Logical and categorial aspects of abstract quadratic forms theories
Algebra: celebrating Paulo Ribenboim's 90th birthday

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Abstract

Part (I): A historical development

Part (II): New (ongoing) results

(I.1) Orderings, quadratic forms, Galois group

• 17th Hilbert problem on the resolution of definite rational functions as sums of squares

• Artin-Schreier theory (1920's)

• Formally real field F: $\sum_{i=1}^{n} x_i^2 = 0 \Rightarrow x_1 = \cdots = x_n = 0$ $(\Rightarrow char(F) = 0)$

• (linear, compatible with $+,.$) orderings on $F \leftrightsquigarrow P \subseteq F$ (positive cones)

• the space of orderings of F: X_F Harrison topology (boolean space)

• $X_F \neq \emptyset$ iff F is formally real

E. Witt (late 1930's): the algebraic theory of quadratic forms (ATQF)
[**Lam**]

- F field, $char(F) \neq 2$
- S symmetric matrix: $S = S^t$
- classes of isometry of symmetric bilinear forms over F^n
- diagonalization: $S = T^t Diag(a_1, \dots, a_n) T$, T invertible
- slogan: "binary isometry determines n-ary isometry"

• $W(F)$, the Witt ring of F: classifies (classes of isometry of) regular anisotropic quadratic forms

• $I(F) = \{[\psi] \in W(F) : dim(\psi)$ is even$\}$: the fundamental ideal of $W(F)$

• Sylvester's Inertia Law (*real closed fields*, euclidean fields)

• $X_F \approx minSpec(W(F)) \hookrightarrow Spec(W(F))$

• $Hom(W(F), \mathbb{Z}) \approx X_F$

$Gal(F^s|F)$: the absolute Galois group of F

- F is real closed iff $Gal(F^s|F) \cong \mathbb{Z}_2$
- $\sigma \in Gal(F^s|F)$ involution ($=$ an element of order 2) $\mapsto$ $Fix(\sigma) =$ real closure of F
- F is formally real iff $Gal(F^s|F)$ contains an involution
- $X_F \approx \{$conjugacy classes of involutions in $Gal(F^s|F)\}$

(I.2) John Milnor's graded rings relations

(late 1960's and 1970's):

• Deeper questions: $I^n(F) \subseteq W(F)$, $n \in \mathbb{N}$
(Marshall's conjecture, Lam's conjecture, Aranson-Pfister property $AP_n(F)$)

• Graded Witt ring of F: $W_*(F) = \bigoplus_{n\in\mathbb{N}} I^n(F)/I^{n+1}(F)$

• (graded) cohomology ring of F: $H^*(F) = H^*(Gal(F^s|F), \mathbb{Z}_2) = \bigoplus_{n\in\mathbb{N}} H^n(Gal(F^s|F), \mathbb{Z}_2)$

[Mil]

• John Milnor's (mod 2) K-theory graded ring of F:
$k_*(F) = K_*(F)/2K_*(F)$ (*interpolation*)

• graded ring morphisms: $W_*(F) \stackrel{s_*}{\leftarrow} k_*(F) \stackrel{h_*}{\rightarrow} H^*(F)$

More involved questions on quadratic forms

$$
\begin{array}{ccccc}
\cdots & I^n/I^{n+1}(F) & \xrightarrow{(\ )\otimes\langle 1,1\rangle} & I^{n+1}/I^{n+2}(F) & \cdots \\
 & \uparrow{\scriptstyle s_n} & & \uparrow{\scriptstyle s_{n+1}} & \\
\cdots & k_n(F) & \xrightarrow{(\ )\otimes l(-1)} & k_{n+1}(F) & \cdots \\
 & \downarrow{\scriptstyle h_n} & & \downarrow{\scriptstyle h_{n+1}} & \\
\cdots & H^n(F) & \xrightarrow[(\ )\cup(-1)]{} & H^{n+1}(F) & \cdots
\end{array}
$$

More involved questions on quadratic forms

• Milnor's graded ring conjectures: s_*, h_* are isomorphisms

(late 1990's, 2000's)
• V. Voevodsky et al: solved (positively) Milnor's conjectures

