From Orderings to Valuation Fans and Related Topics

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Overview

Background in Real Algebra

Some history

Artin-Schreier paper in Crelle Journal (1927) defines real fields and real-closed fields. Is a basis for Real Agebra and Real Algebraic geometry.

Almost no change after ; see Moderne Algebra by Van der Waerden (1930), Lectures in Abstract Algebra by N. Jacobson (1964), and Algebra by S. Lang (1965).

Nevertheless the notion of cone associated to an order came later : J.P. Serre (1949).

This 1927 paper allowed Artin to solve Hilbert's 17th problem again in Crelle 1927.
This result was reproved by S.Lang in 1953 using valuations and real places and by A. Robinson using the model completeness of the real-closed field theory (1955).

Background in Real Algebra

Real Fields : Artin-Schreier (1927)

A real field K is a commutative field such that there exists a total order compatible with field structure.

Theorem

K is a real field $\Leftrightarrow -1 \notin \sum K^2$

From order $\leq_P$ to positive cone P : $a \leq b \Leftrightarrow b - a \in P$

We call such P an ordering. These orderings P are characterized by:
$P + P \subset P$, $P.P \subset P$, $P \cup -P = K$, $-1 \notin P$
implying : $0, 1 \in P$, $P \cap -P = \{0\}$, $\sum K^2 \subset P$

Any real field must have $charK = 0$
Examples : $\mathbb{Q}(\sqrt{2})$, $\mathbb{R}((X))$, $\mathbb{R}(X)$...

Theorem

$\sum K^2 = \cap P_i$, P_i *orderings of* K

Background in Real Algebra

A Krull valuation v **on a field** K is a surjective map

$$v : K^* \twoheadrightarrow \Gamma$$

(Γ a totally ordered abelian group) such that:
(1) $v(xy) = v(x) + v(y)$ for any x, y in K^*
(2) $v(x + y) \geq \min\{v(x), v(y)\}$, for any x, y in K^*, with $x + y$ in K^*

The valuation ring of v is:

$$A := \{x \in K \mid x = 0 \text{ or } v(x) \geq 0\}$$

The maximal ideal of A is:

$$I := \{x \in K \mid x = 0 \text{ or } v(x) > 0\}$$

$U := A \setminus I$ is the group of units

$k_v := A/I$ is the residue field

Background in Real Algebra

Definition

A subring A of a field K is a **valuation ring** if for any $x \in K$, either x or x^{-1} belongs to A.

Definition

The **valuation** v **associated to a valuation ring** A of K, with maximal ideal I, is given by the canonical quotient map $v : K^* \to \Gamma$, where $\Gamma := K^*/(A \setminus I)$ is ordered by $v(x) \leq v(y) \Leftrightarrow yx^{-1} \in A$.

Definition

A valuation v on a field K, with valuation ring A_v, is **henselian**if it satisfies Hensel's lemma : *"For any monic polynomial $f \in A_v[X]$, if $\overline{f}$ has a simple root $\beta \in k_v$, then f has a root $b \in A_v$ such that $\overline{b} = \beta$ "*.

Background in Real Algebra

Definition

A valuation v is real if and only if the residue field k_v is real.

Theorem

A field admits real valuations if and only if it is real.

If an ordering P is given then:

$$A(P) = \{x \in K \mid \exists r \in \mathbb{Q} \quad r \pm x \in P\}$$

is a valuation ring with maximal ideal:

$$I(P) = \{x \in K \mid \forall r \in \mathbb{Q}^*, \quad r \pm x \in P\}$$

$\overline{P}$ induced by P on the residue field $k_v = A(P)/I(P)$ is an archimedean ordering.

Background in Real Algebra

K a real field, P an ordering; a valuation v **is compatible with** P
$\Leftrightarrow 1 + I_v \subset P$
$\Leftrightarrow A_v$ is convex with respect to P
$\Leftrightarrow I_v$ is convex with respect to P
$\Leftrightarrow 0 < a < b \Rightarrow v(a) \geq v(b)$ in Γ.

The trivial valuation is compatible with any P

Given an ordering P the valuation associated to

$$A(P) = \{x \in K \mid \exists r \in \mathbb{Q},\ \ r \pm x \in P\}$$

is compatible with P.

For any P compatible with v, $\overline{P}$ induced by P in the residue field k_v is an ordering of k_v (and for v associated to $A(P)$ this ordering is archimedean).

The valuation rings compatible with a given P form a chain under inclusion with smallest element $A(P)$.

Valuation Fans

General preorderings

K a real field, $T \subset K$ is a preordering iff:

$T + T \subset T$, $T.T \subset T$

$0, 1 \in T$ and $-1 \notin T$

$T^* = T \setminus \{0\}$ is a subgroup of K^*

T is a **quadratic** preordering $\Leftrightarrow K^2 \subset T$

T is a **level** n preordering $\Leftrightarrow K^{2n} \subset T$

T is possibly without level $T = (A(P)^* \cap P) \cup \{0\}$

T is **compatible with a valuation** $v \Leftrightarrow 1 + I_v \subset T$

Valuation Fans

Definition

Fans : Let T be a preordering
T is a fan $\Longleftrightarrow \forall a \notin -T$ holds $T + aT \subset T \cup aT$

These fans are well behaved for compatibility :

Given v a valuation of K a real field, T a preordering compatible with v:
T is a fan in $K \Leftrightarrow \overline{T}$ is a fan in k_v.

