Geometric properties of the Markov and Lagrange spectrum

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Algebra - celebrating Paulo Ribenboim's ninetieth birthday
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Diophantine approximations: the Markov and Lagrange spectra

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Hurwitz, Markov: $|\alpha - \frac{p}{q}| < \frac{1}{\sqrt{5}q^2}$ also has infinitely many rational solutions $\frac{p}{q}$ for any irrational α. Moreover, $\sqrt{5}$ is the largest constant for which such a result is true for any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

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More precisely, we define $k(\alpha) := \sup\{k > 0 \mid |\alpha - \frac{p}{q}| < \frac{1}{kq^2}$ has infinitely many rational solutions $\frac{p}{q}\} =$
$= \limsup_{p,q \to +\infty} (q|q\alpha - p|)^{-1}$.

We have $k(\alpha) \geq \sqrt{5}$, $\forall \alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $k\left(\frac{1+\sqrt{5}}{2}\right) = \sqrt{5}$.

We will consider the set

$$L = \{k(\alpha) \mid \alpha \in \mathbb{R} \setminus \mathbb{Q}, k(\alpha) < +\infty\}.$$

This set is called the **Lagrange spectrum**.

Hurwitz-Markov theorem determines the smallest element of L, which is $\sqrt{5}$. This set L encodes many diophantine properties of real numbers. It is a classical subject the study of the geometric structure of L.

Markov (1879)

$$L \cap (-\infty, 3) = \{k_1 = \sqrt{5} < k_2 = 2\sqrt{2} < k_3 = \frac{\sqrt{221}}{5} < \dots\}$$

where k_n is a sequence (of irrational numbers whose squares are rational) converging to 3

This means that the "beginning" of the set L is discrete. As we will see, this is not true for the whole set L.

The elements of the Lagrange spectrum which are smaller than 3 are exactly the numbers of the form $\sqrt{9 - \frac{4}{z^2}}$ where z is a positive integer for which there are other positive integers x, y such that $1 \leq x \leq y \leq z$ and (x, y, z) is a solution of the *Markov equation* $x^2 + y^2 + z^2 = 3xyz$.
$\bullet (x, y, z)$ solution $\implies (y, z, 3yz - x), (x, z, 3xz - y)$ solutions.

$$\begin{tikzcd}[column sep=tiny]
 & & & (1,1,1) \arrow[d] & & & \\
 & & & (1,1,2) \arrow[d] & & & \\
 & & & (1,2,5) \arrow[dll] \arrow[drr] & & & \\
 & (1,5,13) \arrow[dl] \arrow[dr] & & & & (2,5,29) \arrow[dl] \arrow[dr] & \\
(1,13,34) & & (5,13,194) & & (2,29,169) & & (5,29,433)
\end{tikzcd}$$

An important open problem related to Markov's equation is the *Unicity Problem*, formulated by Frobenius about 100 years ago: for any positive integers x_1, x_2, y_1, y_2, z with $x_1 \leq y_1 \leq z$ and $x_2 \leq y_2 \leq z$ such that (x_1, y_1, z) and (x_2, y_2, z) are solutions of Markov's equation we always have $(x_1, y_1) = (x_2, y_2)$?
If the Unicity Problem has an affirmative answer then, for every real $t < 3$, its pre-image $k^{-1}(t)$ by the function k above consists of a single $GL_2(\mathbb{Z})$-equivalence class (this equivalence relation is such that

$$\alpha \sim \frac{a\alpha + b}{c\alpha + d}, \forall a, b, c, d \in \mathbb{Z}, |ad - bc| = 1.)$$

M. Hall proved in 1947 that if $C(4)$ is the regular Cantor set formed by the numbers in $[0,1]$ whose coefficients in the continued fractions expansion are bounded by 4, then one has

$$C(4) + C(4) = \{x + y; x, y \in C_4\} = [\sqrt{2} - 1, 4(\sqrt{2} - 1)].$$

This implies that L contains a whole half line (for instance $[6, +\infty)$).

