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Automorphisms of polynomials and of free algebras

1. Elementary and tame automorphisms

F, a field of characteristic 0.
$A_n = F[x_1, \ldots, x_n]$, a polynomial algebra,
$GA_n = Aut\, A_n$,
$\varphi \in GA_n \longrightarrow \varphi = (f_1, \ldots, f_n)$,
$f_i = \varphi(x_i) \in A_n$.

Elementary automorphisms:

$$\sigma(i, \alpha, f) := (x_1, \ldots, \alpha x_i + f, \ldots, x_n),$$
$$0 \neq \alpha \in F, \ f \in F[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n].$$

Tame automorphisms group:

$$TA_n := group \langle\, \sigma(i, \alpha, f)\, \rangle.$$

elements of the group TA_n are called *tame automorphisms*.

H. W. E. Jung, (1942), Van der Kulk (1953)
$GA_2 = TA_2$

$n > 3$???? *Tame Generators Problem*

Nagata automorphism (1972):

$$\sigma = \sigma_\alpha = (x + \alpha z, y + 2\alpha x + \alpha^2 z, z),$$
$$\alpha = yz - x^2, \quad \sigma(\alpha) = \alpha, \; \sigma^{-1} = \sigma_{-\alpha}.$$

$\sigma \notin TA_3$ (I.Sh. - U.Umirbaev, 2002)

Van den Essen:
1942 – 1972 – 2002... – 2032 - ???

2. Elementary reductions.

$\varphi \in GA_3$, $\varphi = (f, g, h)$,
$\deg \varphi = \deg f + \deg g + \deg h$.
It is natural to use induction on $\deg \varphi$.

Assume that there exists $f_1 \in F[g, h]$ such that $\bar{f}_1 = \bar{f}$, where $\bar{f}$ denotes the leading term of f. Consider the automorphism
$\varphi_1 = (f - f_1, g, h) = \varepsilon \circ \varphi \in GA_3$,
Then $\deg \varphi_1 < \deg \varphi$ and by induction $\varphi_1 \in TA_3$, hence $\varphi = \varepsilon^{-1} \circ \varphi_1 \in TA_3$.

We will say in this case that φ admits *an elementary reduction*, that is, there exists an elementary automorphism ε such that $\deg(\varepsilon \circ \varphi) < \deg \varphi$.

Example 1: $n = 2$. **Van der Kulk:**
Every automorphism $\varphi = (f, g) \in GA_2$ admits an elementary reduction, therefore $GA_2 = TA_2$. In this case either $\bar{f} = \alpha \bar{g}^k$ or $\bar{g} = \beta \bar{f}^m$.

In general case, an automorphism $(f_1, \dots, f_n)$ admits an elementary reduction if and only if there exists a coordinate f_i such that

$$\bar{f}_i \in \overline{F[f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_n]},$$

where $\bar{A}$ denotes the subalgebra generated by the leading terms of elements from the subalgebra A.

Lemma. If $\bar{g}_1, \dots, \bar{g}_k$ are algebraically independent over F then $\overline{F[g_1, \dots, g_k]} = F[\bar{g}_1, \dots, \bar{g}_k]$.

Examle 2: The Nagata automorphism σ does not admit elementary reductions.

The leading terms of the components of the Nagata automorphism $\alpha z, \alpha^2 z, z$ are pairwise algebraically independent, and no one of them lies in the subalgebra generated by the other two. Therefore σ does not admit elementary reductions.

"Conjecture". Every tame automorphism is elementary reducible.

I.Sh. - U.Umirbaev:

• *Elementary reducibility is algorithmically recognizable in* A_3.

The proof is based on the study of the structure of the subalgebra of leading terms $\overline{F[f,g]}$ for algebraically independent elements f, g. A certain analogue of Groebner basis is constructed for the subalgebra $F[f,g]$. In particular, an estimate $N(f,g,h)$ is found such that if $\bar{h} = \overline{G(f,g)}$ for some $G(x,y)$ then $\deg G < N(f,g,h)$.

In fact, the estimate $N(f,g,h)$ depends only on degrees $\deg f, \deg g, \deg h$, and $\deg[f,g]$, where $[f,g]$ denotes the *Poisson bracket* of the polynomials f, g. For this, we include the polynomial algebra $F[x,y,z]$ into the free Poisson algebra $P[x,y,z]$.