Theorem

Trivialization theorem : *T a fan in K then there exists a valuation v compatible with T inducing in the residue field k_v a trivial fan $\overline{T}$ (means an ordering P, or the intersection of two orderings $P_1 \cap P_2$)*

Quadratic case proved by Bröcker

Torsion preordering case proved by Becker

Valuation Fans

Definition

(Bill Jacob 1981) : **a valuation fan is a preordering** T such that: there exists a real valuation v, compatible with T, (meaning $1 + I_v \subset T$), inducing an archimedean ordering on the residue field k_v.

Example : higher level orderings are valuation fans (Becker 1978).

K commutative real field, $P \subset K$ is an **ordering of exact level** n iff : $\sum K^{2n} \subset P$, $P + P \subset P$, $P.P \subset P$ (hence P^* is a subgroup of K^*) and we have $K^*/P^* \simeq \mathbb{Z}/2n\mathbb{Z}$. (Level 1 orderings are total usual orders).

Theorem

Theorem (Becker) : $\sum K^{2n} = \bigcap_{\text{level of } P \text{ divides } n} P$

Theorem

Theorem (Becker) : *Let p be a prime,*
$\sum K^2 \neq \sum K^{2p} \iff K$ *admits orderings of level p.*

Valuation Fans

Example in $K = R((X))$
The two usual orders are given by :
$P_+ = K^2 \cup XK^2$ and $P_- = K^2 \cup -XK^2$
For any prime p there exist two orderings of level p :
$P_{p,+} = K^{2p} \cup X^p K^{2p}$ and $P_{p,-} = K^{2p} \cup -X^p K^{2p}$

Another look for orderings of higher level:

A signature of level n is a morphism of abelian groups

$$\sigma : K^* \to \mu_{2n}$$

Where μ_{2n} *set of 2n-roots of 1, such that the kernel is additively closed.*

Then $P = \ker \sigma \cup \{0\}$ is an ordering of higher level, and its level divides n.

Valuation Fans

Definition

(N. Schwartz 1990) : **a generalized signature** is a morphism of abelian groups

$$\sigma : K^* \to G$$

such that **the kernel is a valuation fan**

There exist many notions of real closure under algebraic extensions of either higher level orderings or signatures, generalized signatures, chains of signatures, valuation fans, chains of valuation fans.

All these can be unified in one theory :

Henselian Residually Real-Closed Fields (HRRC fields)

In the litterature there exist other names for the same theory :
- real henselian fields in Brown [Br];
- real-closed with respect to a signature in Schwartz [S];
- almost real-closed fields in Delon-Farre [DF].

Henselian Residually Real Closed Fields

Henselian Residually Real-Closed Fields (HRRC)
v henselian valuation , k_v real-closed field
closed for generalized signature or for valuation fan
HRRC field of type S ($p \notin S \Rightarrow \Gamma_v$ p-divisible)

$S-$generalized real-closed field	Rolle field
$p \notin S \Rightarrow$ Γ_v p-divisible	Γ_v odd divisible
$p \in S$ and Γ_v non p-divisible	HRRC field of type $\{2\}$
$\Rightarrow \Gamma_v/p\Gamma_v \simeq \mathbb{Z}/p\mathbb{Z}$	
closed for an higher level ordering	

$\downarrow$ $\searrow$ $\swarrow$ $\downarrow$

Real-closed field	Chain-closed field
$\emptyset$-generalized real-closed	$\{2\}$-generalized real-closed
Γ_v divisible	$\Gamma_v/2\Gamma_v \simeq \mathbb{Z}/2\mathbb{Z}$
closed for a usual order	closed for an ordering of level 2^k

Henselian Residually Real Closed Fields

(1) $\forall n \in \mathbb{N}$, K is pythagorean :
$K^{2n} + K^{2n} = K^{2n}$

(2) K is hereditarily pythagorean :
every real algebraic extension is again a pythagorean field

(3) $\forall n \in \mathbb{N}$, K^{2n} is a fan:

$0, 1 \in K^{2n}$, $-1 \notin K^{2n}$, $K^{2n} + K^{2n} = K^{2n}$,
K^{2n*} is a subgroup of K^*,
$\forall x \notin -K^{2n}$ holds $K^{2n} + xK^{2n} = K^{2n} \cup xK^{2n}$

(4) all real valuations of K are henselian

(5) The set of real valuation rings is totally ordered by inclusion.

Henselian Residually Real Closed Fields

(6) The *smallest* real valuation ring is called Becker's ring:
$A(K^2) = A(K^{2n}) = H(K)$
where $A(T) = \{x \in K \mid \exists n \in \mathbb{N} \quad n \pm x \in T\}$
with T a valuation fan

$H(K)$ is the real holomorphy ring equal to the intersection of all real valuation rings

(7) This ring is associated to a valuation v corresponding to the unique $\mathbb{R} - \textit{place}$ of K

(8) The Jacob's ring $J(\cap K^{2n})$ is the *biggest* valuation ring with real-closed residue field. This ring is defined as follows : if T is a valuation fan, the ring $J(T)$ is equal to $J_1(T) \cup J_2(T)$ with

$$\begin{aligned} J_1(T) &= \{x \in K \mid x \notin \pm T \text{ and } 1 + x \in T\} \\ \text{and } J_2(T) &= \{x \in K \mid x \in \pm T \text{ et } xJ_1(T) \subset J_1(T)\}) \end{aligned}$$

