# On the application of Gröbner basis theory to the study of certain evaluation codes 

Cícero Carvalho<br>Faculdade de Matemática - UFU

Celebrating Paulo Ribenboim's ninetieth birthday


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## Facts from Gröbner bases theory

Let $\mathcal{M}$ be the set of monomials of $k\left[X_{1}, \ldots, X_{n}\right]=: k[X]$ and endow $\mathcal{M}$ with a monomial order $\prec$. Given $f \in k[\mathbf{X}] \backslash\{0\}$ the leading monomial of $f(\operatorname{Im}(f))$ is the greatest monomial appearing in $f$. Let $I \subset k[X]$ be an ideal, we say that $\left\{g_{1}, \ldots, g_{s}\right\} \subset I$ is a Gröbner basis for $I($ w.r.t. $\prec)$ if the leading monomial of any nonzero polynomial in $I$ is multiple of $\operatorname{Im}\left(g_{i}\right)$ for some $i=1, \ldots, s$. One can prove that such a set is a basis for $l$ in the usual sense $I=\left(g_{1}, \ldots, g_{s}\right)$ and that any ideal has a Gröbner basis. Let
$\Delta(I):=\{M \in M \mid M$ is not the leading monomial of any polynomial in $/\}$ be the footprint of $I($ w.r.t. $\prec)$. Observe that if $I \subset J$ then $\Delta(J) \subset \Delta(I)$. Buchberger (1965) proved that $\{M+I \mid M \in \Delta(I)\}$ is a basis for $k[\mathbf{X}] / I$ (considered as a $k$-vector space).
If $I$ is homogeneous then $\{M+I \mid M \in \Delta(I), \operatorname{deg}(M)=d\}$ is a basis for $k[\mathbf{X}]_{d} / I(d)$.
We also have that $M \in \triangle(I)$ if and only if $M$ is not a multiple of $1 \mathrm{~m}\left(g_{i}\right)$ for all $i=1, \ldots, s$.

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## Gröbner bases and coding theory

A (linear error correcting) code of length $N$ defined over $\mathbb{F}_{q}$ is an $\mathbb{F}_{q}$-vector subspace $C \subset \mathbb{F}_{q}^{N}$. Given an $N$-tuple $\alpha=\left(\alpha_{1}, \ldots, \alpha_{N}\right) \in \mathbb{F}_{q}^{N}$ the weight of $\boldsymbol{\alpha}$ is $w(\boldsymbol{\alpha})=\#\left\{i \mid \alpha_{i} \neq 0\right\}$ and the minimum distance of $C$ is $d_{\min }(C):=\min \{w(\alpha) \mid \alpha \in C \backslash\{0\}\}$.
One may use a set $X=\left\{P_{1}, \ldots, P_{N}\right\} \subset \mathbb{A}^{m}\left(\mathbb{F}_{q}\right)$ to construct a code in the following way. Let $I_{X} \subset \mathbb{F}_{q}\left[X_{0}, \ldots, X_{m}\right]=\mathbb{F}_{q}[X]$ be the ideal of $X$ anc let $\varphi: \mathbb{F}_{q}[X] / I_{X} \rightarrow \mathbb{F}_{q}^{N}$ be given by $\varphi\left(f+I_{X}\right)=\left(f\left(P_{1}\right) \ldots, f\left(P_{N}\right)\right)$. It is not difficult to show that $\varphi$ is an isomorphism of $\mathbb{F}_{q}$-vector spaces. Thus, for any subspace $L \subset \mathbb{F}_{q}[X] / I_{X}$ we have a code $C_{L}:=\varphi(L)$.
Let $d$ be a nonnegative integer and let
$L_{d}:=\left\{f+I_{X} \mid f=0\right.$ or $\left.\operatorname{deg}(f) \leq d\right\}$. In this case we say that $C_{L_{d}}$ is "of Reed-Muller type" and has order $d$. From Buchberger's result we know that the classes of the monomials in $\Delta\left(I_{X}\right)$ ) form a basis for $\mathbb{F}_{q}[\mathbf{X}] / I_{X}$ (in particular $\left.\#\left(\Delta\left(I_{X}\right)\right)=N\right)$ and one may prove that the set $\Delta\left(I_{X}\right)_{d}:=\left\{M+I_{X} \mid M \in \Delta\left(I_{X}\right), \operatorname{deg}(M) \leq d\right\}$ is a basis for $L_{d}$, so that $\operatorname{dim}\left(C_{L_{d}}\right)=\#\left(\Delta\left(I_{X}\right)_{d}\right)$.

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A (linear error correcting) code of length $N$ defined over $\mathbb{F}_{q}$ is an $\mathbb{F}_{q}$-vector subspace $C \subset \mathbb{F}_{q}^{N}$. Given an $N$-tuple $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{N}\right) \in \mathbb{F}_{q}^{N}$ the weight of $\boldsymbol{\alpha}$ is $w(\boldsymbol{\alpha})=\#\left\{i \mid \alpha_{i} \neq 0\right\}$ and the minimum distance of $C$ is $d_{\text {min }}(C):=\min \{w(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in C \backslash\{\mathbf{0}\}\}$.
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## Gröbner basis methods and the parameters of $C_{L}$

As for the minimum distance $d_{\min }\left(C_{L}\right)$, we would like to estimate the number of zero entries in $\varphi\left(f+I_{X}\right)=\left(f\left(P_{1}\right), \ldots, f\left(P_{N}\right)\right)$. Let $I_{X, f}:=I_{X}+(f)$, we want to estimate $N-\#\left(V\left(I_{X, f}\right)\right)$. From $I_{X} \subset I_{X, f}$ we get $\Delta\left(I_{X, f}\right) \subset \Delta\left(I_{X}\right)$, in particular $\Delta\left(I_{X, f}\right)$ is finite which implies $\#\left(V\left(I_{X, f}\right)\right) \leq \#\left(\Delta\left(I_{X, f}\right)\right)$ and we get $N-\#\left(V\left(I_{X, f}\right)\right) \geq N-\#\left(\Delta\left(I_{X, f}\right)\right)$.

