Free group algebras in division rings with valuation

Javier Sánchez

Departamento de Matemática - IME
Universidade de São Paulo, Brazil

Algebra: celebrating Paulo Ribenboim's 90th birthday
São Paulo, October 24–27, 2018

- J. S., *Free group algebras in division rings with valuation I*, accepted for publication in Journal of Algebra.
- J. S., *Free group algebras in division rings with valuation II*, to be submitted for publication.

Supported by FAPESP and CNPq

Notation

- Rings and algebras are associative with 1.
- Homomorphisms, subrings, subalgebras and embeddings of these objects preserve 1.
- A division ring is a nonzero ring such that every nonzero element is invertible.
- If $\mathbb{k}$ is a commutative ring, free $\mathbb{k}$-algebras $\mathbb{k}\langle X \rangle$ are supposed to be noncommutative, i.e. $|X| \geq 2$.
- The free group $\mathbb{k}$-algebra on the set X is the group $\mathbb{k}$-algebra $\mathbb{k}[G]$ where G is the free group on the set X, $|X| \geq 2$.

A conjecture

Let D be a division ring with center Z.

Conjecture A (L. Makar-Limanov, T. Stafford)

(A) If D is finitely generated (as a division Z-algebra) over Z and $[D : Z] = \infty$, then D contains a free Z-algebra

- If $\mathbb{k} < Z$, **(P. Malcolmson and L. Makar-Limanov)**

$$D \text{ contains free (group) } Z\text{-algebras} \Updownarrow D \text{ contains free (group) } \mathbb{k}\text{-algebras}$$

- $\mathbb{k}\langle X \rangle \subseteq D \not\Rightarrow \mathbb{k}\langle X, X^{-1} \rangle$ is a free group $\mathbb{k}$-algebra.

- Important examples of division rings D satisfying the conjecture contain free group $\mathbb{k}$-algebras.

Makar-Limanov

- char $\mathbb{k} = 0$, $A_1 = \mathbb{k}\langle x, y \mid yx - xy = 1\rangle$ is a noetherian domain, thus an Ore domain with Ore ring of fractions D_1.
  - D_1 contains a free group $\mathbb{k}$-algebra.
- G a nonabelian torsion-free nilpotent group.
  The group $\mathbb{k}$-algebra $\mathbb{k}[G]$ is an Ore domain with Ore ring of fractions $\mathbb{k}(G)$.
  - $\mathbb{k}(G)$ contains a free group $\mathbb{k}$-algebra for any field $\mathbb{k}$,

- We give sufficient conditions for the existence of free group algebras in certain division rings.

Some definitions

- An ordered group is a pair $(G,<)$ where
  - G is a group with operation denoted additively.
  - $<$ is a strict total ordering such that $\forall\, g_1, g_2, h \in G$

$$g_1 < g_2 \implies g_1 + h < g_2 + h \text{ and } h + g_1 < h + g_2$$

Example

- $\mathbb{Z}$, $\mathbb{R}$, any torsion-free abelian group can be ordered.
- torsion-free nilpotent groups can be ordered.
- residually $\{$torsion-free nilpotent$\}$ groups can be ordered.

Some definitions

- Let D be a division ring and $(G,<)$ be an ordered group. A map $v\colon D \to G \cup \{\infty\}$ is a valuation if it satisfies
  - $v(f) = \infty \iff f = 0$.
  - $v(f_1 + f_2) \geq \min\{v(f_1), v(f_2)\}\ \forall\ f_1, f_2 \in D$.
  - $v(f_1 f_2) = v(f_1) + v(f_2),\ \forall\ f_1, f_2 \in D$.

- Valuations induce descending filtrations.
  For all $g, h \in G$:

  $$D_{\geq g} = \{x \in D\colon v(x) \geq g\},\ \ D_{>g} = \{x \in D\colon v(x) > g\},\ \ D_g = \frac{D_{\geq g}}{D_{>g}}.$$

  A multiplication can be defined by

  $$D_g \times D_h \longrightarrow D_{g+h},\quad (x + D_{>g})(y + D_{>h}) = (xy) + D_{>g+h}.$$

  The associated graded ring of v on D is

  $$\mathrm{gr}_v(D) = \bigoplus_{g \in G} D_g.$$

Main result for real valued valuations

- A map $D \to \mathrm{gr}_\upsilon(D) = \bigoplus_{g \in G} D_g$, $x \mapsto x + D_{>\upsilon(x)} \in \frac{D_{\geq \upsilon(x)}}{D_{>\upsilon(x)}}$.

Theorem (S.)

Let D be a division ring with prime subring P. Let $\upsilon\colon D \to \mathbb{R} \cup \{\infty\}$ be a nontrivial valuation. Let X be a subset of D satisfying the following three conditions.