G. Freiman determined in 1975 the biggest half line that is contained in L, which is $[c, +\infty)$, with

$$c = \frac{2221564096 + 283748\sqrt{462}}{491993569} \cong 4,52782956616\ldots.$$

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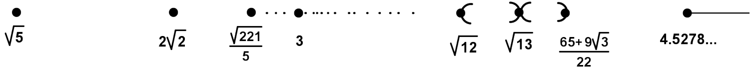


If the continued fraction of α is

$$\alpha = [a_0; a_1, a_2, \ldots] \stackrel{\text{def}}{=} a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}.$$

then we have the following formula for $k(\alpha)$:

$$k(\alpha) = \limsup_{n \to \infty} (\alpha_n + \beta_n),$$

where $\qquad \alpha_n = [a_n; a_{n+1}, a_{n+2}, \ldots] \qquad$ and

$$\beta_n = [0; a_{n-1}, a_{n-2}, \ldots, a_1].$$

The previous formula follows from the equality

$$|\alpha - \frac{p_n}{q_n}| = \frac{1}{(\alpha_{n+1} + \beta_{n+1})q_n^2}, \quad \forall n \in \mathbb{N},$$

where $p_n/q_n = [a_0; a_1, a_2, \ldots, a_n], n \in \mathbb{N}$ are the convergents of the continued fraction of α.

Remark: If $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$ then $\frac{p}{q}$ is a convergent $\frac{p_n}{q_n}$ of the continued fraction of α.

This formula for $k(\alpha)$ implies that we have the following alternative (dynamical) definition of the Lagrange spectrum L:

Let $\Sigma = (\mathbb{N}^*)^{\mathbb{Z}}$ be the set of all bi-infinite sequences of positive integers. If $\underline{\theta} = (a_n)_{n\in\mathbb{Z}} \in \Sigma$, we define $f(\underline{\theta}) = \alpha_0 + \beta_0 = [a_0; a_1, a_2, \dots] + [0; a_{-1}, a_{-2}, \dots]$. We have

$$L = \{\limsup_{n\to\infty} f(\sigma^n \underline{\theta}), \underline{\theta} \in \Sigma\}$$

where $\sigma\colon \Sigma \to \Sigma$ is the shift defined by $\sigma((a_n)_{n\in\mathbb{Z}}) = (a_{n+1})_{n\in\mathbb{Z}}$.

The Markov spectrum M is the set

$$M = \{\sup_{n\in\mathbb{Z}} f(\sigma^n \underline{\theta}), \underline{\theta} \in \Sigma\}.$$

It also has an arithmetical interpretation, namely

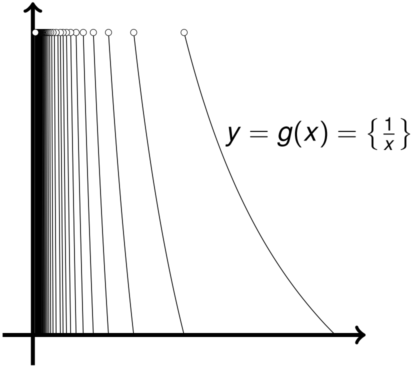
$$M = \{(\inf_{(x,y)\in\mathbb{Z}^2\setminus(0,0)} |f(x,y)|)^{-1},$$
$$f(x,y) = ax^2 + bxy + cy^2, \quad b^2 - 4ac = 1\}.$$

It follows from the dynamical characterization above that M and L are closed sets of the real line and $L \subset M$.

Regular Cantor sets

Regular Cantor sets on the line are one-dimensional hyperbolic sets, defined by expanding maps and have some kind of self-similarity property: small parts of them are diffeomorphic to big parts with uniformly bounded distortion. Sets of real numbers whose continued fraction representation has bounded coefficients with some combinatorial constraints are often regular Cantor sets, which we call Gauss-Cantor sets (since they are defined by the Gauss map $g(x) = \{1/x\}$ from $(0, 1)$ to $[0, 1)$).

We represent below the graphics of the Gauss map $g(x) = \{\frac{1}{x}\}$.



Remark:

In general, we say that a set $X \subset \mathbb{R}$ is a Cantor set if X is compact, without isolated points and with empty interior. Cantor sets in $\mathbb{R}$ are homeomorphic to the classical ternary Cantor set $K_{1/3}$ of the elements of $[0, 1]$ which can be written in base 3 using only digits 0 and 2. The set $K_{1/3}$ is itself a regular Cantor set, defined by the map $\psi : [0, 1/3] \cup [2/3, 1] \to \mathbb{R}$ given by $\psi(x) = 3x$ for $x \in [0, 1/3]$ and $\psi(x) = 3x - 2$ for $x \in [2/3, 1]$.

