SOME RESULTS ON THE ARITHMETIC BEHAVIOR OF TRANSCENDENTAL FUNCTIONS

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October 25, 2018

My first transcendental steps

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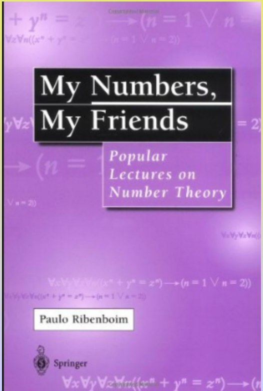
After I found the famous Ribenboim's book (**Chapter 10:** What kind of number is $\sqrt{2}^{\sqrt{2}}$?):



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- In 1976, Mahler wrote a book entitled "Lectures on Transcendental Numbers". In its chapter 3, he left three problems, called A,B, and C.
- The goal of this lecture is to talk about these problems...

Algebraic and transcendental numbers

Algebraic ($\overline{\mathbb{Q}}$): A complex number which is root of a nonzero polynomial with integer coefficients.

A number which is not algebraic is called Transcendental (Euler and Leibniz, XVIII Century).

The first examples of transcendental numbers

In 1844, Liouville proved that

Theorem (Liouville)

If α *is an algebraic number of degree* $n > 1$*, then there exists a constant* $A > 0$*, such that*

$$\left|\alpha - \frac{p}{q}\right| > \frac{A}{q^n},$$

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The number

$$\sum_{n=1}^{\infty} 10^{-n!} = 0.110001000000000000000001000000\ldots$$

is transcendental.

Transcendence of some constants

- 1874, Charles Hermite: transcendence of e.
- 1882, Ferdinand Lindemann: transcendence of π.
- 1934, Aleksandr Gelfond: transcendence of

$$e^{\pi} = 23.1406926327792690057290863679\ldots$$

- 1996, Yuri Nesterenko: transcendence of

$$\pi + e^{\pi} = 26.2822852863690622 44\ldots$$

Hermite-Lindemann Theorem

Theorem (Hermite-Lindemann)

If $\alpha \in \overline{\mathbb{Q}}$ is nonzero, then e^{α} is transcendental.

Consequences: For all $\alpha \in \overline{\mathbb{Q}}$, nonzero, $\cos\alpha$, $\sin\alpha, \log\alpha(\alpha \neq 1)$ are transcendental (Euler formula: $e^{i\alpha} = \cos\alpha + i\sin\alpha$)

The Gelfond-Schneider Theorem

At the 1900 International Congress of Mathematicians in Paris, as the seventh in his famous list of 23 problems, Hilbert gave a big push to transcendental number theory with his question of the arithmetic nature of the power α^β of two algebraic numbers α and β. In 1934, Gelfond and Schneider, independently, completely solved the problem

Theorem

If $\alpha \in \overline{\mathbb{Q}} \backslash \{0, 1\}$, *and* $\beta \in \overline{\mathbb{Q}} \backslash \mathbb{Q}$, *then* α^β *is transcendental.*

Consequences: The numbers $2^{\sqrt{2}}, 2^i$ and e^π are transcendental.
$(e^\pi = (-1)^{-i})$

Baker's Theorem

Theorem (Baker (Fields Medal - 1970))

Let $\alpha_1, \ldots, \alpha_n$ *be nonzero algebraic numbers and let* $\beta_1, \ldots, \beta_n$ *be algebraic numbers, then*

$$\Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n \neq 0$$

is a transcendental number.

Consequences: The numbers $\log 2 + \sqrt{3} \log 3$ and $\pi + \log 2$ are transcendental. $(-i \log(-1) = \pi)$

Algebraic and transcendental functions

A function $f : \Omega \to \mathbb{C}$ is called algebraic (over $\mathbb{C}$), if there exists $P(x, y) \in \mathbb{C}[x, y]$, nonzero, such that

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On the contrary f is said to be transcendental.

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An entire function f is transcendental if and only if it is not a polynomial.

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- In 1896, Stäckel proved that for each countable subset $\Sigma \subseteq \mathbb{C}$ and each dense subset $T \subseteq \mathbb{C}$, there exists a transcendental entire function f such that $f(\Sigma) \subseteq T$

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- In 1896, Stäckel proved that for each countable subset $\Sigma \subseteq \mathbb{C}$ and each dense subset $T \subseteq \mathbb{C}$, there exists a transcendental entire function f such that $f(\Sigma) \subseteq T$(Weiestrass assertion: $\Sigma = T = \overline{\mathbb{Q}}$).

A brief history: Mahler's question B

- In 1902, Stäckel produced a transcendental function $f(z)$, analytic in a neighbourhood of the origin, and with the property that both $f(z)$ and its inverse function assume, in this neighbourhood, algebraic values at all algebraic points (his proof depends on the implicit function theorem)

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- In 1902, Stäckel produced a transcendental function $f(z)$, analytic in a neighbourhood of the origin, and with the property that both $f(z)$ and its inverse function assume, in this neighbourhood, algebraic values at all algebraic points (his proof depends on the implicit function theorem)
- Based on this result, in his 1976 book, Mahler suggested the following question

Problem B

Does there exist a transcendental entire function

$$f(z) = \sum_{n=0}^{\infty} f_n z^n,$$

where $f_n \in \mathbb{Q}$ and such that both $f(z)$ and its inverse function are algebraic at all algebraic points?