In fact, only the degree $\deg[f,g]$ is important for calculations, which can be calculated as
$\deg[f,g] = \max\{\deg((\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i})x_i x_j) \mid 1 \leq i < j \leq n\}$.

Example 3:
There exists a tame automorphism φ of multi-degree $(9,6,8)$ which does not admit elementary reductions.

2. Nonelementary reductions.

Therefore, in order to reduce degree of an automorphism from TA_3, we need *non-elementary reductions*.

We define four type of reductions (types I, II, III and IV) which are compositions of $k \leq 4$ elementary automorphisms of the form:

$$\begin{aligned} r_1 &= \varepsilon \circ l, \\ r_2 &= \varepsilon \circ l_2 \circ l_1, \\ r_3 &= \varepsilon \circ q \circ l, \\ r_4 &= \varepsilon \circ q_2 \circ q_1 \circ l, \end{aligned}$$

where $l, l_i \; q, q_i$ are linear and quadratic automorphisms of types

$$(x, y - \alpha z, z), \; (x, y - \beta z - \gamma z^2, z),$$

and ε is an elementary automorphism.

The exact definitions of reductions $r_1 -$ $-r_4$ are rather tecnical and used Poisson brackets.

If an automorphism $\varphi = (f_1, f_2, f_3)$ admits a non-elementary reduction r then

$$\deg(r \circ \varphi) < \deg \varphi,$$
$$\deg f_i > 1,\ i = 1, 2, 3.$$

I.Sh. - U.Umirbaev.:

$\varphi \in TA_3,\ \deg \varphi > 3$	$\Rightarrow$	φ admits either elementary reduction or one of reductions $r_1 - r_4$.

The proof is performed by a double induction: by degree $\deg\varphi$ of a tame automorphism φ and by the minimal possible number m of elementary automorphisms in the representation $\varphi = \varepsilon_1 \circ \cdots \circ \varepsilon_m$. By induction, the automorphism $\varphi_1 = \varepsilon_1 \circ \cdots \circ \varepsilon_{m-1}$ admits either elementary reduction or one of reductions $r_1 - r_4$. We consider all possible cases and prove that in every case φ satisfies the conlcusion of the theorem as well. The most difficult case here is when φ_1 admits an elementary reduction.

Corollary. *The Nagata automorphism σ is not tame.*

For every $\varphi \in TA_3$, the type i of the first non-elementary reduction r_i needed to reduce its degree is defined UNIQUELY, and we say that such φ belongs to the class R_i. Let E denotes the set of automorphisms that can be reduced to linear only by elementary reductions. Then we have the disjoint union

$$\boxed{TA_3 = E \cup R_1 \cup R_2 \cup R_3 \cup R_4.}$$

The mentioned above tame non-elementary reducible automorphism with degrees (6,9,8) is of type R_1.

Sh. Kuroda:

- $R_4 = \emptyset$

Kuroda's Conjecture:

- $R_2 = R_3 = \emptyset$

3. The algorithm.

- **I.Sh: - U.Umirbaev**

Tame automorphisms are algorithmically recognizable in A_3.

Kuroda has essentially simplified the algorithm by founding the following sufficient condition for an automorphism to be wild:

- *An automorphism $(f_1, f_2, f_3) \in GA_3$ is wild if the following conditions are verified for some monomial order in A_3:*

(1) The leading terms $\bar{f}_1, \bar{f}_2, \bar{f}_3$ are algebraically dependent over F and pairwise algebraically independent,

(2) $\bar{f}_i \notin F[\bar{f}_j \,|\, j \neq i]$ for $i = 1, 2, 3$.

A concrete algorithm was implemented in Magma by S. Uchiyama et al. (Tokyo Metropolitan University).

4. Wild coordinates.

U.Umirbaev - J.Yu:
If $\varphi = (f_1, f_2, f_3)$ is wild then at least two of its coordinates are wild. In particular, coordinates f_1, f_2 of the Nagata automorphism are wild.

