Group rings over Frobenius rings

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Algebra: celebrating Paulo Ribenboim's ninetieth birthday

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Basic results

Definition

A ring R is called **quasi-Frobenius** *(QF ring for short) if R is right noetherian and right self-injective.*

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Theorem

Let R be a ring. Then, R is QF if and only if R is (2-sided) artinian and the following conditions hold:

1. $ann_r(ann_l(A)) = A$ for any right ideal $A \subseteq R$;
2. $ann_l(ann_r(B)) = B$ for any left ideal $B \subseteq R$.

Basic results

Proposition

For any QF ring R, we have

$$ann_l(J(R)) = soc(R_R) = soc({}_RR) = ann_r(J(R)).$$

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In the case of group rings, Nakayama and Connel proved the following

Proposition

Let R be a ring and let G be a group. Then RG is QF if and only if G is finite group and R is QF.

Basic results

QF rings have been also used in coding theory. For instance, J. Wood proved that *a finite ring R has the extension property for Hamming weight if and only if R is Frobenius.*

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Definition

Let R be a QF ring. We say that R is a **Frobenius ring** *if $soc({}_R R) \cong_R (R/J(R))$ as R-modules.*

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- Group algebras $\mathbb{F}G$ of finite groups G over a field $\mathbb{F}$ are Frobenius rings;
- if R is commutative ring, G finite group, then RG is Frobenius if and only if R is Frobenius;
- if R is a finite Frobenius ring and G a finite group, then RG is Frobenius. (J. Wood)

New results

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Definition

Given a finite group G, we say that an artinian ring R is a **Jacobson ring for G** if the equality $J(R)G = J(RG)$ holds.

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New results

Proposition

Given a finite group G, an artinian ring R is a Jacobson ring for G if and only if $|G| \in \mathcal{U}(R/J(R))$.

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Corollary

Let R be an artinian local ring and G be a finite group such that $m = char(R)$ does not divide $|G|^k$ where k denotes the nilpotency index of $J(R)$. Then R is a Jacobson ring for G.

New results

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Proposition

Let G be a finite group and R a Jacobson ring for G. Then, the following equality holds:

$$soc(RG) = soc(R)G = \left\{ \sum_{g \in G} a_g g \mid a_g \in soc(R) \right\}.$$

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New results

Now, we are going to present the main result of this talk.

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Theorem

Let G be a finite group and R be an artinian ring. If R is Frobenius and a Jacobson ring for G, then RG is Frobenius.

New results

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New results

Corollary

Let R be a commutative artinian ring with char(R)=0 and let G be a finite group. The following conditions are equivalent:

(i) *R is Frobenius;*

(ii) *RG is Frobenius.*

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Let R be a commutative artinian ring with char(R)=0 and let G be a finite group. The following conditions are equivalent:

(i) RG is QF;

(ii) RG is Frobenius.

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Thank you for your attention!!!

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