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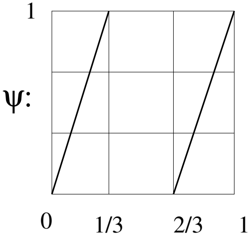
The usual ternary Cantor set is a regular Cantor set:



We have the following result about the Markov and Lagrange spectra:

Theorem

Given $t \in \mathbb{R}$ we have

$$HD(L \cap (-\infty, t)) = HD(M \cap (-\infty, t)) =: d(t)$$

and $d(t)$ is a continuous surjective function from $\mathbb{R}$ to $[0,1]$.
Moreover:
i) $d(t) = \min\{1, 2D(t)\}$, where
$D(t) := HD(k^{-1}(-\infty, t)) = HD(k^{-1}(-\infty, t])$ is a continuous function from $\mathbb{R}$ to $[0,1)$.
ii) $\max\{t \in \mathbb{R} \mid d(t) = 0\} = 3$.
iii) There is $\delta > 0$ such that $d(\sqrt{12} - \delta) = 1$.

In this work we also proved that:

- $\lim_{t \to +\infty} HD(k^{-1}(t)) = 1$
- L' is a perfect set, i.e., $L' = L''$.

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In collaboration with C. Matheus, we proved that
$0.53 < HD(M \setminus L) < 0.888$.
We also found the currently largest known element in M, namely
$[3; 2, 2, 2, 1, 2, 3, 3, 2, 2, 2, 1, 2, 2, 1, 2, 1, 2, 1, \overline{1, 2}] +$
$[0; 3, \overline{2, 1, 2, 2, 2, 3, 3}] =$

$$= \frac{7940451225305 - \sqrt{3}}{2326589591051} + \frac{-483 + \sqrt{330629}}{310} = 3.70969985975\ldots.$$

A fundamental tool in the proof of this results is related to the techniques of the proof, in collaboration with Jean-Christophe Yoccoz, of a conjecture by J. Palis on arithmetic sums and differences of regular Cantor sets.

Given two subsets K, K' of the real line, we define

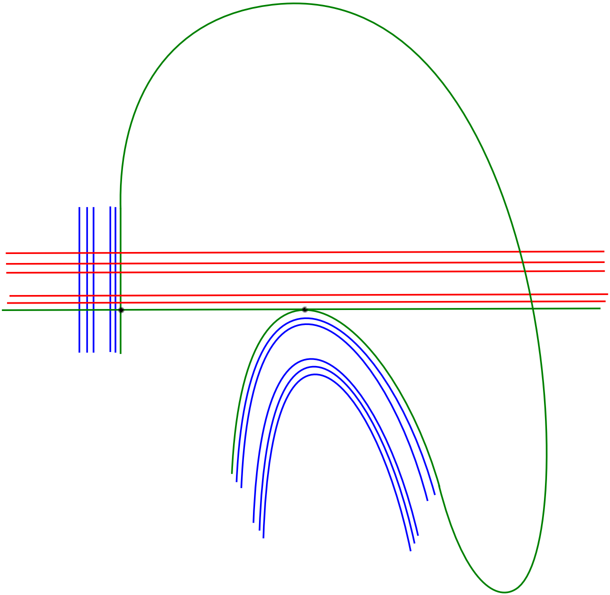
$$K - K' = \{x - y \mid x \in K, y \in K'\} = \{t \in \mathbb{R} | K \cap (K' + t) \neq \emptyset\}$$

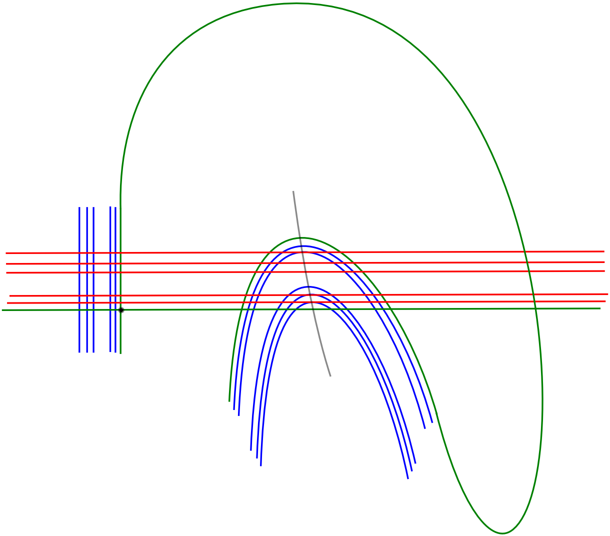
(the *arithmetic difference* between K and K').