Henselian Residually Real Closed Fields

Examples of HRRC fields

$R((\Gamma)) = \{\sum_{\gamma} a_\gamma t^\gamma \mid \gamma \in \Gamma, a_\gamma \in R\}$

where support of $\sum_{\gamma} a_\gamma t^\gamma$ is well ordered, R is a real-closed field, and Γ is a totally ordered abelian group

In $K = R((\Gamma))$ define :

- product $t^\gamma t^\delta = t^{\gamma+\delta}$
- sum $\sum_{\gamma} a_\gamma t^\gamma + \sum_{\delta} b_\delta t^\delta = \sum_{\alpha} (a_\alpha + b_\alpha) t^\alpha$
- order $\sum_{\gamma} a_\gamma t^\gamma >_K 0 \Leftrightarrow a_m >_R 0$

where $m = \min(\text{support} \sum_{\gamma} a_\gamma t^\gamma)$

- valuation $v : R((\Gamma)) \to \Gamma$

defined by $v(\sum_{\gamma} a_\gamma t^\gamma) = m = \min(\text{support} \sum_{\gamma} a_\gamma t^\gamma)$

Then $R((\Gamma))$ is an HRRC field, admitting v as henselian valuation with real-closed residue field R and value group Γ .

Henselian Residually Real Closed Fields

For a field equipped with a usual ordering it is well known that the real closure under algebraic extension is unique up to K-isomorphism.

Definition

(Becker-Berr-Gondard, [BBG]). A chain of valuation fans in a field K is defined as $(T_n)_{n\in\mathbb{N}}$ such that:
(1) $K^{2n} \subset T_n$
(2) $T_{n.m} \subset T_n$
(3) $(T_n)^m \subset T_{n.m}$
(4) $T_n^*/T_{n.m}^* \subset T_1^*/T_{n.m}^*$ is the subgroup of elements of exponent m.

Theorem

(Becker-Berr-Gondard, [BBG]). Any field K, equipped with a chain of valuation fans $(T_n)_{n\in\mathbb{N}}$, admits a closure under algebraic extensions R, unique up to K-isomorphism. Then R is a HRRC field, and R induces on K a chain of valuation fans $(T_n)_{n\in\mathbb{N}}$ (i.e. $T_n = R^{2n} \cap K$ for all n).

Some Model Theory

First order language : Variables, symbols for relations, for functions, for constants and logic symbols. $L(=,+,.,0,1)$ is the language of rings with unit and $L(=,<,+,.,0,1)$ the language used for ordered fields.
Atomic formula : basic bricks using variables, symbols for relations and functions, and constants;
for instance $x > y$, $x + y = 1$, ...
First order formula : made from atomic fomulae using conjunctions, disjunctions, negations and quantifiers.
$(non(x^2 + y^2 > 1) \wedge y > 0) \vee (2y > 1)$
Closed formula(or sentence) : formula with no free variable meaning that there is a quantifier on any variable:
$\forall x \exists y \; x + y = 0$
Elementary theory : a theory admitting an axiomatization with only first order closed fomulae (including axiom schemes)
Model of a theory : a structure satifying some axiomatization of the theory.

Some Model Theory

Any **commutative field** is a model of the following *First order axiomatization written in the language L(=,+,.,0,1)*

$\forall x \forall y \forall z\ (x + y) + z) = x + (y + z)$

$\forall x \forall y\ x + y = y + x$

$\forall x\ x + 0 = x$

$\forall x \exists y\ x + y = 0$

$\forall x \forall y \forall z\ (x.y).z = x.(y.z)$

$\forall x \forall y\ x.y = y.x$

$\forall x\ x.1 = x$

$\forall x((x = 0) \vee \exists y\ x.y = 1)$

$\forall x \forall y \forall z\ (x + y).z = x.z + y.z$

$non\ 0 = 1$

To be a real field just add an axiom scheme : for each $n \geqslant 1$

$$\forall x_1 ... \forall x_n\ non\ (0 = 1 + x_1^2 + ... + x_n^2)$$

Some Model Theory

First order axiomatization in $\mathcal{L} = (=,+,\ .\ ,0\ ,1)$, the language of rings with unit, for the **real-closed field theory**:

(1) *axioms for commutative fields* ;

(2) "*K is real* " : the axiom scheme for each $n \geqslant 1$

$$\forall x_1 ... \forall x_n non(0 = 1 + x_1^2 + ... + x_n^2)$$

(3) "*K does not have any algebraic extension of odd degree*" : the axiom scheme for each $p \geqslant 0$

$$\begin{aligned} &\forall x_0 ... \forall x_{2p+1} \exists y \\ (x_{2p+1} \quad = \quad &0 \vee x_0 + x_1 y + ... + x_{2p+1} y^{2p+1} = 0) \end{aligned}$$

(4) "*Every element of K is a square or minus a square*"

$$\forall x \exists y (x = y^2 \vee x + y^2 = 0)$$

Some Model Theory

First Order Axiomatization for Rolle fields [G.]

(Brown, Craven and Pelling: ordered fields where the Rolle property holds for polynomials) [BCP] 1980.

