ji}, $b_j \in \mathbb{R}$, $|J| < \infty$ and we assume P is bounded.

- The sinc function

$$\hat{1}_{[-\frac{1}{2},\frac{1}{2}]}(\xi) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i x\xi} dx = \frac{\sin(\pi\xi)}{\pi\xi} =: \mathrm{sinc}(\xi)$$

- Fourier transform of a parallelepiped

- The stretch lemma

$$(\widehat{f \circ M})(\xi) = \frac{1}{|\det(M)|}\hat{f}\left(M^{-\mathsf{T}}\xi\right).$$

$$(\widehat{f \circ M})(\xi) = \int_{\mathbb{R}^d} f(Mx)e^{-2\pi i\langle x,\xi\rangle}dx = \frac{1}{|\det(M)|}\int_{\mathbb{R}^d} f(y)e^{-2\pi i\langle M-1y,\xi\rangle}dx$$
$$= \frac{1}{|\det(M)|}\int_{\mathbb{R}^d} f(y)e^{-2\pi i\langle y,M^{-\mathsf{T}}\xi\rangle}dx,$$

  where we use the change of variables $y = Mx$.

- The translation lemma ($f \circ T_v(x) := f(x + v)$):

$$(\widehat{f \circ T_v})(\xi) = e^{2\pi i\xi v}\hat{f}(\xi).$$

Exercise 2.2. Show that the Fourier transform of the unit cube $C := [0,1]^d \subset \mathbb{R}^d$ is:

$$\hat{1}_C(\xi) = \frac{1}{(2\pi i)^d} \prod_{k=1}^{d} \frac{1 - e^{-2\pi i \xi_k}}{\xi_k},$$

valid for all $\xi \in \mathbb{R}^d$, except for the union of hyperplanes defined by
$H := \{x \in \mathbb{R}^d \mid \xi_1 = 0 \text{ or } \xi_2 = 0, \ \ldots \ \text{or } \xi_d = 0\}$.

Solution:

$$\hat{1}_C(\xi) = \int_{[0,1]^d} e^{-2\pi i \langle x, \xi \rangle} dx = \prod_{k=1}^{d} \int_0^1 e^{-2\pi i x_k \xi_k} dx_k = \prod_{k=1}^{d} \frac{1}{(-2\pi i \xi_k)} e^{-2\pi i x_k \xi_k} \Big|_{x_k=0}^{1}$$
$$= \prod_{k=1}^{d} \frac{e^{-2\pi i \xi_k} - 1}{(-2\pi i \xi_k)} = \frac{1}{(2\pi i)^d} \prod_{k=1}^{d} \frac{1 - e^{-2\pi i \xi_k}}{\xi_k}.$$

Exercise 2.4. To gain some facility with generating functions, show by a brute-force computation with Taylor series that the coefficients on the right-hand-side of equation

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}, \tag{2.13}$$

which are called $B_n(x)$ by definition, must in fact be polynomials in x.

Solution:

$$te^{xt} = (e^t - 1) \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!},$$
$$\sum_{n=0}^{\infty} \frac{x^n}{n!} t^{n+1} = \sum_{m=1}^{\infty} \frac{t^m}{m!} \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}, \qquad \text{next we use } n = k + m - 1,$$
$$\sum_{n=0}^{\infty} \frac{x^n}{n!} t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{B_k(x)}{k!(n-k+1)!} t^n,$$
$$\frac{B_n(x)}{n!} + \sum_{k=0}^{n-1} \frac{B_k(x)}{k!(n-k+1)!} = \frac{x^n}{n!}$$
$$B_n(x) = x^n - \frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k(x).$$

(This is not the formula asked, but also shows that $B_n(x)$ is a polynomial of degree n in x.)

