A GENTLE INTRODUCTION TO FOURIER ANALYSIS ON POLYTOPES - EXERCISES 5

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Exercises from the book: Robins, S., "A friendly introduction to Fourier analysis on polytopes", available at `https://arxiv.org/abs/2104.06407`. This solution sheet is available at `https://www.ime.usp.br/~fabcm/33coloquio-impa/`.

Lecture 5

- The geometry of numbers: Siegel's formula.

  Let $B \subset \mathbb{R}^d$ be a d-dimensional convex body, symmetric about the origin. If the only integer point in its interior is the origin, then

$$2^d = \operatorname{vol} B + \frac{4^d}{\operatorname{vol} B} \sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} \left| \hat{1}_{\frac{1}{2}B}(\xi) \right|^2.$$

  (Poisson summation applied to $1_{\frac{1}{2}B} * 1_{-\frac{1}{2}B}$.)

  Lemma. Let $B \subset \mathbb{R}^d$ be a d-dimensional convex body, symmetric about the origin. If the only integer point in its interior is the origin, then

$$\operatorname{vol} B = 2^d \iff \hat{1}_{\frac{1}{2}B}(\xi) = 0, \quad \forall \xi \in \mathbb{Z}^d \setminus \{0\}.$$

- The geometry of numbers: tiling of $\mathbb{R}^d$ with translations of a convex body.

$$\sum_{n \in \Lambda} 1_{\frac{1}{2}B}(x - n) = \sum_{n \in \Lambda} 1_{\frac{1}{2}B + n}(x) = 1, \quad \text{for almost all } x \in \mathbb{R}^d.$$

  If Λ is a lattice, the function above is periodic in x and has a Fourier series:

$$1 = \frac{1}{\det \Lambda} \sum_{\xi \in \Lambda^*} \hat{1}_{\frac{1}{2}B}(\xi) e^{2\pi i \langle \xi, x \rangle},$$

  by uniquiness of the Fourier series:

$$\begin{cases} 1 = \frac{1}{\det \Lambda} \hat{1}_{\frac{1}{2}B}(0) = \frac{\operatorname{vol}(\frac{1}{2}B)}{\det \Lambda} = \frac{1}{2^d} \frac{\operatorname{vol}(B)}{\det \Lambda}, \\ 0 = \hat{1}_{\frac{1}{2}B}(\xi) \quad \forall \xi \in \Lambda^* \setminus \{0\}. \end{cases}$$

  Theorem. Let B be any convex, centrally symmetric subset of R^d. If the only integer point in the interior of B is the origin, then $2^d = \operatorname{vol} B$ if and only if $\frac{1}{2}B$ tiles $\mathbb{R}^d$ by integer translations.

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Let $◺ = \operatorname{conv}\{(0,0),(1,0),(0,1)\}$. We compute its Fourier transform (see Formula (2.30) in the book):

$$\begin{aligned}
\hat{1}_{◺}(\xi_1,\xi_2) &= \int_{◺} e^{-2\pi i(\xi_1 x_1+\xi_2 x_2)} dx_1 dx_2 \\
&= \int_0^1 e^{-2\pi i\xi_1 x_1} \int_0^{1-x_1} e^{-2\pi i\xi_2 x_2} dx_2 dx_1 \\
&= \int_0^1 e^{-2\pi i\xi_1 x_1} \frac{e^{-2\pi i\xi_2(1-x_1)}-1}{(-2\pi i\xi_2)} dx_1 \\
&= \frac{1}{(-2\pi i\xi_2)} \int_0^1 (e^{-2\pi i((\xi_1-\xi_2)x_1+\xi_2)} - e^{-2\pi i\xi_1 x_1}) dx_1 \\
&= \frac{1}{(-2\pi i)^2\xi_2}\Big(e^{-2\pi i\xi_2}\frac{e^{-2\pi i(\xi_1-\xi_2)}-1}{\xi_1-\xi_2} - \frac{e^{-2\pi i\xi_1}-1}{\xi_1}\Big) \\
&= \frac{1}{(2\pi i)^2}\Big(\frac{e^{-2\pi i\xi_1}-e^{-2\pi i\xi_2}}{(\xi_1-\xi_2)\xi_2} - \frac{e^{-2\pi i\xi_1}}{\xi_1\xi_2} + \frac{1}{\xi_1\xi_2}\Big) \\
&= \frac{1}{(2\pi i)^2}\Big(\frac{e^{-2\pi i\xi_1}}{(\xi_1-\xi_2)\xi_1} - \frac{e^{-2\pi i\xi_2}}{(\xi_1-\xi_2)\xi_2} + \frac{1}{\xi_1\xi_2}\Big).
\end{aligned}$$

Let $\Diamond := \operatorname{conv}\{(0,0),(1,0),(0,1),(-1,0),(0,-1)\}$. To compute its Fourier transform, we decompose it as the union of 4 triangles: ◺ , ⊿ , ◸ , and ◹ and sum its Fourier transforms.