(1) axioms for commutative fields;

(2) " K is real " : for each $n \geqslant 1$ axiom

$$\forall x_1 \dots \forall x_n \rceil(-1 = x_1^2 + \dots + x_n^2)$$

(3) " K has no algebraic extension of odd degree" : for each $p \geqslant 0$ axiom

$$\begin{aligned} & \forall x_0 \dots \forall x_{2p+1} \exists y \\ (x_{2p+1} &= 0 \vee x_0 + x_1 y + \dots + x_{2p+1} y^{2p+1} = 0) \end{aligned}$$

(4) " K^2 is a fan " :

$$\forall x \forall y \forall z \exists t (x = -t^2 \vee y^2 + xz^2 = t^2 \vee y^2 + xz^2 = xt^2)$$

(5)" K is pythagorean at level 2 " :

$$\forall x \forall y \exists z (x^4 + y^4 = z^4)$$

Some Model Theory

Elementary theory of HRRC fields
(1) R is a commutative field ;
(2) R is a hereditarily pythagorean field ;
(3) for all $n \in \mathbb{N}$, R^{2n} is a valuation fan.

It is known from B. Jacob (Pacific J. Math. 93, n°1(1981),95-105), and also from Becker (IMPA Lectures Notes, vol.29, Rio de Janeiro, 1978, thm. 4, p. 94) that the class of Hereditarily Pythagorean Fields is elementary.

Being a valuation fan can also be first order written (see next slide)

The class of HRRC fields of type S is also an elementary class, just add
(4) for all $p \in \mathbb{P} \backslash S$, $K^2 = K^{2p}$.

Some Model Theory

Original and alternative definition for valuation fan (Bill Jacob)

T preordering such that
$\forall x \notin \pm T$ holds
$1 \pm x \in T$ or $1 \pm x^{-1} \in T$

This allow to say that the property for R^{2n} to be a valuation fan is elementary :

$\forall x(\exists y \;\; x = y^{2n} \vee x = -y^{2n})$
$\vee[\exists z \exists t(1 + x = z^{2n} \wedge 1 - x = t^{2n})$
$\vee(1 + x^{-1} = z^{2n} \wedge 1 - x^{-1} = t^{2n})]$

Some Model Theory

"The class of Hereditarily-Pythagorean Fields is elementary".
From Becker (IMPA Lectures Notes, vol.29, Rio de Janeiro, 1978, thm. 4, p. 94) a hereditarily pythagorean field is characterized by :

$$\sum K(X)^2 = K(X)^2 + K(X)^2$$

Equivalently it is characterized by :
$\sum K[X]^2 \subset K(X)^2 + K(X)^2$
By Cassel's theorem this is also equivalent to :

$$\sum K[X]^2 = K[X]^2 + K[X]^2 \quad (*)$$

Remark that if $f, g, h \in K[X]$ satisfy $f^2 = g^2 + h^2$, the degrees of g and h are less or equal to the degree of f as K is formally real.
Hence $(*)$ is expressible by an infinite sequence of first order sentences in the language of fields.

Some Model Theory

A theory T **is complete** *if any sentence - without parameters - true in one model of* T *is true in any model of* T*.*
Example: theory of algebraically closed field of given characteristic p ; for instance $\forall x(p+1)x = x$ is true in any model .
Counter- example: theory of commutative fields is not complete ;
$\exists x(1+x^2=0)$ is true in the non real model $\mathbb{C}$ and false in the real one $\mathbb{R}$)
When this holds *for a given pair of models* we say that the models are **elementary equivalent and denote** $M \equiv M'$.

A theory T **is model complete** *if for any two models* $M \subset M'$ *any closed first order sentence with parameters in the small model is true in one model if and only if it is true in the other.*
Example : theory of real-closed fields
When this holds *for a given pair of models* we say that there is an **elementary inclusion and denote** $M \prec M'$.
This allows the use of the socalled "transfer principle".

Some Model Theory

Note that there is no relation in general between being complete and being model complete.

Real-Closed Field theory (RCF) is complete in the language of rings with unit $L(=,+,.,0,1)$.
RCF is also model complete. This follows from elimination of quantifiers in the language used for ordered fields $L(=,<,+,.,0,1)$ as first shown by Tarski.

Elimination of quantifiers : any formula is equivalent to a quantifierfree formula ($\mathcal{F}$ equivalent to $\mathcal{F}'$meaning $\mathcal{F} \leftrightarrow \mathcal{F}'$ is deducible from the axioms).
- For RCF in $L(=,<,+,.,0,1)$:
$\exists x \; ax^2 + bx + c = 0$ is equivalent to
$(nona = 0 \wedge (b^2 - 4ac > 0 \vee b^2 - 4ac = 0)) \vee (a = 0 \wedge (nonb = 0 \vee c = 0))$
- Whereas for Algebraically Closed fields (ACF) in $L(=,+,.,0,1)$:
$\exists x \; ax^2 + bx + c = 0$ is equivalent to $nona = 0 \vee nonb = 0 \vee c = 0$

Some Model Theory

Well-known Hilbert 17th problem in real-closed field R:
"Let $f \in R[X_1, ..., X_n]$ such that $\forall (x_1, ..., x_n) \in R^n$ $f(x_1, ..., x_n) \geqslant 0$.
Is f a sum of squares in $R(X_1, ..., X_n)$?"
A proof using Model Theory and Algebra:
Model completeness of real-closed field theory allowing to transfer formulae from one model to another
Artin-Schreier characterization of elements positive in every order of the field as sums of squares.