Exercise 2.6. Show that for all $n \geq 1$, we have

$$B_n(x+1) - B_n(x) = n x^{n-1}.$$

Answer: Using Exercise 2.4,

$$
\begin{aligned}
B_n(x+1) - B_n(x) &= (x+1)^n - \frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k(x+1) - x^n + \frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k(x) \\
&= (x+1)^n - x^n - \frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}(B_k(x+1) - B_k(x)) \quad (*) \\
&= (x+1)^n - x^n - \frac{1}{n+1}\sum_{k=1}^{n-1}\frac{(n+1)!}{k!(n+1-k)!}kx^{k-1} \\
&= (x+1)^n - x^n - \sum_{k=0}^{n-2}\frac{n!}{k!(n-k)!}x^k \\
&= nx^{n-1}
\end{aligned}
$$

In $(*)$, we use that $B_0(x) = 1$, recognize that the term $k = 0$ is 0 and apply induction for the other terms. We also use

$$
(x+1)^n = \sum_{k=0}^{n}\binom{n}{k}x^k.
$$

Exercise 2.9. Show that the periodic Bernoulli polynomials $P_n(x) := B_n(\{x\})$, for all $n \geq 2$, have the following Fourier series:

$$
P_n(x) = -\frac{n!}{(2\pi i)^n}\sum_{k\neq 0}\frac{e^{2\pi ikx}}{k^n}, \tag{2}
$$

valid for all $x \in \mathbb{R}$. For $n \geq 2$, these series are absolutely convergent. We note that from the definition above, $B_n(x) = P_n(x)$ when $x \in (0,1)$.

Answer: We use the orthogonality relations of the exponential functions and the fact that $P_n(x)$ has a Fourier series for $x \in (0,1)$:

$$
P_n(x) = \sum_{k\in\mathbb{Z}} f_k e^{2\pi ikx} \implies \int_0^1 P_n(x)e^{-2\pi ikx}dx = f_k.
$$

We want to show that $f_0 = 0$ and for all $k \in \mathbb{Z}\setminus\{0\}$, $n \in \mathbb{Z}$, $n \geq 2$:

$$
f_k = \frac{-n!}{(2\pi i)^n k^n}.
$$

If $k = 0$, we show that $\int_0^1 B_n(x)dx = 0$ for all $n \geq 1$ using the generating function (2.13):

$$
\begin{aligned}
\int_0^1 \frac{te^{xt}}{e^t - 1}dx &= \sum_{n=0}^{\infty}\int_0^1 B_n(x)dx\frac{t^n}{n!} \\
\frac{e^t - 1}{e^t - 1} &= \sum_{n=0}^{\infty}\int_0^1 B_n(x)dx\frac{t^n}{n!} \\
1 &= \sum_{n=0}^{\infty}\int_0^1 B_n(x)dx\frac{t^n}{n!}
\end{aligned}
$$

If $k \neq 0$ and $n = 1$,

$$\begin{aligned}
\int_0^1 B_1(x) e^{-2\pi i k x} dx &= \int_0^1 \left(x - \frac{1}{2}\right) e^{-2\pi i k x} dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} x e^{-2\pi i k x} e^{-\pi i k} dx \\
&= e^{-\pi i k} \frac{x e^{-2\pi i k x}}{(-2\pi i k)} \Big|_{x=-\frac{1}{2}}^{\frac{1}{2}} + \frac{e^{-\pi i k}}{2\pi i k} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i k x} dx \\
&= e^{-\pi i k} \frac{e^{\pi i k}}{(-2\pi i k)} = -\frac{1}{2\pi i k}.
\end{aligned}$$

Note that using Exercise 2.6 with $x = 0$, for $n \geq 2$ we have $B_n(1) = B_n(0)$.

We need the following identity:

$$\sum_{k=1}^{\infty} B_k'(x) \frac{t^k}{k!} = \frac{t^2 e^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^{k+1}}{k!} = \sum_{k=1}^{\infty} k B_{k-1}(x) \frac{t^k}{k!}$$

$$B_k'(x) = k B_{k-1}(x) \tag{1}$$

Next we integrate by parts and use (1) (assume $k \neq 0$ and $n \geq 2$):

$$\begin{aligned}
\int_0^1 B_n(x) e^{-2\pi i k x} dx &= B_n(x) \frac{e^{-2\pi i k x}}{(-2\pi i k)} \Big|_{x=0}^{1} + \int_0^1 B_n'(x) \frac{e^{-2\pi i k x}}{2\pi i k} dx \\
&= \frac{n}{2\pi i k} \int_0^1 B_{n-1}(x) e^{-2\pi i k x} dx \qquad \text{using induction,} \\
&= \frac{n}{2\pi i k} \frac{-(n-1)!}{(2\pi i)^{n-1} k^{n-1}} \\
&= \frac{-n!}{(2\pi i)^n k^n}.
\end{aligned}$$