- *The map $X \to \mathrm{gr}_\upsilon(D)$, $x \mapsto x + D_{>\upsilon(x)}$, is injective.*
- *For each $x \in X$, $\upsilon(x) > 0$.*
- *The P_0-subalgebra of $\mathrm{gr}_\upsilon(D)$ generated by the set $\{x + D_{>\upsilon(x)}\}_{x \in X}$ is the free P_0-algebra on the set $\{x + D_{>\upsilon(x)}\}_{x \in X}$, where $P_0 := P_{\geq 0}/P_{>0} \subseteq D_0$.*

Then, for any central subfield $\mathbb{k}$, the k-subalgebra of D generated by $\{1 + x, (1 + x)^{-1}\}_{x \in X}$ is the free group $\mathbb{k}$-algebra on $\{1 + x\}_{x \in X}$.

Main result for general ordered groups

Let $(G, <)$ be an ordered group.
Let $g, h \in G$, $g, h > 0$.

- We say $g \sim h$ if there exist positive integers m and n such that

$$g < mh \text{ and } h < ng.$$

- We say $g \ll h$ if

$$ng < h \text{ for all positive integers } n.$$

- For positive elements $g, h \in \mathbb{R}$, $g \sim h$.
- $\mathbb{Z} \times \mathbb{Z}$,

$$(a, b) < (a', b') \iff \{b < b'\} \text{ or } \{b = b' \text{ and } a < a'\}.$$

  Then $(1, 0) \ll (0, 1)$.

Main result for general ordered groups

Theorem (S.)

Let D be a division ring with prime subring P. Let $(G,<)$ be an ordered group and $\upsilon\colon D \to G \cup \{\infty\}$ be a valuation. Let X be a subset of D satisfying

- *The map $X \to \mathrm{gr}_\upsilon(D)$, $x \mapsto x + D_{>\upsilon(x)}$, is injective.*
- *$\upsilon(x) > 0$ for all $x \in X$.*
- *$\upsilon(x) \sim \upsilon(x')$ for all $x, x' \in X$.*
- *The P_0-subalgebra of $\mathrm{gr}_\upsilon(D)$ generated by $\{x + D_{>\upsilon(x)}\}_{x\in X}$ is the free P_0-algebra on $\{x + D_{>\upsilon(x)}\}_{x\in X}$*

Then the following hold true.

- *If there does not exist $z \in P$ such that $\upsilon(z) \gg \upsilon(x)$ for all $x \in X$, then the P-subalgebra of D generated by $\{1 + x,\ (1 + x)^{-1}\}_{x\in X}$ is the free group P-algebra on $\{1 + x\}_{x\in X}$.*
- *If there exists $z \in P$ such that $\upsilon(z) \gg \upsilon(x)$ for some $x \in X$, then the P-subalgebra of D generated by $\{1 + zx,\ (1 + zx)^{-1}\}_{x\in X}$ is the free group P-algebra on $\{1 + zx\}_{x\in X}$.*

A corollary

Corollary (S.)

Let D be a division ring. Let $(G, <)$ be an ordered group and $v\colon D \to G \cup \{\infty\}$ be a surjective valuation.
If G contains a noncommutative free monoid, then D contains a free group $\Bbbk$-algebra for any central subfield $\Bbbk$ of D.

We single out three types of ordered groups $(G, <)$.

Type 1: For all convex jumps (H, N), $[G, H] \subseteq N$.

Type 2: Every convex subgroup of G is normal in G, but G not of Type 1.

Type 3: There exists a convex subgroup of G which is not normal.

Applications

- L a Lie algebra over a field $\mathbb{k}$.
- $U(L)$ universal enveloping algebra of L.
- There is a construction of a division ring $\mathfrak{D}(L)$ that contains $U(L)$ and is generated by it **(Cohn, Lichtman, Huishi Li)**
- If $U(L)$ is an Ore domain, then $\mathfrak{D}(L)$ is the Ore ring of fractions of $U(L)$.

Theorem (S.)

Let $\mathbb{k}$ be a field of characteristic zero. Let L be a nonabelian Lie $\mathbb{k}$-algebra with universal enveloping algebra $U(L)$. If one of the following conditions is satisfied

- *L a residually nilpotent Lie $\mathbb{k}$-algebra;*
- *$U(L)$ is an Ore domain and with Ore ring of fractions $\mathfrak{D}(L)$.*

Then $\mathfrak{D}(L)$ contains a free group $\mathbb{k}$-algebra.

The case $U(L)$ Ore was first proved by **A. I. Lichtman**.

Algebras with involutions

- Let $\mathbb{k}$ be a field.
- A $\mathbb{k}$-involution on a $\mathbb{k}$-algebra R is a $\mathbb{k}$-linear map $R \to R$, $x \mapsto x^*$,
$$(xy)^* = y^* x^*, \quad (x^*)^* = x \quad \text{for all } x, y \in R$$
- A $\mathbb{k}$-involution on a Lie $\mathbb{k}$-algebra L is a $\mathbb{k}$-linear map $L \to L$, $x \mapsto x^*$,
$$[x, y]^* = [y^*, x^*], \quad (x^*)^* = x \quad \text{for all } x, y \in L$$
The map $L \to L$, $x \mapsto -x$, is a $\mathbb{k}$-involution for any L.
- Any $\mathbb{k}$-involution on L can be extended to a $\mathbb{k}$-involution of its universal enveloping algebra $U(L)$.
- An involution on a group G is a map $G \to G$, $x \mapsto x^*$,
$$(xy)^* = y^* x^*, \quad (x^*)^* = x \quad \text{for all } x, y \in G$$
The map $G \to G$, $g \mapsto g^{-1}$, is an involution for any G.
- Any involution on G can be extended to a $\mathbb{k}$-involution on the group ring $\mathbb{k}[G]$.