(I.3) Abstract quadratic forms theories

(dually) equivalent (sub)categories:

• Quadratic forms with invertible coefficients in rings with "many invertible"
1980's: Cordes Schemes, Quaternionic Structures,
Abstract Witt Rings [**Mar1**], Abstract Order Spaces [**Mar2**](Marshall)
1990's: **Special Groups Theory** (**SG**, Dickmann-Miraglia, [**DM1**])

• • Quadratic forms with general coefficients in "general rings"
1990's: Abstract Real Spectra [**Mar2**], 2000's: MultiRings [**Mar5**] (Marshall)
2000's: **Real Semigroups** (**RS**, Dickmann-Petrovich, [**DP1**])

• **SG, RS**: *first-order* theories $\rightsquigarrow$ model-theoretic methods in ATQF

Special groups

[DM1]
SG-theory: a first-order axiomatization of ATQF (1990's)

- $\mathcal{G} = (G, \cdot, 1, -1, \equiv)$
- $(G, \cdot, 1, -1)$ is a*pointed* exponent 2 group ($g^2 = 1$)
- $\langle a_1, a_2 \rangle \equiv \langle b_1, b_2 \rangle$, (or $a_1 \in D_G(b_1, b_2)$)
- Axioms for SG (respectively RSG) are L_{SG}-formulas:

$\forall \vec{x}(\psi_0(\vec{x}) \to \psi_1(\vec{x}))$ where $\psi_i(\vec{x})$ is positive primitive (p.p.) (respectively, $\psi_i(\vec{x})$ is p.p. or the negation of atomic formula)
- L_{SG}-morphisms

• some functors:
$\mathcal{G} : Fields_{1/2} \to SG, \; F \mapsto \mathcal{G}(F) = (\dot{F}/\dot{F}^2, ., 1, -1, \equiv)$
boolean algebras $\leftrightarrows$ (reduced) special groups

• Marshall's and Lam's conjectures for fields (1970's) $\rightsquigarrow$
(positively) solved in [**DM2**], [**DM3**] ($\approx$ 2000's), combining:
- SG-theory (boolean hull of a RSG);
- Milnor's graded ring isomorphisms;
- Galois cohomological methods.

(II.1) Graded rings for quadratic forms

- $G \in SG \mapsto W(G) \mapsto W_*(G) = \bigoplus_{n\in\mathbb{N}} I^n(G)/I^{n+1}(G)$

- $G \in pSG \mapsto k_*(G) = \bigoplus_{n\in\mathbb{N}} G^{\otimes^n}/Q_n(G)$
$Q_n(G) : [\{l(a_1) \otimes ... \otimes l(a_n) : a_{i+1} = b.a_i, a_i \in D_G(1, b)\}]$

- $W(\mathcal{G}(F)) \cong W(F)$, **[DM1]**
$k_*(\mathcal{G}(F)) \cong k_*(F)$, **[DM6]**

IGR: inductive graded ring

IGR:adequate category of graded rings, adequate first-order language

$\mathcal{R} = ((\mathcal{R}_n)_{n\in\omega}, (*_{n,m})_{n,m\in\omega})$

- $\mathcal{R}_n = (R_n, +_n, 0_n, \top_n)$ is a pointed $\mathbb{F}_2$-module ;
- $R_0 \cong \mathbb{Z}_2$
- $R_n \times R_m \stackrel{*_{n,m}}{\to} R_{n+m}$ is:

- bilinear;
- $\top_n *_{n,m} \top_m = \top_{n+m}$;
- $R := \bigoplus_{n\in\omega} R_n$, endowed with $R \times R \stackrel{*}{\to} R$ is such that:

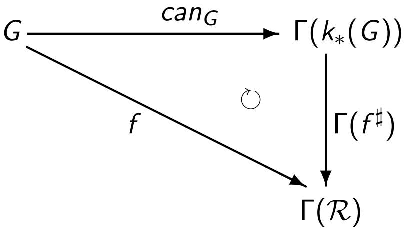
$(R, +, *, 0, 1)$ is a commutative ring with unity $1 = \top_0$;

- $\forall a \in R_1,\ a * a = \top_1 * a \in R^2$

- arrows $\mathcal{R} \to \mathcal{R}'$: (pointed) graded ring morphisms
- **Theorem** ([**MR1**]): IGR is a complete and cocomplete category.

