A survey on simple derivations and their isotropy groups

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Summary

d-simplicity

Let K be a field of characteristic zero A be a $K-$algebra and d a K-derivation of A.
$(d(a+b) = d(a) + d(b); d(ab) = d(a)b + ad(b); d(K) = 0)$.

An ideal I of A is a *d*-ideal (or *d*-stable) if $d(I) \subseteq I$. The algebra A is *d*-simple if (0) e A are the only *d*-ideals of A.

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$\mathcal{D}$-simplicity

Let $\emptyset \neq \mathcal{D} \subset \mathrm{Der}(A)$ be a family (finite or not) of derivations de A. The ring A is $\mathcal{D}$-simple (or $\mathcal{D}$-stable) if it is d-simple for all derivation $d \in \mathcal{D}$. Finally, A is differentially simple when it is $\mathcal{D}$-simple for $\mathcal{D} = \mathrm{Der}\,(A)$.

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K[X] is NOT d'-simple.
In fact, (X) is a d'-ideal, since

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Let $A := K[x_1, \dots, x_n]$ be an affine domain.

Theorem

(Seidenberg, 1967)
A is differentially simple $\Leftrightarrow$ A is regular.

BUT, Hart (1975) showed that when $K[x_1, \cdots, x_n]$ is differentially simple, it may not exist a single derivation d such that $k[x_1, \cdots, x_n]$ is d-simple.
For example

$$A := \frac{\mathbb{Q}[X, Y, Z]}{(X^2 + Y^2 + Z^2 - 1)}$$

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Let A be any commutative ring that contains $\mathbb{Q}$ and let d be a derivation of A. Given any polynomial $f(Y) \in A[Y]$, we can extend d to a derivation of $A[Y]$ putting:

$$d(Y) := f(Y).$$

We are interested when $f(Y)$ has degree one, that is:

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Then,

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Other example of simple ´´ Shamsuddin´´ derivations

Coutinho (Collier) (1999):

$$R := K[X, Y, Z]$$

$$d := \frac{\partial}{\partial X} + (a(X)Y + b(X))\frac{\partial}{\partial Y} + (c(X)Z + d(X))\frac{\partial}{\partial Z}$$

Shamsuddin derivations

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Shamsuddin derivations that are NOT simple

Not all Shamsuddin derivations are simple:

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It is possible do decide (effectively) when a Shamsuddin derivation

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is simple or not.

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Simple derivations that are not Shamsuddin

Example

(Archer, 1981) (based in Shamsuddin)

$$A = K[X_1, \ldots, X_n]$$

$$d = \partial_1 + \sum_{i=2}^{n} (1 + X_{i-1} X_i) \partial_i$$

Example

(Maciejewski, Moulin-Ollagnier, Nowicki, 2001). They studied when a derivation of the form

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Observation and Problem

Observation: Archer showed that there always exists a base of the free module of derivations

$$Der_K(K[X_1, \ldots, X_n])$$

formed by simple derivations.

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Definition

Let d be a K-derivation of a K-algebra A. Consider the full group $\mathrm{Aut}_K(A)$ of K-automorphisms of A.
The isotropy group of d is the subgroup of $\mathrm{Aut}_K(A)$ defined by

$$Aut(A)_d = \{\rho \in Aut_K(A) |\, \rho d \rho^{-1} = d\}.$$

Note that the isotropy group is just de stabilizer subgroup of the action by conjugation:

$$\mathrm{Aut}_K(A) \times \mathrm{Der}_K(A) \to \mathrm{Der}_K(A)$$

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He considered the polynomial ring $K[X, Y]$.

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He computed:

$$Aut(K[X, Y])_d = \{\rho : (X, Y) \mapsto (X + p(Y), aY + b)\},$$

where $a, b \in K, a \neq 0$
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Baltazar's theorem

Theorem

(Baltazar, 2014) Let d be a Shamsuddin derivation of $K[X, Y]$. Suppose that d is simple. Then its isotropy group $Aut(K[X, Y])_d$ is trivial.

Then Baltazar and Pan (his supervisor) conjectured:

Conjecture (Baltazar and Pan): If d is a simple derivation of an affine K-algebra, then its isotropy group is finite.

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Simple Shamsuddin derivations

Theorem

(Bertoncello,—, 2016.) Let d be a simple Shamsuddin derivation of $K[X_1, \ldots, X_n], n \geq 2$. Then its isotropy group $Aut(K[X_1, \ldots, X_n])_d$ is trivial.

Remark: The proof of this theorem is heavily based in Lequain's characterization of simple Shamsuddin derivations.

