# A GENTLE INTRODUCTION TO FOURIER ANALYSIS ON POLYTOPES - EXERCISES FROM CHAPTER 1

FABRÍCIO C. MACHADO

Exercises from the book: Robins, S., "A friendly introduction to Fourier analysis on polytopes", available at `https://arxiv.org/abs/2104.06407`. This solution sheet is available at `https://www.ime.usp.br/~fabcm/33coloquio-impa/`.

1.1. Show that if $x \in \mathbb{C}$, then $e^{2\pi i x} = 1$ if and only if $x \in \mathbb{Z}$.

1.3. Here we prove the orthogonality relations for the exponential functions defined by $e_n(x) := e^{2\pi i n x}$ for each integer $n$. Recall that the complex conjugate of any complex number $x + iy$ is defined by $\overline{x+iy} := x - iy$, so that $\overline{e^{i\theta}} = e^{-i\theta}$ for real $\theta$. Prove that for all integers $a, b$:

$$\int_0^1 e_a(x)\overline{e_b(x)}dx = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if not.} \end{cases}$$

1.5. We recall that the $N$'th roots of unity are by definition the set of $N$ complex solutions to $z^N = 1$, and are given by the set $\{e^{2\pi i k/N} : k = 0, 1, 2, \ldots, N-1\}$ of points on the unit circle. Prove that the sum of all of the $N$'th roots of unity vanishes. Precisely, fix any positive integer $N \geq 2$, and show that

$$\sum_{k=0}^{N-1} e^{\frac{2\pi i k}{N}} = 0.$$

1.6. Prove that, given positive integers $M, N$, we have

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{\frac{2\pi i k M}{N}} = \begin{cases} 1 & \text{if } N \mid M \\ 0 & \text{if not.} \end{cases}$$

Notes. This result is sometimes referred to as "the harmonic detector" for detecting when a rational number $\frac{M}{N}$ is an integer; that is, it assigns a value of 1 to the sum if $\frac{M}{N} \in \mathbb{Z}$, and it assigns a value of 0 to the sum if $\frac{M}{N} \notin \mathbb{Z}$.

1.8. Show that for any positive integer $n$, we have

$$n = \prod_{k=1}^{n-1}(1 - \zeta^k),$$

where $\zeta := e^{2\pi i/n}$.

1.13. Thinking of the function $\sin(\pi z)$ as a function of a complex variable $z \in \mathbb{C}$, show that its zeros are precisely the set of integers $\mathbb{Z}$.

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In this course we assume that we know

- Trigonometry,
- Calculus,
- Arithmetic with complex numbers,

but not much from complex analysis.

The *Fourier transform* of a function $f\colon \mathbb{R}^d \to \mathbb{C}$ is the function $\hat{f}\colon \mathbb{R}^d \to \mathbb{C}$ defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \xi \rangle} dx.$$

If $P \subset \mathbb{R}^d$ is bounded and $1_P$ is the indicator funtion of $P$,

$$1_P(x) = \begin{cases} 1, & \text{if } x \in P, \\ 0, & \text{otherwise,} \end{cases}$$

we are interested in the function $\hat{1}_P$ and in how we can retrieve properties of $P$ from $\hat{1}_P$.

**Exercise 1.1.** Show that if $x \in \mathbb{C}$, then $e^{2\pi i x} = 1$ if and only if $x \in \mathbb{Z}$.

*Solution:* First, lets assume that $x \in \mathbb{R}$.

$$e^{2\pi i x} = \cos(2\pi x) + i \sin(2\pi x)$$

$e^{2\pi i x} = 1 \iff \cos(2\pi x) = 1$ and $\sin(2\pi x) = 0$.

$\cos(2\pi x) = 1 \iff x \in \mathbb{Z}$, while $\sin(2\pi x) = 0 \iff 2x \in \mathbb{Z}$.

$\therefore\ e^{2\pi i x} = 1 \iff x \in \mathbb{Z}$.

Now we consider $z \in \mathbb{C}$, $z = x + iy$, $x, y \in \mathbb{R}$.

$$e^{2\pi i z} = e^{2\pi i (x+iy)} = e^{2\pi i x - 2\pi y} = e^{-2\pi y} e^{2\pi i x}$$

$|e^{2\pi i z}| = e^{-2\pi y}$, $e^{2\pi i z} = 1 \implies e^{-2\pi y} = 1 \implies y = 0 \implies z \in \mathbb{R}$.

**Exercise 1.3.** Prove that for all integers $a, b$:

$$\int_0^1 e_a(x) \overline{e_b(x)} dx = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if not.} \end{cases}$$

*Solution:*

$$\begin{aligned}
\int_0^1 e_a(x)\overline{e_b(x)}dx &= \int_0^1 e^{2\pi i a x}e^{-2\pi i b x}dx \\
&= \int_0^1 e^{2\pi i(a-b)x}dx \quad (= 1 \text{ if } a = b) \\
&= \frac{1}{2\pi i(a-b)}e^{2\pi i(a-b)x}\Big|_{x=0}^{1} \\
&= 0.
\end{aligned}$$

since $e^{2\pi i(a-b)x} = 1$ for every $x \in \mathbb{Z}$.

**Exercise 1.5.** Fix any positive integer $N \geq 2$, and show that

$$\sum_{k=0}^{N-1} e^{\frac{2\pi i k}{N}} = 0.$$

*Solution:*

$$\sum_{k=0}^{N-1} e^{\frac{2\pi i k}{N}} = \frac{e^{\frac{2\pi i N}{N}} - 1}{e^{\frac{2\pi i}{N}} - 1} = 0.$$

Where we have used

$$\sum_{k=0}^{n-1} r^k = \frac{r^n - 1}{r - 1}, \quad \text{for } r \neq 1.$$

**Exercise 1.8.** Show that for any positive integer $n$, we have

$$n = \prod_{k=1}^{n-1}(1 - \zeta^k),$$

where $\zeta := e^{2\pi i/n}$.

*Solution:*

$$x^n - 1 = (x-1)\sum_{k=0}^{n-1} x^k = (x-1)\prod_{k=1}^{n-1}(x - \zeta^k)$$

$$\implies \sum_{k=0}^{n-1} x^k = \prod_{k=1}^{n-1}(x - \zeta^k) \implies n = \sum_{k=0}^{n-1} 1 = \prod_{k=1}^{n-1}(1 - \zeta^k).$$

**Exercise 1.13.** Thinking of the function $\sin(\pi z)$ as a function of a complex variable $z \in \mathbb{C}$, show that its zeros are precisely the set of integers $\mathbb{Z}$.

*Solution:*

$$\sin(\pi z) := \frac{e^{i\pi z} - e^{-i\pi z}}{2i}$$

$$\sin(\pi z) = 0 \iff e^{\pi i z} - e^{-\pi i z} = 0 \iff e^{-\pi i z}(e^{2\pi i z} - 1) = 0 \iff e^{2\pi i z} = 1 \iff z \in \mathbb{Z}.$$