$\overline{R}$ real closed field, unique ordering $\overline{R}^2$ (extends P)
real closure of $R(X_1, ..., X_n)$ equipped with P

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$R(X_1, ..., X_n)$ rational function field; P an ordering in $R(X_1, ..., X_n)$.
$P = \overline{R}^2 \cap R(X_1, ..., X_n)$

|

R a real closed field

Some Model Theory

$(*)\ \forall x_1, ..., \forall x_n \exists y\ f(x_1, ..., x_n) = y^2$,
$(*)$ is a closed first order formula which holds in R.

From the model theory of real closed fields there is an elementary inclusion $R \prec \overline{R}$; hence by transfer principle $(*)$ also holds in $\overline{R}$.

In $\overline{R}$ choose $x_i = X_i$ for all $i = 1, ...n$, we deduce $f(X_1, ..., X_n) \in \overline{R}^2$.

Hence $f(X_1, ..., X_n) \in \overline{R}^2 \cap R(X_1, ..., X_n)$ which is equal to the chosen ordering P. This proves $f(X_1, ..., X_n) \in P$.

Doing that for any ordering P in $R(X_1, ..., X_n)$, one gets that $f(X_1, ..., X_n)$ is a totally positive element of $R(X_1, ..., X_n)$, so by Artin-Schreier theory $f(X_1, ..., X_n)$ is a sum of squares in $R(X_1, ..., X_n)$.

Using density the proof works also for any field with only one ordering and dense in its real-closure (for instance $\mathbb{Q}$).

Some Model Theory

In Delon-Farre it is again proved that the theory of HRRC fields is elementary, and the authors established a bijection between theories of HHRC fields and certain theories of ordered abelian groups. This bijection preserves completeness and sometimes decidability.

Finally they proved that the only model-complete theory among these theories of HRRC fields is that of real-closed fields.

They also characterized definable real valuation rings in such fields and have shown that they were in bijection with the definable convex subgroups of the value group of the Becker ring $(A(K^2) = A(K^{2n}) = H(K))$.
(**Definable set :** set in a model which can be described by one first order formulae)

In case we have only one real (henselian) valuation ring with real-closed residue field, i.e. the Becker ring equals the Jacob ring, then the model theory works well, and we have been able to get real algebraic results such as a Hilbert's 17th problem at level n.

Some Model Theory

In [BBDG] we have searched, depending on K and V, for which $n \in \mathbb{N}$ holds

- strong property Q_n : $\forall f \in K[V]$

$$(f \in \sum K(V)^{2n} \Longleftrightarrow \forall x \in V_{reg}(K)\ f(x) \in \sum K^{2n})$$

- weak property Q'_n : $\forall f \in K[V)]$

$$(f \in \sum K(V)^{2n} \Longleftrightarrow \forall x \in V_{reg}(\overline{K})\ f(x) \in \sum K(x)^{2n})$$

We got for instance :

Theorem

(Becker, Berr, Delon, Gondard [BBDG]) :
$R((G))$, where $G = \{\frac{r}{s} \mid r, s \in Z$ and $p \nmid s\}$, is a $\{p\}$-generalized real-closed field with only one henselian valuation with real-closed residue field. For any variety V, property Q'_n holds for $R((G))$ if and only if $n \in < p >$ the multiplicative semi-group with 1.

Some Model Theory

The Jacob's ring $J(\cap K^{2n})$ is the *biggest* valuation ring with real-closed residue field. This ring is defined as follows : if T is a valuation fan, the ring $J(T)$ is equal to $J_1(T) \cup J_2(T)$ with

$$\begin{aligned} J_1(T) &= \{x \in K \mid x \notin \pm T \text{ et } 1 + x \in T\} \\ \text{and } J_2(T) &= \{x \in K \mid x \in \pm T \text{ et } xJ_1(T) \subset J_1(T)\}) \end{aligned}$$

The importance of the Jacob's ring is seen with the following theorem (Delon- Farre JSL 61 #4, 1996, 1121-1151)

Theorem

Let K and L be a HRRC fields. Then :
(i) $K \equiv L \Leftrightarrow \Gamma_{J(K)} \equiv \Gamma_{J(L)}$;
(ii) if $K \subset L$ then
$K \prec L \Leftrightarrow \Gamma_{J(L)}$ extends $\Gamma_{J(K)}$, and $\Gamma_{J(K)} \prec \Gamma_{J(L)}$;
where the Γ's are the value groups of the Jacob rings of K and L.

Some Model Theory

This theorem recalls the well known Ax-Kochen-Ershov theorem

Ax-Kochen-Ershov theorem (equicharacteristic zero case)**:**

Let (K, v) *and* (L, w) *be henselian valued fields such that char* $k_v = 0$ *and char* $l_v = 0$ *(hence also K and L have characteristic 0),*

$(K, v) \equiv (L, w)$ *(as valued fields, L(=,+,.,A,0,1))*
if and only if
$\Gamma_v \equiv \Gamma w$ *(as ordered abelian groups, L(=,+,* $\leq, 0)$
and $k_v \equiv k_w$ *(as fields, L(=,+,.,0,1))*

An old joint theorem with new developments

A symetric matrix with entries in a real closed field R is positive whenever the associated quadratic form is positive , or whenever its eigenvalues are nonnegative.

Theorem (Gondard-Ribenboim 1974)

Let A be a symetric matrix with entries in $R(X_1, ..., X_n)$. Then A is positive for all substitutions $(x_1, ..., x_n) \in R^n$ for which A is defined if and only if A is a sum of squares of symetric matrices with entries in $R(X_1, ..., X_n)$.