Kuroda:
A coordinate f is called *totally wild* if $\varphi(f) \neq f$ for any nontrivial tame automorphism φ, and f is called *quasi-totally wild* if $\varphi(f) = f$ only for finite number of tame automorphisms φ,

$$\begin{aligned} \emptyset &\neq \{\text{totally wild coordinates in} A_3\} \\ &\subsetneq \{\text{quasi-totally wild coordinates in} A_3\} \\ &\subsetneq \{\text{ wild coordinates in } A_3\} \end{aligned}$$

For example, let $f(x) \in F[x]$. $\deg f > 12$, $\alpha = f(x) - yz$. Then the mapping
$\varphi = (x + \alpha z, y + \frac{f(x+\alpha z)-f(x)}{z}, z)$
is a wild automorphism in GA_3 with the second coordinate quasi-totally wild. With some additional conditions on f it is totally wild. On the other hand, the first coordinate of the Nagata automorphism is wild but not quasi-totally wild.

5. Multi-degrees of tame automorphisms.

For an automorphism $\varphi = (f_1, f_2, f_3) \in GA_3$, define its multi-degree as $\mathrm{mdeg}\,\varphi = (\deg \mathrm{f}_1, \deg \mathrm{f}_2, \deg \mathrm{f}_3)$.
Problem: For which triples (d_1, d_2, d_3) there exists $\varphi \in TA_3$ with $mdeg\,\varphi = (d_1, d_2, d_3)$?
Let us write $m \in \langle m_1, \ldots, m_k\rangle$ if $m = m_1 l_1 + \cdots + m_k l_k$, $l_i \geq 0$.

M. Karas' et al.:
• Let $d_n \geq d_{n-1} \geq \cdots \geq d_1$. If $d_i \in \langle d_1, \ldots, d_{i-1}\rangle$ for some $i > 1$ then there exists $\varphi \in TA_n$ with $\mathrm{mdeg}\,\varphi = (\mathrm{d}_1, \ldots, \mathrm{d}_\mathrm{n})$.
• Let $d_3 \geq d_2 > d_1 \geq 3$, $(d_1, d_2) = 1$ and d_1, d_2 are odd. Then there exists $\varphi \in TA_3$ with $mdeg\,\varphi = (d_1, d_2, d_3)$ if and only if $d_3 \in \langle d_1, d_2\rangle$.
• On the other hand, for any $d \geq 6$ there exists $\varphi \in TA_3$ with $\mathrm{mdeg}\,\varphi = (4, 6, \mathrm{d})$.

For example, there exsts $\varphi \in TA_3$ with $\mathrm{mdeg}\,\varphi = (4, 6, 7)$but there are no $\varphi \in TA_3$ with $\mathrm{mdeg}\,\varphi = (3, 5, 7)$.

6. Exponential automorphisms

A derivation D of an algebra A is called *locally nilpotent*, if for any $a \in A$ there exists n such that $D^n(a) = 0$. For example, the derivation $\frac{\partial}{\partial x_i}$ of the algebra A_n is locally nilotent. For locally nilpotent derivation D, one may define the automorphism

$$\exp D(f) = f + D(f) + \frac{D^2(f)}{2!} + \cdots$$

• **(Exponential Generators Conjecture)** The group GA_n is generated by affine automorphisms and exponential automorphisms.

7. Generators and relations in $\mathrm{TA_n}$.

The following relations are true in any free algebra:

$$\sigma(i,\alpha,f)\sigma(i,\beta g) = \sigma(i,\alpha\beta,\beta f+g). \quad (1)$$

If $i \neq j,\ f \in F[X\backslash\{x_i,x_j\}]$ then

$$\sigma(i,\alpha,f)^{-1}\sigma(j,\beta,g)\sigma(i,\alpha,f) = \sigma(j,\beta,\sigma(i,\alpha,f)^{-1}(g)). \quad (2)$$

Let $(ks): x_k \leftrightarrow x_s$ be a transposition automorphism, permutting the variables x_s and x_k. It is clear that it is tame and may be written via elementary automorphisms. Then

$$\sigma(i,\alpha,f)^{(ks)} = \sigma(j,\alpha,(ks)(f)), \quad (3)$$

where $x_j = (ks)(x_i)$.

U.Umirbaev:
The defining relations of the group TA_3 are given by (1), (2) and (3).

$n > 3$???...

8. Other open questions:

- The Nagata automorphism in positive characteristics.
- Does the normal closure of TA_3 in GA_3 coincides with GA_3? ($TA_3 \ntrianglelefteq GA_3$).
- GA_n and TA_n for $n > 3$.
- (Stable Tameness Conjecture). *For any $\varphi = (f_1, \ldots, f_n) \in GA_n$ there exists $k > 0$ such that*

$\tilde{\varphi} = (f_1, \ldots, f_n, x_{n+1}, \ldots, x_{n+k}) \in TA_{n+k}$.