From Buchberger's result we can assume that $f$ is a linear combination of monomials in $\Delta\left(I_{X}\right)$ so that $\operatorname{Im}(f) \in \Delta\left(I_{X}\right)$. One may prove that $N-\#\left(\Delta\left(I_{X, f}\right)\right) \geq \#\left(\left\{M^{\prime} \in \Delta\left(I_{X}\right)|\operatorname{Im}(f)| M^{\prime}\right\}\right)$, so the idea now is to determine for each monomial $M \in \Delta\left(I_{X}\right)$ the cardinality of the set $\left\{M^{\prime} \in \Delta\left(I_{X}\right)|M| M^{\prime}\right\}$, and from this determine a lower bound for $d_{\min }\left(C_{L}\right)$. Moreover, it is true that if $\left\{g_{1}, \ldots, g_{s}\right\}$ is a Gröbner basis for $I_{X}$ and $\left\{f, g_{1}, \ldots, g_{s}\right\}$ is a Gröbner basis for $I_{X, f}$ then this bound is the true value of the minimum distance.

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Let $f \in \mathbb{F}_{q}[\mathbf{X}]$ with $\operatorname{deg}(f) \leq d$ and let $M:=\operatorname{Im}(f)$.


We have $\Delta\left(I_{X}+(f)\right) \subset \Delta\left(I_{X}\right)$
and $\#\left(V\left(I_{X}+(f)\right)\right) \leq \#\left(\Delta\left(I_{X}+(f)\right)\right)$.
Hence $N-\#\left(V\left(I_{x}+(f)\right)\right)$

$$
\geq \#\left(\left\{M^{\prime} \in \Delta\left(I_{X}\right)|M| M^{\prime}\right\}\right)
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Writing $d=\sum_{i=1}^{k}\left(d_{i}-1\right)+\ell$ with $0 \leq \ell<d_{k+1}$

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$$
d_{\min }\left(C\left(L_{d}\right)\right)=\left(d_{k+1}-\ell\right) \prod_{i=k+2}^{n} d_{i}
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One may also use a set $X=\left\{P_{1}, \ldots, P_{N}\right\} \subset \mathbb{P}^{l}\left(\mathbb{F}_{q}\right)$ to construct a code in the following way.
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## $\mathbb{F}_{q}$-vector space


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To determine the dimension of $\psi\left(\mathbb{F}_{q}[\mathbf{X}]_{d} / I_{X}(d)\right)=: \mathcal{C}_{d}$ we can, as in the affine case, find a Gröbner basis for $I_{X}$, determine the footprint of $I_{X}$ and count the elements in the set $\Delta(/ X) d$ of monomials of degree $d$ in $\Delta(/ X)$, whose classes form a basis for $\mathbb{F}_{q}[\mathbf{X}]_{d} / I_{X}(d)$. As for the minimum distance, we can adapt the method used on affine variety codes.

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where $\ell=n+m+1$ with $1 \leq m \leq n$. We proved that the set $\mathcal{G}=\left\{X_{i} X_{j}-X_{i+1} X_{j-1} \mid 0 \leq i \leq \ell-2, i \neq n, i+1<j \leq \ell, j \neq n+1\right\}$ is a Gröbner basis, w.r.t. the graded lexicographic order such that $X_{\ell}<\cdots<X_{0}$, for the ideal $I$ that it defines in $\mathbb{F}_{q}\left[X_{0}, \ldots, X_{\ell}\right]$

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## In 2018 in a joint work with V.G.L. Neumann, X. Mondragon and H.

 Tapia-Recillas we generalized the above results for codes defined over scrolls of higher dimension.Let $e_{0} \geq e_{1} \geq e_{2} \geq \ldots \geq e_{n} \geq 1$ be integers, and let
$\ell=\left(e_{0}+1\right)+\left(e_{1}+1\right)+\ldots+\left(e_{n}+1\right)-1=\sum_{i=0}^{n} e_{i}+n$. Let $S$ be the set of zeros in $\mathbb{P}\left(\mathbb{F}_{q}\right)^{\ell}$ of the homogeneous ideal generated by the minors of $\mathcal{M}=\left(\begin{array}{cccccccccc}x_{0,0} & \ldots & x_{0, e_{0}-1} & x_{1,0} & \ldots & x_{1, e_{1}-1} & \ldots & x_{n, 0} & \ldots & x_{n, e_{n}-1} \\ x_{0,1} & \ldots & x_{0, e_{0}} & x_{1,1} & \ldots & x_{1, e_{1}} & \ldots & x_{n, 1} & \ldots & X_{n, e_{n}}\end{array}\right)$

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## THANK YOU!


[^0]:    A. Sørensen, Projective Reed-Muller codes (1991)
    C. Rentería H. Tania-Recillas Reed-Muller codes: An ideal theory approach (1997) M. González-Sarabia et al., Reed-Muller-Type Codes Over the Segre Variety, (2002)

[^1]:    This can be used to obtain a lower bound for $d_{\min }\left(C_{d}\right)$