This is a generalization of the famous Artin's theorem solving Hilbert's 17th problem.
It gives algebraic certificate to matrix non negativity. The paper is published in Bull. Sc. Math. 2^e série, 98, 49-56 (1974).

Recently this theorem has been quoted by several authors including Konrad Schmüdgen,Yuri Savchuk, Jaka Cimpric, Chistopher Hillar, Jiawang Nie, Igor Klep, Thomas Unger and others...

An old joint theorem with new developments

In the Banff conference on "Positive Polynomials" held in october 2006, Konrad Schmüdgen gave a first constructive proof for our 1974 theorem. Konrad Schmüdgen published the proof in *"Emerging applications of Algebraic Geometry"*, IMA Vol. Math. Appl., 149, 325-350, Springer (2009) with title:
Non Commutative real algebraic geometry - some basic concepts and first ideas.
Another short paper appeared giving also a constructive proof : *"An elementary and constructive solution to Hilbert's 17th problem for matrices"* written by Christopher J. Hillar and Jiawang Nie and published in Proceedings of the American Mathematical Society Vol. 136, #1, 73-76 (2008).
Note that by "constructive" we mean modulo the fact that you can get effective decompositions for the positive scalars.

An old joint theorem with new developments

This second one is the source for the following
In the Hillar and Nie's paper there is a more general theorem :

Theorem

Let F be a real field and let A be in $S_n(F)$. If the principal minors of A can be expressed as sums of squares in F, then A is a sum of squares in $S_n(F)$.

From matrix theory it is known that the set of symetric matrices with all principal minors being nonnegative coincides with the set of positive matrices. Hence the 1974 theorem follows from this one.
Proof of theorem needs a **lemma** :
Let A be as in the theorem, then the minimal polynomial $p(t) \in F[t]$ of A is of the form $p(t) = \sum^{s}(-1)^{s-i} a_i t^i = t^s - a_{s-1}t^{s-1} + \ldots + (-1)^s a_0$ for a_i that are sums of squares of elements of F. Moerover $a_1 \neq 0$.

An old joint theorem with new developments

Let $A \in S_n(F)$ such that its principal minors can be expressed as sums of squares. By the lemma the minimal polynomial is
$p(t) = \sum^{s}(-1)^{s-i}a_i t^i = t^s - a_{s-1}t^{s-1} + ... + (-1)^s a_0$ for a_i that are sums of squares of elements of F. Moerover $a_1 \neq 0$.
By Cayley Hamilton theorem $p(A) = 0$, hence:
$A^s - a_{s-1}A^{s-1} + ... + (-1)^{s-1}a_1 A + (-1)^s a_0 I = 0$
Considering the case s is odd (same argument for s even) we get:
$(A^{s-1} + ... + a_3 A^2 + a_1 I)A = a_{s-1}A^{s-1} + ... + a_0 I$
Let $B = A^{s-1} + ... + a_3 A^2 + a_1 I$, hence B is a sum of squares.
B is invertible, because in any real closure R, B is diagonalizable with stricly positive eigenvalues since $a_1 \neq 0$.
Hence $BA = a_{s-1}A^{s-1} + ... + a_0 I$ and multiplying by B^{-1} we get :
$A = B^{-1}(a_{s-1}A^{s-1} + ... + a_0 I)$
$A = B(a_{s-1}B^{-2}A^{s-1} + ... + a_0 B^{-2})$
B being a sum of squares involving A, and B^{-1} commuting with A, it follows that A is a sum of squares in $S_n(F)$.

An old joint theorem with new developments

Proof of lemma.

Proof of $\Leftarrow$.
Let $p(t)$ be as in lemma and R be any real closure of F. In R the principal minors are nonnegative being sums of squares. Since A is diagonalizable over R and has nonnegative eigenvalues it follows that each $a_i \geq 0$ and also that $p(t)$ has no repeated root.

Proof of $\Rightarrow$.
Suppose some a_i is not a sum of squares. Then there exist an ordering of F in which a_i is negative. Let R be a real closure of F with this ordering, since A is diagonalizable with positive eigenvalues we get a contradiction. To prove $a_1 \neq 0$ remark that A being diagonalizable $p(t)$ has no repeated root hence t^2 does not divide $p(t)$ and a_0 and a_1 cannot be both 0. Then a_1 being a sum of products of non negative eigenvalues in R if a_1 was 0 then we should have $p(0) = 0 = (-1)^s a_0$ which is impossible.

An old joint theorem with new developments

Example : Let A be the following symetric positive definite matrix :

$$A = \begin{bmatrix} 1 & X_1X_2 \\ X_1X_2 & 1 + X_1^4X_2^2 + X_1^2X_2^4 \end{bmatrix}$$

$p(A) = A^2 - tr(A)A + \det(A)I = 0$

$A^2 + \det(A)I = tr(A)A$

Hence $A = tr(A)^{-1}[A^2 + \det(A)I]$

$A = tr(A)[(tr(A)^{-1}A)^2 + \det(A)(tr(A)^{-1}I)^2]$

$tr(A) = 2 + X_1^4X_2^2 + X_1^2X_2^4$ is a sum of polynomial squares.
$\det(A) = 1 + X_1^4X_2^2 + X_1^2X_2^4 - X_1^2X_2^2$ can be written as a sum of rational squares :

$(X_1^2 + X_2^2)\det(A) =$
$(X_2 - \frac{1}{2}X_1^2X_2)^2 + (X_1 - \frac{1}{2}X_1X_2^2)^2 + 2(X_1X_2 - \frac{1}{2}X_1^2X_2 - \frac{1}{2}X_1^2X_2)^2 +$
$+\frac{3}{4}(X_1^4X_2^2 + X_1^2X_2^4) + \frac{1}{2}(X_1X_2^3 + X_1^3X_2).$

The abstract side of orderings and Murray Marshall

The space of orderings of a field, studied in relation with quadratic forms and real valuations, have been the origin of the theory of abstract spaces of orderings (1979-80) and of Marshall's problem:

"Is every abstract space of orderings the space of orderings of some field ?"