Recall the stretch lemma $(\widehat{f\circ M})(\xi) = \frac{1}{|\det(M)|}\hat{f}\big(M^{-\mathsf{T}}\xi\big)$. So taking $M = \left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$, we have that $\hat{1}_{⊿}(\xi_1,\xi_2) = \hat{1}_{◺}(-\xi_1,\xi_2)$, and similarly, $\hat{1}_{◸}(\xi_1,\xi_2) = \hat{1}_{◺}(\xi_1,-\xi_2)$ and $\hat{1}_{◹}(\xi_1,\xi_2) = \hat{1}_{◺}(-\xi_1,-\xi_2)$.

$$\begin{aligned}
1_{\Diamond}(\xi_1,\xi_2) &= \frac{1}{(2\pi i)^2}\Big(\frac{e^{-2\pi i\xi_1}}{(\xi_1-\xi_2)\xi_1} - \frac{e^{-2\pi i\xi_2}}{(\xi_1-\xi_2)\xi_2} + \frac{1}{\xi_1\xi_2}\Big) + \frac{1}{(2\pi i)^2}\Big(\frac{e^{2\pi i\xi_1}}{(\xi_1+\xi_2)\xi_1} + \frac{e^{-2\pi i\xi_2}}{(\xi_1+\xi_2)\xi_2} - \frac{1}{\xi_1\xi_2}\Big) \\
&\quad \frac{1}{(2\pi i)^2}\Big(\frac{e^{-2\pi i\xi_1}}{(\xi_1+\xi_2)\xi_1} + \frac{e^{2\pi i\xi_2}}{(\xi_1+\xi_2)\xi_2} - \frac{1}{\xi_1\xi_2}\Big) + \frac{1}{(2\pi i)^2}\Big(\frac{e^{2\pi i\xi_1}}{(\xi_1-\xi_2)\xi_1} - \frac{e^{2\pi i\xi_2}}{(\xi_1-\xi_2)\xi_2} + \frac{1}{\xi_1\xi_2}\Big) \\
&= \frac{2}{(2\pi i)^2}\Big(\frac{e^{-2\pi i\xi_1}+e^{2\pi i\xi_1}}{(\xi_1-\xi_2)(\xi_1+\xi_2)} - \frac{e^{-2\pi i\xi_2}+e^{2\pi i\xi_2}}{(\xi_1-\xi_2)(\xi_1+\xi_2)}\Big) \\
&= -\frac{1}{\pi^2}\left(\frac{\cos(2\pi\xi_1)-\cos(2\pi\xi_2)}{(\xi_1-\xi_2)(\xi_1+\xi_2)}\right).
\end{aligned}$$

(See Equation (2.61) in the book.)

Exercise 2.21. Using the formula for the Fourier transform of the 2-dimensional cross-polytope $\Diamond := \text{conv}\{(0,0),(1,0),(0,1),(-1,0),(0,-1)\}$, namely

$$\hat{1}_{\Diamond}(\xi) = -\frac{1}{\pi^2}\left(\frac{\cos(2\pi\xi_1) - \cos(2\pi\xi_2)}{\xi_1^2 - \xi_2^2}\right),$$

find the area of $\Diamond$ by letting $\xi \to 0$ in the latter formula.

Solution: We let $\xi = (t, 0)$ for $t > 0$ and compute the limit $t \to 0$:

$$\begin{aligned}
\lim_{\xi \to 0} \hat{1}_{\Diamond}(\xi) &= \lim_{t \to 0} -\frac{1}{\pi^2}\left(\frac{\cos(2\pi t) - 1}{t^2}\right) \\
&= -\frac{1}{\pi^2}\lim_{t \to 0}\left(\frac{-2\pi\sin(2\pi t)}{2t}\right) \\
&= \frac{1}{\pi}\lim_{t \to 0} 2\pi\cos(2\pi t) \\
&= 2.
\end{aligned}$$