# Group rings over Frobenius rings

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# Algebra: celebrating Paulo Ribenboim's ninetieth birthday

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# Definition

A ring R is called quasi-Frobenius (QF ring for short) if R is

right noetherian and right self-injective.



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# Definition

A ring R is called quasi-Frobenius (QF ring for short) if R is

right noetherian and right self-injective.

#### Theorem

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Let R be a ring. Then, R is QF if and only if R is (2-sided) artinian

and the following conditions hold:

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$$ann_r(ann_l(A)) = A$$
 for any right ideal  $A \subseteq R$ ;

ann<sub>l</sub>(ann<sub>r</sub>(B)) = B for any left ideal 
$$B \subseteq R$$
.



# Proposition

For any QF ring R, we have

$$ann_{I}(J(R)) = soc(R_{R}) = soc(R_{R}) = ann_{r}(J(R)).$$



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# Proposition

For any QF ring R, we have

$$ann_I(J(R)) = soc(R_R) = soc(R_R) = ann_r(J(R)).$$

In the case of group rings, Nakayama and Connel proved the

# following

# Proposition

Let R be a ring and let G be a group. Then RG is QF if and only if

 $\frac{centro}{univers}$  G is finite group and R is QF.



QF rings have been also used in coding theory. For instance, J. Wood proved that a finite ring R has the extension property for Hamming weight if and only if R is Frobenius.



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#### Definition

Let R be a QF ring. We say that R is a Frobenius ring if  $soc(_RR) \cong_R (R/J(R))$  as R-modules.



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# Basic results



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 Group algebras 𝔽G of finite groups G over a field 𝖳 are Frobenius rings;



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- if *R* is commutative ring, *G* finite group, then *RG* is Frobenius if and only if *R* is Frobenius;



- Group algebras 𝔽G of finite groups G over a field 𝖳 are Frobenius rings;
- if *R* is commutative ring, *G* finite group, then *RG* is Frobenius if and only if *R* is Frobenius;
- if *R* is a finite Frobenius ring and *G* a finite group, then *RG* is Frobenius. (J. Wood)



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• J(R) is nilpotent ideal;



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• 
$$J(R)$$
 is nilpotent ideal;  
•  $J(R)G = \left\{ \sum_{g \in G} a_g g \mid a_g \in J(R) \right\} \subseteq J(RG);$ 



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• 
$$J(R)$$
 is nilpotent ideal;  
•  $J(R)G = \left\{ \sum_{g \in G} a_g g \mid a_g \in J(R) \right\} \subseteq J(RG);$ 

## Definition

Given a finite group G, we say that an artinian ring R is a

**Jacobson ring for G** if the equality J(R)G = J(RG) holds.



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## Proposition

Given a finite group G, an artinian ring R is a Jacobson ring for G if and only if  $|G| \in U(R/J(R))$ .



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#### Proposition

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### Corollary

Let R be an artinian local ring and G be a finite group such that m = char(R) does not divide  $|G|^k$  where k denotes the nilpotency index of J(R). Then R is a Jacobson ring for G.



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The next result shows that we can give a precise description of the socle of RG if R is a Jacobson ring for G.



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## Proposition

Let G be a finite group and R a Jacobson ring for G. Then, the following equality holds:

$$soc(RG) = soc(R)G = \left\{ \sum_{g \in G} a_{gg} \mid a_{g} \in soc(R) \right\}.$$



## Now, we are going to present the main result of this talk.



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#### Theorem

Let G be a finite group and R be an artinian ring. If R is Frobenius

and a Jacobson ring for G, then RG is Frobenius.



# New results



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# New results

## Corollary

Let R be a commutative artinian ring with char(R)=0 and let G be a

finite group. The following conditions are equivalent:

(i) R is Frobenius;

(ii) RG is Frobenius.



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# New results

#### Corollary

Let R be a commutative artinian ring with char(R)=0 and let G be a

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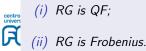
(i) R is Frobenius;

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#### Corollary

Let R be a commutative artinian ring with char(R)=0 and let G be a

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# Thank you for your attention!!!



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