The conjecture by J. Palis, stated in 1983, is the following:

Conjecture (Palis)

For typical pairs of regular Cantor sets (K, K'),
$HD(K) + HD(K') > 1 \Rightarrow \mathrm{int}(K - K') \neq \emptyset$.





We say that a C^2-regular Cantor set on the real line is *essentially affine* if there is a C^2 change of coordinates for which the dynamics that defines the corresponding Cantor set has zero second derivative on all points of that Cantor set.
Typical C^2-regular Cantor sets are not essentially affine.
The *scale recurrence lemma*, which is the main technical lemma of the work with Yoccoz on Palis' conjecture, can be used in order to prove the following

Theorem

If K and K' are regular Cantor sets of class C^2 and K is non essentially affine, then $HD(K+K') = \min\{HD(K) + HD(K'), 1\}$.

A version of this result was also proved by Hochman and Shmerkin.

As we have seen, the sets M and L can be defined in terms of symbolic dynamics. Inspired by these characterizations, we may associate to a dynamical system together with a real function generalizations of the Markov and Lagrange spectra as follows:

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Definition

Given a map $\psi : X \to X$ and a function $f : X \to \mathbb{R}$, we define the associated dynamical Markov and Lagrange spectra as $M(f, \psi) = \{\sup_{n\in\mathbb{N}} f(\psi^n(x)), x \in X\}$ and $L(f, \psi) = \{\limsup_{n\to\infty} f(\psi^n(x)), x \in X\}$, respectively.

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Given a flow $(\varphi^t)_{t\in\mathbb{R}}$ in a manifold X, we define the associated dynamical Markov and Lagrange spectra as $M(f, (\varphi^t)) = \{\sup_{t\in\mathbb{R}} f(\varphi^t(x)), x \in X\}$ and $L(f, (\varphi^t)) = \{\limsup_{t\to\infty} f(\varphi^t(x)), x \in X\}$, respectively.

We will describe some results obtained in collaboration with S. Romaña, C. Matheus and A. Cerqueira.

Theorem (M., Romaña)

Let Λ be a horseshoe associated to a C^2-diffeomorphism φ such that $HD(\Lambda) > 1$. Then there is, arbitrarily close to φ a diffeomorphism φ_0 and a C^2-neighborhood W of φ_0 such that, if Λ_ψ denotes the continuation of Λ associated to $\psi \in W$, there is an open and dense set $H_\psi \subset C^1(M, \mathbb{R})$ such that for all $f \in H_\psi$, we have

$$\text{int } L(f, \Lambda_\psi) \neq \emptyset \text{ and int } M(f, \Lambda_\psi) \neq \emptyset,$$

where int A denotes the interior of A.

In a work in collaboration with A. Cerqueira and C. Matheus, we prove:

Lemma

Let (φ, f) be a generic pair, where $\varphi\colon M^2 \to M^2$ is a diffeomorphism with $\Lambda \subset M^2$ a hyperbolic set for φ and $f : M \to \mathbb{R}$ is C^2. Let π_s, π_u be the projections of the horseshoe Λ to the stable and unstable regular Cantor sets K^s, K^u associated to it (along the unstable and stable foliations of Λ). Given $t \in \mathbb{R}$, we define

$$\Lambda_t = \bigcap_{m \in \mathbb{Z}} \varphi^m(\{p \in \Lambda | f(p) \leq t\}),$$

$$K_t^s = \pi_s(\Lambda_t), K_t^u = \pi_u(\Lambda_t).$$

Then the functions $d_s(t) = HD(K_t^s)$ and $d_u(t) = HD(K_t^u)$ are continuous and coincide with the corresponding box dimensions.