In [M] it is proved that one can always associate to an abstract space of orderings a "P-structure" (partition of the space of orderings into subspaces which are fans, and such that any fan intersects only one or two classes).
Such a P-structure is a candidate to be analogous to the space of $\mathbb{R}$-places in the field case.
But it appeared that not every P-structure is a Hausdorff space, hence we have to improve this notion to fit with the space of $\mathbb{R}$-places in the field case.

The abstract side of orderings and Murray Marshall

We need a finer theory taking into account the $\mathbb{R}$-places: $\mathbb{Q}(2^{\frac{1}{2}})$ and $\mathbb{R}((X))$ have isomorphic spaces of orderings, but the first one has two $\mathbb{R}$-places and no ordering of level 2, and the second one has only one $\mathbb{R}$-place but has 2-power level orderings.

Definition

The $\mathbb{R}$-place associated to P is $\lambda_P : K \to \mathbb{R} \cup \{\infty\}$ defined by the following commutative diagram, where v is associated to A(P):

$$\begin{array}{ccc} K & \xrightarrow{\lambda_P} & \mathbb{R} \cup \{\infty\} \\ \pi \searrow & & \nearrow i \\ & k_v \cup \{\infty\} & \end{array}$$

Explicitly $\lambda_P(a) = \infty$ when $a \notin A(P)$, and
$\lambda_P(a) = \inf\{r \in \mathbb{Q} \mid a \leq_P r\} = \sup\{r \in \mathbb{Q} \mid r \leq_P a\}$ if $a \in A(P)$.

Any $\mathbb{R}$-place arises in this way from some ordering P (see [L], 9.1).

The abstract side and Murray Marshall

Abstract space of orderings have been introduced using signatures by Marshall in [M]:

Definition

An abstract space of orderings is (X, G), where G is a group of exponent 2 (hence abelian), -1 a distinguished element of G, and X a subset of $Hom(G, \{1, -1\})$ such that:
(1) X is a closed subset of $Hom(G, \{1, -1\})$;
(2) $\forall \sigma \in X \;\; \sigma(-1) = -1$;
(3) $\bigcap_{\sigma \in X} \ker \sigma = 1$ (where $\ker \sigma = \{a \in G \mid \sigma(a) = 1\}$);
(4) For any f and g quadratic forms over G:

$$D_X(f \oplus g) = \cup\{D_X \langle x, y\rangle \mid x \in D_X(f),\; y \in D_X(g)\}.$$

In the above definition $D_X(f)$ denotes the set $\{a \in G$ represented by $f\}$, i.e. there exists g such that $f \equiv_X \langle a\rangle \oplus g$ where $f \equiv_X h$ if and only if f and h have same dimension, and have for any $\sigma \in X$ same signature.

The abstract side of orderings and Murray Marshall

On the side of fans, seen as sets of signatures on a field, a four elements fan of level 1 is characterized by: $\sigma_0\sigma_1\sigma_2\sigma_3 = 1$ and it corresponds to the fan seen as a preordering: $T = \bigcap_{i=0}^{3} \ker \sigma_i \cup \{0\}$.
In the abstract situation abstract fans have been defined by Marshall.

Definition

An abstract fan is an abstract space of orderings (X, G) such that
$X = \{\sigma \in Hom(G, \{1, -1\}) \mid \sigma(-1) = -1\}$.
It is also characterized by: if $\sigma_0, \sigma_1, \sigma_2 \in X$ then the product $\sigma_0\sigma_1\sigma_2 \in X$.

What was expected to correspond to the space of $\mathbb{R}$-places of the field case in the context of abstract spaces of orderings is called a P-structure and has been defined by Marshall in [M].

The abstract side of orderings and Murray Marshall

The space of $\mathbb{R}$-places of a field K is the set $M(K) = \{\lambda_P \mid P \in \chi(K)\}$, where $\chi(K)$ denotes the space of orderings of K. $M(K)$ is equipped with the coarsest topology making continuous the evaluation mappings defined for every $a \in K$ by:

$$e_a : M(K) \longrightarrow \mathbb{R} \cup \{\infty\}$$

$$\lambda_P \mapsto \lambda_P(a)$$

Recall that the usual topology on $\chi(K)$ is the Harrison topology generated by the open-closed Harrison sets:

$$\mathcal{H}(a) = \{P \in \chi(K) \mid a \in P\}.$$

With this topology $\chi(K)$ is a compact totally disconnected space. Craven has shown that every compact totally disconnected space is homeomorphic to the space of orderings $\chi(K)$ of some field K.

The abstract side of orderings and Murray Marshall

Now consider the mapping Λ defined by:

$$\Lambda : \chi(K) \longrightarrow M(K)$$

$$P \mapsto \lambda_P$$

With the previous topologies on $\chi(K)$ and $M(K)$ the mapping Λ is a continuous, surjective and closed mapping.