The complete answer

In 2017, we solved completely this Mahler question by proving that

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Theorem (M., Moreira)

There are uncountable many transcendental entire functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

with rational coefficients a_n *and such that the* $f(\overline{\mathbb{Q}}) \subseteq \overline{\mathbb{Q}}$ *and* $f^{-1}(\overline{\mathbb{Q}}) \subseteq \overline{\mathbb{Q}}$.

A generalization

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Theorem (M., Moreira)

Let X and Y be countable subsets of $\mathbb{C}$ that are dense and closed for complex conjugation. Suppose that either both $X \cap \mathbb{R}$ and $Y \cap \mathbb{R}$ are dense in $\mathbb{R}$ or both intersections are the empty set and that if $0 \in X$, then $Y \cap \mathbb{Q} \neq \emptyset$. Then, there are uncountably many transcendental entire functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

with rational coefficients a_n and such that $f(X) = Y$, $f^{-1}(Y) = X$ and $f'(\alpha) \neq 0$, for all $\alpha \in X$.

Exceptional sets

Definition

Let f be an entire function. The exceptional set of f is defined as

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Weierstrass questions:

(W1) Is there a transcendental entire function f such that $S_f = \overline{\mathbb{Q}}$?
(W2) What are the possible S_f, for f entire and transcendental?

In 1895, Paul Stắckel showed that the answer for (W1) is Yes.

Mahler and the question (W2)

A set $A \subseteq \overline{\mathbb{Q}}$ is closed related to $\overline{\mathbb{Q}}$, if for each $\alpha \in A$, the all its (algebraic) conjugates also become to A.

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In 1965, Kurt Mahler proved that

Theorem

If A is closed related to $\overline{\mathbb{Q}}$, then there exists a transcendental entire function $f \in \mathbb{Q}[[z]]$, such that $S_f = A$.

Consequences: Every subset of $\mathbb{Q}$ (e.g., the prime numbers) and every normal extension of $\mathbb{Q}$ (e.g., $\mathbb{Q}(\sqrt{2})$) are exceptional sets.

Our result: Exceptional sets are not exceptional!

In 2009, it was proved that

Theorem (Huang-M.-Mereb, 2009)

For any $A \subseteq \overline{\mathbb{Q}}$, there exist uncountable many transcendental entire functions f, such that

$$S_f = A.$$

Mahler's problem C

In his 1976 book, Mahler suggested the following question

Problem C

Does there exist for any choice of $\rho \in (0, \infty]$ *and* $S \subseteq \overline{\mathbb{Q}}$ *(closed under complex conjugation) a transcendental function* $f \in \mathbb{Q}[[z]]$ *for which* $S_f = S$ *and with convergence radius* ρ*?*

Solution for $\rho = \infty$

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Theorem (M., Ramirez, 2016)

For any $A \subseteq \overline{\mathbb{Q}}$ (closed under complex multiplication with $0 \in A$), there exist uncountable many transcendental entire functions $f \in \mathbb{Q}[[z]]$, such that

$$S_f = A.$$

The general case

Theorem (M., Moreira, 2018)

There exist for any choice of $\rho \in (0, \infty]$ *and* $S \subseteq \mathbb{Q}$ *(closed under complex conjugation) a transcendental function* $f \in \mathbb{Q}[[z]]$ *for which* $S_f = S$ *and such that its convergence radius* ρ

Current research

In a recent paper, jointly with Gugu, we proved that

Theorem (M., Moreira)

For any $A \subseteq \overline{\mathbb{Q}} \cap B(0,1)$ *(closed under complex multiplication with* $0 \in A$*), there exist uncountable many transcendental functions* $f \in \mathbb{Z}[[z]]$ *analytic inside the unit ball, such that*

$$S_f = A.$$

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Is there a transcendental function $f(z) = \sum_{k\geq 0} a_k z^k \in \mathbb{Z}[[z]]$ *analytic in* $B(0:1)$ *with bounded coefficients and such that* $f(\overline{\mathbb{Q}} \cap B(0:1)) \subseteq \overline{\mathbb{Q}}$*?*

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Is there a transcendental function $f(z) = \sum_{k \geq 0} a_k z^k \in \mathbb{Z}[[z]]$ *analytic in* $B(0:1)$ *with bounded coefficients and such that* $f(\overline{\mathbb{Q}} \cap B(0:1)) \subseteq \overline{\mathbb{Q}}$*?*

Mahler conjectured that the answer is No, and he showed the following evidence.

A result of Mahler

Theorem (Mahler, 1965)

Let $f \in \mathbb{Z}[[z]]$ *be a strongly lacunary power series with bounded coefficients, then* $f(\overline{\mathbb{Q}} \cap B(0:1)) \not\subseteq \overline{\mathbb{Q}}$.

A recent result

Let $P(n)$ the largest prime factor of n. Very recently, with Gugu, we prove that

Theorem (M., Moreira)

There exists $f(z) = \sum_{k\geq 0} a_k z^k \in \mathbb{Z}[[z]]$ *analytic in* $B(0:1)$ *with* $P(a_k) \leq 3$ *and such that* $f(\overline{\mathbb{Q}} \cap B(0:1)) \subseteq \overline{\mathbb{Q}}$.

"May his theorems live forever!"
Paul Erdös, remembering Mahler in one of his works

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Thank you for your attention!