Adjointly related categories

- **Theorem** ([**MR1**]): Adjoint functors: $pSG \underset{k_*}{\overset{\Gamma}{\leftrightarrows}} IGR$



$\forall f \in PSG(G, \Gamma(\mathcal{R}))$
$\exists!\ f^\sharp \in IGR(k_* G, \mathcal{R})$

- $can_G : G \to \Gamma(k_*(G))$ is a pSG-morphism bijective
- **Def.**: G is k-stable when $can_G : G \overset{\cong}{\to} \Gamma(k_*(G))$
- If G is a $SG \Rightarrow AP(1), AP(2)$ and $\Gamma(k_*(G)) \cong \Gamma(W_*(G))$
- If G is a SG and $AP(3) \Rightarrow G$ is k-stable
- If G is RSG or $G \cong \mathcal{G}(F)$ is AP(n)
- AP(n) is a L_{SG}-formula: $\bigwedge \forall (p.p.(L_{SG}) \to p.p.(L_{SG}))$

• $j : G \to G'$ is L_{SG}-pure embedding $\Rightarrow$
$k_*(j) : k_*(G) \to k_*(G')$ is pure embedding

• $[MC(G)]$: Marshall's signature conjecture $(W_*(G))$
$[SMC(G)]$: Strong Marshall's signature conjecture $(k_*(G))$

• For each special group G: $[SMC(G)] = [MC(G)] + [s_*(G) = iso]$

• $j : G \to G'$ is L_{SG}-pure embedding, $SMC(G') \Rightarrow SMC(G)$

(II.2) Galois groups in ATQF

• quadratic forms over F and Galois cohomology of $Gal(F^s|F)$: late 1960's, early 1970's ([**Sch**], [**Mil**]).

• quadratic forms over F and Galois cohomology of $Gal(F^q|F)$: 1980's ([**War**], [**AEJ**]).

• *W-groups*, a "nice" small quotient $Gal(F^q|F) \twoheadrightarrow Gal(F^{\{3\}}|F)$: late 1990's ([**MiSp**], [**AKM**])

• recent mathematical developments: [**CEM**], [**EM1**], [**EM2**], [**EM3**] (2010's):
$F^{\{3\}}|F$ is minimal such that determines and is determined by $H^*(F)$

Galois groups of pSG

[MR2]

• a category of "pointed" profinite 2-groups $\mathcal{G}$:

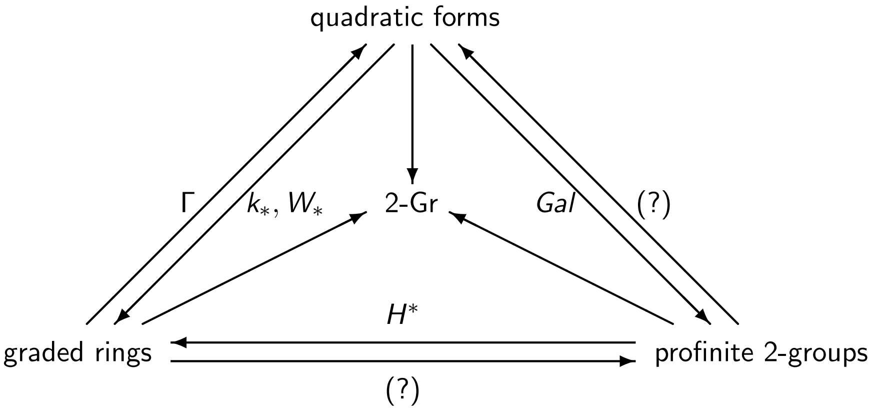
• Galois group of a (k-stable) pSG: description by "generators and relations":

• $Gal(G)$ encodes information on G: formally real, reduced, space of orderings,...

• cohomological methods in SG-theory...

• adequate category of abstract quadratic "Galois groups", adequate multi-sorted language ([**Cha**])

Transferring information: adjointly related (first-order) settings?



Thank You!

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