$M(K)$ equipped with the above topology is a compact Hausdorff space. Remark that this topology on $M(K)$ is also the quotient topology inherited from the above topology on $\chi(K)$.

The abstract side of orderings and Murray Marshall

Definition

A P-structure is an equivalence relation on an abstract space of orderings (X, G) such that the canonical mapping $\Lambda : X \to M$, where M is the set of equivalence classes, satisfies:

(1) Each fiber is a fan;
(2) If $\sigma_0\sigma_1\sigma_2\sigma_3 = 1$ then $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ has a non empty intersection with at most two fibers.

Marshall has proved that every abstract space of orderings has a P-structure, generally not unique. But unlike the case of the space of $\mathbb{R}$-places in a field, this P-structure M equipped with the quotient topology, is not always Hausdorff.

The abstract side of orderings and Murray Marshall

In fields, the space of $\mathbb{R}$-places is known as soon as one knows the usual orderings and the orderings of level 2. Using this idea in the abstract situation we have been able to obtain a theorem which can be seen as the first case of a P-structure which looks like an abstract space of $\mathbb{R}$-places.

Theorem

(Gondard-Marshall, [GM]). Let (X, G) be a subspace of a space of signatures (X', G') with 2-power exponent.
For $\sigma_0, \sigma_1 \in X$, define $\sigma_0 \sim \sigma_1$ if $\sigma_0\sigma_1 = \tau^2 \in X'^2$.
Then the followings are equivalent:
(1) If $\sigma_0\sigma_1\sigma_2\sigma_3 = 1$, then either σ_0 is in relation by $\sim$ with exactly one of the $\sigma_1, \sigma_2, \sigma_3$, or σ_0 is in relation by $\sim$ with everyone of the $\sigma_1, \sigma_2, \sigma_3$.
(2) $\sim$ defines a P-structure on X.

Abstract 2^n-level signatures

Definition **:** (X, G) ,
G abelian group of exponent 2^n
$X \subset Hom(G, \mu_{2^n}) = \chi(G)$ such that :
(0)$\forall \sigma \in X$, $\forall k \in \mathbb{N}$ with k odd, $\sigma^k \in X$;
(1) X is a closed subset of $\chi(G)$;
(2) $\forall \sigma \in X \quad \sigma(-1) = -1$ (-1 distinguished element) ;
(3) $\sigma \in X \cap \ker \sigma = 1$ (where $\ker \sigma = \{a \in G \mid \sigma(a) = 1\}$)
(4) f, g forms over G

$$\begin{aligned} D_X(f \oplus g) &= \\ \cup\{D_X \langle x, y\rangle &\quad | \quad x \in D_X(f), y \in D_X(g)\} \end{aligned}$$

The abstract side of orderings and Murray Marshall

We denote the real spectrum of the real holomorphy ring of K by:

$$Sper(H(K)) = \{\alpha = (\mathfrak{p},\overline{\alpha}),\ \mathfrak{p} \in Spec(H(K)),\ \overline{\alpha} \text{ ordering of } quot(H(K)/\mathfrak{p})\}$$

Relations between $\chi(K)$, $M(K)$ and $H(K)$ are given by the next theorem.

Theorem

(Becker-Gondard, [BG]). The following diagram is commutative:

$$\begin{array}{ccccc} \chi(K) & & \xrightarrow{sper\ i} & & MinSperH(K) \\ \downarrow \Lambda & & & & \downarrow sp \\ M(K) & \xrightarrow{res} & Hom(H(K),\mathbb{R}) & \xrightarrow{j} & MaxSperH(K) \end{array}$$

Let λ be a $\mathbb{R}-place$ on a field K, let $\Lambda^{-1}(\lambda) = \{P_i \mid \lambda_{P_i} = \lambda\}$, then $T = \cap P_i$ is a valuation fan and it is a minimal valuation fan of level 1.

The abstract side of orderings and Murray Marshall

Horizontal mappings are homeomorphisms, vertical ones continuous surjective mappings (see definitions below).

$\chi(K)$ the space of orderings of K is homeomorphic to $MinSperH(K)$, $M(K)$ the space of $\mathbb{R}$-places on K is homeomorphic to $MaxSperH(K)$.

The mappings in the above diagram are defined as follows:
$\Lambda : \chi(K) \longrightarrow M(K)$ is given by $P \mapsto \lambda_P$ (see 2.2).
sper i : $\chi(K) \longrightarrow MinSperH(K)$ is given by $P \mapsto P \cap H(K)$.
sp : $MinSperH(K) \longrightarrow MaxSperH(K)$ is given by $\alpha \longmapsto \alpha^{\max}$, where $\alpha^{\max}$ is the unique maximal specialization of α.
res : $M(K) \longrightarrow Hom(H(K), \mathbb{R})$ is given by $\lambda \mapsto \lambda_{|H(K)}$.
j : $Hom(H(K), \mathbb{R}) \longrightarrow MaxSperH(K)$ is given by $\varphi \mapsto \alpha_\varphi$, where $\alpha_\varphi = \varphi^{-1}(\ \mathbb{R}^2)$ or, using the notation for the real spectrum, $\alpha_\varphi = (\ker\varphi, \overline{\alpha})$ with $\overline{\alpha} = \mathbb{R}^2 \cap quot(\varphi(H(K))$.
All the spaces in the diagram are compact and the topologies of $M(K)$ and $MaxSperH(K)$ are the quotient topologies inherited through Λ and *sp*.

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Thank you for your attention !