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A new conjecture

Conjecture (Baltazar, Bertoncello,–, Mendes, Pan): Let d be a Shamsuddin derivation of the polynomial ring

$$K[X_1, \ldots, X_n], n \geq 2.$$

Then d is simple if, and only if, its isotropy group is trivial.

Definition

Let $K[X, Y]$ be the polynomial ring in two variables over a field K of characteristic zero. A derivation d of $K[X, Y]$ has Y-degree n if $d(X) = 1$ and $d(Y)$ has degree n as a polynomial in Y with coefficients in $K[X]$.

Notation: $d(Y) = h_n Y^n + h_{n-1} Y^{n-1} + \cdots + h_1 Y + h_0$.
Then $d = \frac{\partial}{\partial X} + (h_n Y^n + h_{n-1} Y^{n-1} + \cdots + h_1 Y + h_0)\frac{\partial}{\partial Y}$, where $h_i(X) \in K[X]$.

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A quadratic derivation has Y-degree 2:
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A cubic derivation has Y-degree 3:
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- **(i)** $\rho(X) = X + \alpha$, $(\alpha \in K)$ *and* $\rho(Y) = b_0 + b_1 Y$ *with* $b_0 \in K[X], b_1 \in K^{\star}$ *and satisfies* $b_1^{n-1} = 1$.
- **(ii)** *If* $h_n(X) \in K[X] \setminus K$ *and* $b_1 = 1$, *then* $b_0 = 0$. *In this case* $\rho =$*id.*
- **(iii)** *If If* $h_n(X) \in K[X] \setminus K$, $h_0 \neq 0$ *and* $b_0 = 0$, *then* $b_1 = 1$. *In this case* $\rho =$*id.*
- **(iv)** *If If* $h_n(X) \in K[X] \setminus K$, d *is simple and* $b_0 = 0$, *then* $b_1 = 1$. *In this case* $\rho =$*id.*

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(iv) *If If* $h_n(X) \in K[X] \setminus K$, *d is simple and* $b_0 = 0$, *then* $b_1 = 1$. *In this case* $\rho =$*id.*

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(Bertoncello,—, 2016.) Let d be a derivation of the polynomial ring in two variables $K[X, Y]$ of Y-degree $n \geq 2$. Let $\rho \in (K[X, Y])_d$ be in the isotropy group of d. Then,

- **(i)** $\rho(X) = X + \alpha$, *($\alpha \in K$) and* $\rho(Y) = b_0 + b_1 Y$ *with* $b_0 \in K[X], b_1 \in K^\star$ *and satisfies* $b_1^{n-1} = 1$.
- **(ii)** *If* $h_n(X) \in K[X] \setminus K$ *and* $b_1 = 1$, *then* $b_0 = 0$. *In this case* $\rho =$*id.*
- **(iii)** *If If* $h_n(X) \in K[X] \setminus K$, $h_0 \neq 0$ *and* $b_0 = 0$, *then* $b_1 = 1$. *In this case* $\rho =$*id.*
- **(iv)** *If If* $h_n(X) \in K[X] \setminus K$, *$d$ is simple and* $b_0 = 0$, *then* $b_1 = 1$. *In this case* $\rho =$*id.*

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Corollaries

Corollary

Let d be a derivation in two variables of Y-degree $n \geq 2$ with $h_n(X) \in K[X] \setminus K$. Let $\mu_{n-1}(K)$ denote the cyclic group of $n-1$ roots of unity in K. Then

$$Aut(K[X, Y])_d \hookrightarrow \mu_{n-1}(K).$$

In particular it is a finite cyclic group.

Proof.

Consider the map $\varphi : K[X, Y]_d \to \mu_{n-1}(K)$ given by $\varphi(\rho) = b_1$ where $\rho(Y) = b_0 + b_1 Y$. It is a group homomorphisms. By (i) of the theorem above, it is well defined; by (ii) it is injective. Then the result follows. □

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(i) *If d is a quadratic derivation in two variables with $h_2(X) \in K[X] \setminus K$, then its isotropy group is trivial.*

(ii) *If d is a cubic derivation in two variables with $h_3(X) \in K[X] \setminus K$, then its isotropy group is either trivial or a group of order 2 (and both cases occur).*

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Examples

Example

Let d be a cubic derivation in two variables given by:

$$d = \frac{\partial}{\partial X} + (h_1 Y + h_3 Y^3)\frac{\partial}{\partial Y}$$

Let

$$\rho : X \mapsto X, Y \mapsto -Y$$

Then,

$$K[X, Y]_d = \{\text{id}, \rho\}$$

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Let d be the derivation in two variables given by:

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