The following result is a consequence of the scale recurrence lemma:

Lemma

Let (φ, f) be a generic pair, where $\varphi \colon M^2 \to M^2$ is a diffeomorphism with $\Lambda \subset M^2$ a hyperbolic set for φ and $f : M \to \mathbb{R}$ is C^2. Then

$$HD(f(\Lambda)) = min(HD(\Lambda), 1).$$

Moreover, if $HD(\Lambda) > 1$ then $f(\Lambda)$ has persistently non-empty interior.

Using the previous lemmas we prove a generalization of the results on dimensions of the dynamical spectra:

Theorem

Let (φ, f) *be a generic pair, where* $\varphi\colon M^2 \to M^2$ *is a* **conservative** *diffeomorphism with* $\Lambda \subset M^2$ *a hyperbolic set for* φ *and* $f : M \to \mathbb{R}$ *is* C^2*. Then*

$$HD(L(f,\Lambda) \cap (-\infty, t)) = HD(M(f,\Lambda) \cap (-\infty, t)) =: d(t)$$

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is a continuous real function whose image is $[0, min(HD(\Lambda), 1)]$.

It is also possible to prove ([M17]) the following: let (φ, f) be a generic pair, where $\varphi\colon M^2 \to M^2$ is a diffeomorphism with $\Lambda \subset M^2$ a hyperbolic set for φ and $f : M \to \mathbb{R}$ is C^2. Then $\min L(f, \Lambda_\psi) = \min M(f, \Lambda_\psi) = f(p)$, for a periodic point $p \in \Lambda_\psi$, and is an isolated point in both $L(f, \Lambda_\psi)$ and $M(f, \Lambda_\psi)$.

In this context, in collaboration with D. Lima, we proved the following result on the topological structure of typical dynamical Markov and Lagrange spectra:

Theorem (M., D. Lima)

Let Λ be a horseshoe associated to a conservative C^2 diffeomorphism φ such that $HD(\Lambda) > 1$. Then there is, arbitrarily close to φ a diffeomorphism φ_0 and a residual set R in a C^2-neighborhood W of φ_0 such that, if Λ_ψ denotes the continuation of Λ associated to $\psi \in R$, there is a residual set $H_\psi \subset C^1(M, \mathbb{R})$ such that for all $f \in H_\psi$, we have

$$\sup\{t \in \mathbb{R} | d(t) < 1\} = \inf \mathit{int}\ L(f, \Lambda_\psi) = \inf \mathit{int}\ M(f, \Lambda_\psi) \neq \emptyset.$$

The classical Markov and Lagrange spectra can also be characterized as sets of maximum heights and asymptotic maximum heights, respectively, of geodesics in the modular surface $N = \mathbb{H}^2/PSL(2, \mathbb{Z})$.

A small movie by Pierre Arnoux and Edmund Harriss

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We extend the fact that these spectra have non-empty interior to the context of negative, non necessarily constant curvature as follows:

Theorem (M., Romaña)

Let N provided with a metric g_0 be a complete surface with finite Gaussian volume and Gaussian curvature bounded between two negative constants, i.e., if K_N denotes the Gaussian curvature, then there are constants $a, b > 0$ such that

$$-a^2 \leq K_N \leq -b^2 < 0.$$

SN is its unitary tangent bundle and ϕ its geodesic flow.

Theorem (M., Romaña, continuation)

Then there is a metric g close to g_0 and a dense and C^2-open subset $\mathcal{H} \subset C^2(SN, \mathbb{R})$ such that

$$int\ M(f, \phi_g) \neq \emptyset \text{ and } int\ L(f, \phi_g) \neq \emptyset$$

for any $f \in \mathcal{H}$, where ϕ_g is the vector field defining the geodesic flow of the metric g.
Moreover, if X is a vector field sufficiently close to ϕ_g then

$$int\ M(f, X) \neq \emptyset \text{ and } int\ L(f, X) \neq \emptyset$$

for any $f \in \mathcal{H}$.

We proved analogous results for geometric Lorenz attractors in collaboration with M. J. Pacífico and S. Romaña.

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Muito obrigado!
Muchas gracias!
Thank you very much!
Merci beaucoup!
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