Involutional version of conjecture (A)

Question

Let D be a division $\mathbb{k}$-algebra equipped with a $\mathbb{k}$-involution.

(A) If D is finitely generated (as a division ring) over its center Z and $[D:Z]=\infty$, does D contains a free Z-algebra generated by symmetric elements, i.e. $x^*=x$?

- Results where the answer is affirmative have been given by **{Ferreira, Gonçalves} + Fornaroli or S.**
- Any $\mathbb{k}$-involution on L can be extended to a $\mathbb{k}$-involution of $\mathfrak{D}(L)$. **(Cimprič)**
- Let G be an orderable group.

  The group ring $\mathbb{k}[G]$ can be embedded in a division ring $\mathbb{k}(G)$. **(Malcev, Neumann)**

  Any involution on G can be extended to a $\mathbb{k}$-involution $\mathbb{k}(G)$ **(Ferreira-Gonçalves-S.)**

Division rings with involution

Theorem (S.)

Let $\Bbbk$ be a field of characteristic zero and L be a nonabelian Lie $\Bbbk$-algebra endowed with a $\Bbbk$-involution $*\colon L \to L$. Suppose that one of the following conditions is satisfied.

- *L is residually nilpotent;*
- *The universal enveloping algebra $U(L)$ is an Ore domain and either*
    - *there exists $x \in L$ such that $[x^*, x] \neq 0$ and the Lie k-subalgebra of L generated by $\{x, x^*\}$ is of dimension at least three, or*
    - *$[x^*, x] = 0$ for every $x \in L$, but there exist $x, y \in L$ with $[y, x] \neq 0$ and the k-subspace of L spanned by $\{x, x^*, y, y^*\}$ is not equal to the Lie k-subalgebra of L generated by $\{x, x^*, y, y^*\}$.*

Then $\mathfrak{D}(L)$ contains a (noncommutative) free group $\Bbbk$-algebra whose free generators are symmetric with respect to the extension of $*$ to $\mathfrak{D}(L)$.

About the proofs

Argument by **Lichtman**

- $H := \langle x, y \colon [y,[y,x]] = [x,[y,x]] = 0 \rangle$.
- Obtain *suitable* free (group) algebras in $\mathfrak{D}(H)$.
- Obtain *suitable* free (group) algebras in $\mathfrak{D}(L)$ for L a residually nilpotent Lie $\Bbbk$-algebra.
- Suppose L is generated by two elements a, b. Construct a filtration as

$$F_{-1}L = \Bbbk a + \Bbbk b, \quad F_{-(n+1)}L = \sum_{n_1+\cdots+n_r \geq -(n+1)} [F_{n_1}L, [F_{n_2}L, \dots] \cdots]$$

- $\operatorname{gr}(L) = \bigoplus_{n \leq -1} L_n$ is a residually nilpotent Lie $\Bbbk$-algebra
- It induces a filtration in $U(L)$ such that $\operatorname{gr}(U(L)) \cong U(\operatorname{gr}(L))$. So it comes from a valuation.
- This valuation can be extended to $\mathfrak{D}(L)$
- $\operatorname{gr}(\mathfrak{D}(L)) \cong \mathcal{H}^{-1}U(\operatorname{gr}(L))$, where $\mathcal{H}$ is the set of homogeneous elements of $U(\operatorname{gr}(L))$.

Division rings with involution

Theorem (S.)

Let $\mathbb{k}$ be a field of characteristic zero and G be a nonabelian residually torsion-free nilpotent group endowed with an involution $*\colon G \to G$. Then $\mathbb{k}(G)$ contains a free group $\mathbb{k}$-algebra whose free generators are symmetric with respect to the extension of $*$ to $\mathbb{k}(G)$.

Proof.

- $\mathbb{H} := \langle u, v\colon (v,(v,u)) = (u,(v,u)) = 1\rangle$
- There is a valuation of $\mathbb{k}[\mathbb{H}]$ such that $\mathrm{gr}_v(\mathbb{k}[\mathbb{H}]) \cong U(H)$.
- $\mathrm{gr}_v(\mathbb{k}(\mathbb{H})) \cong \mathfrak{D}(H)$
- There is a way to obtain free group algebras in $\mathbb{k}(G)$ from $\mathbb{k}(\mathbb{H})$ **(Ferreira-Gonçalves-S.)**

Thank you!