- $Aut\, F(x, y, z) = TAut\, F(x, y, z)$? ($Cr_3 = TCr_3$?)

9. Nielsen-Schreier varieties.

$GLie_n = TLie_n$ (P.Cohn, 1964.)

A variety $\mathfrak{M}$ of linear algebras is called *Nielsen-Schreier*, if any subalgebra of a free algebra of this variety is free, i.e. an analog of the classical Nielsen-Schreier theorem is true.

Nielsen-Schreier examples: *All nonassociative algebras, commutative and anticommutative algebras, Lie algebras, Akivis alebras, Sabinin algebras*.

Non-(Nielsen-Schreier) examples: polynomials, associative and alternative algebras, Jordan algebras, Malcev algebras, Poisson algebras.

If $\mathfrak{M}$ is Nielsen-Schreier then $G\mathfrak{M}_n = T\mathfrak{M}_n$.
(J.Lewin, 1968).

If $\mathfrak{M}$ is Nielsen-Schreier then the group $G\mathfrak{M}_n$ is generated by elementary automorphisms via relations (1) - (3) .
(U.Umirbaev, 2006).

10. Associative and Jordan algebras.

$GAs_2 = TAs_2 = GA_2$
(Makar-Limanov – Czerniakiewicz) .

$GJord_2 = TJord_2 = GA_2$.

$GAlt_2 = TAlt_2 = GA_2$.

U.Umirbaev: *The Anick automorphism*

$$\delta = \varphi_\beta = (x + z\beta, y + \beta z, z),$$

where $\beta = xz - zy$*, is wild in the free associative algebra* $As[x, y, z]$*.*

I.Sh.: *The automorphism*

$$\varphi = \varphi_\gamma = (x + \gamma \cdot z^2, y + \gamma \cdot z, z),$$

where $\gamma = x \cdot z - y \cdot z^2$*, is wild in the free Jordan algebra* $Jord[x, y, z]$*.*

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10. Poisson algebras.

Poisson algebra $\langle P, +, \cdot, \{,\}\rangle$:
$\langle P, +, \cdot, \rangle$ is associative and commutative,
$\langle P, +, \{,\}\rangle$ is a Lie algebra,
$\{a \cdot b, c\} = a \cdot \{b, c\} + \{a, c\} \cdot b.$

Lie Poisson algebra $P(L)$:
L is a Lie algebra over a field F with multiplication $[a, b]$ and a base $x_1, \ldots, x_n$,
$P(L) = F[x_1, \ldots, x_n]$ is the algebra of polynomials on L with the Poisson bracket

$$\{f, g\} = \sum_{i<j} (f'_i \cdot g'_j - f'_j \cdot g'_i) \cdot [x_i, x_j].$$

I.Sh., 1993: Let $Lie[X]$ be a free Lie algebra on X, then $P[X] = P(Lie[X])$ is a *free Poisson algebra* on the set of generators X.

Free Poisson algebras are closely connected with polynomial, free associative, and free Lie algebras, and are useful in their study.

We have natural Poisson algebra epimorphisms:

$$P_n \overset{\pi}{\twoheadrightarrow} A_n, \quad P_n \overset{\tau}{\twoheadrightarrow} Lie_n,$$

which induce group epimorphisms

$$GP_n \overset{\pi^*}{\twoheadrightarrow} GA_n, \quad GP_n \overset{\tau^*}{\twoheadrightarrow} GLie_n = TLie_n,$$

$$TP_n \overset{\pi^*}{\twoheadrightarrow} TA_n, \quad TP_n \overset{\tau^*}{\twoheadrightarrow} TLie_n,$$

Makar-Limanov+Tursunbekova+Umirbaev, 2007:
$GP_2 = TP_2 \cong GA_2 \cong GB_2$.

It is not true if $char\, F = p > 0$!!!!

Since the Nagata automorphism σ can be lifted to P_3, we have $GP_3 \neq TP_3$. It was interesting to find a non-tame automorphism in P_3 whose polynomial counterpart is tame.

Shestakov (2017): The automorphism

$\varphi = (x + \{xz - \{y, z\}, z\}, y + (xz - \{y, z\})z, z).$

is wild.

Observe that the abelization of φ : $\{x, y + xz^2, z\}$ is an elementary automorphism.