Rational algebras with the hyperbolic property

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Algebra: celebrating Paulo Ribenboim's ninetieth birthday
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October 25^{th} São Paulo

Join work with S. O. Juriaans
Rational algebras with the hyperbolic property, submited

subject

If G is a hyperbolic group then it is known that $\mathbb{Z}^2$ does not embed into G. Here we investigate the structure of a finite-dimensional alternative algebra $\mathcal{A}$ subject to the condition that for an order $\Gamma \subset \mathcal{A}$, and thus for every order of $\mathcal{A}$, the loop of units of $\mathcal{U}(\Gamma)$ does not contain $\mathbb{Z}^2$.

We first give a full classification of their Wedderburn building blocks and then give their Wedderburn decomposition. In the nonassociative case, we prove that the radical J of such an algebra associates with the whole algebra. In any case, $J^2 = (0)$ and J is one dimensional over $\mathbb{Q}$. We also determine which of these building blocks are isomorphic to a Wedderburn component of the rational group algebra $\mathbb{Q}G$ of a finite group G or the rational loop algebra of a finite alternative loop L.

These results are then used to give a complete classification of those RA-loops L for which $\mathcal{U}(\mathbb{Z}L)$ has this property. We highligth that such a classification for group rings of infinite groups is still an open problem.

I. B. Passi proposed the classification of those groups G such that the unit group of $\mathbb{Z}G$ is a hyperbolic group (in the sense of Gromov). This was done in [14] in the case when G is polycyclic-by-finite.

A similar question was considered for RG, R being the ring of algebraic integers of $\mathbb{K} = \mathbb{Q}(\sqrt{-d}), d > 0$ square free and G a finite group (see [15]). In [11, 13] these results were extended to characterize associative algebras $\mathcal{A}$, finite-dimensional over the rational numbers, containing an order $\Gamma \subset \mathcal{A}$ whose unit group $\mathcal{U}(\Gamma)$ does not contain a subgroup isomorphic to a free abelian group of rank two. An algebra $\mathcal{A}$ with this property is said to have the *hyperbolic property*.

In this context, the finite semigroups S and the fields $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$, with $d > 0$ square free, such that $\mathbb{K}S$ has the hyperbolic property were classified.

1. [9, 1940], Higman classifies the finite groups with $\mathcal{U}(\mathbb{Z}G)$ finite. Also if G is finite abelian the $\mathcal{U}(\mathbb{Z}G) \cong \mathcal{U}(\mathbb{Z}) \times G \times L$ with L a free Abelian group.
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Some nonassociative algebras also arise from group ring like structures. If L is a loop and R a ring, one can obtain a loop algebra RL using the multiplication table of L. To obtain an algebra which is closer to the associative case one can impose, for example, that the loop algebra is alternative.

In this direction Chein and Goodaire (see [4]) defined alternative loops: a loop L is alternative if RL is alternative over some ring of characteristic not equal to two. It turns out that these loops have a very nice structure and that the loop algebras of these loops can be obtained using the Cayley-Dickson duplication process applied to a group algebra.

In what follows, k will denote a number field and $R = I_k = \mathfrak{o}_k$ its ring of algebraic integers. We will look only at finite-dimensional alternative algebras over k. In particular, we address the problem posed by I. B. Passi in the context of nonassociative algebras. More precisely, we classify the finite-dimensional alternative algebras over k whose I_k-orders does not contain an embedding of $\mathbb{Z}^2$.

Let $\mathcal{A}$ be a finite dimensional k-algebra. An R-order of $\mathcal{A}$ is a subring Λ of $\mathcal{A}$ satisfying the conditions:

1. The centre of Λ is R.
2. Λ is a finitely generated R module.
3. $k \cdot \Lambda = \mathcal{A}$

subject

Let G be a δ-hyperbolic group in the Gromov sense ([8]). The Flat Plane Theorem ([3, Corollary $III.\Gamma.3.10.(2)$]) states that if G is a δ-hyperbolic group than it does not contain a copy of $\mathbb{Z}^2$.

We use this as a starting point for a definition which we call the hyperbolic property. For an associative finite-dimensional algebra over $\mathbb{Q}$ the hyperbolic property was defined in [11].

Definition

Let L be a finitely generated diassociative loop. We say that L has the hyperbolic property if it does not contain a free abelian subgroup of rank two.

Definition

Let k be a field of characteristic zero, I_k its ring of algebraic integers and let $\mathfrak{A}$ be an alternative k-algebra. We say that $\mathfrak{A}$ has the *hyperbolic property* if there exists an I_k-order $\Gamma \subset \mathfrak{A}$ whose unit loop $\mathcal{U}(\Gamma)$ has the hyperbolic property.

The classification we proposed follows according to the steps below

1. We first fix a gap in the classification of the associative algebras with the hyperbolic property. We give the correct classification and give a self-contained proof. A typical missing object is a central $\mathbb{Q}$ division algebra $\mathcal{A}$ of degree four.
2. In degree four, $\mathcal{A}_{\mathbb{R}} = \mathcal{A} \bigotimes_{\mathbb{Q}} \mathbb{R} = M_2(\mathbf{H}(\mathbb{R}))$. Thus the image of $\mathcal{A}$ in the Brauer group $Br(\mathbb{R})$ of $\mathbb{R}$ is $[\mathbf{H}(\mathbb{R})]$, the class of the quaternion algebra over the real numbers. Using the Albert-Hasse-Brauer-Noether description of the Brauer group $Br(\mathbb{Q})$ of $\mathbb{Q}$, one can give the invariants of such an algebra.
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In the associative case we fix a gap of a theorem in [11].

Theorem (Main Associative Theorem)

An associative finite-dimensional indecomposable central k-algebra $\mathcal{A}$ has the hyperbolic property if and only if it is one of the following algebras:

1. $k = \mathcal{A} \in \{\mathbb{Q}, \mathbb{Q}(\sqrt{d})\}$, *with* $d \in \mathbb{Z}^*$ *square free.*
2. $k = \mathcal{A}$ *is a non-totally real field with* $[\mathcal{A} : \mathbb{Q}] = 3$.
3. $k = \mathcal{A}$ *is a totally complex field with* $[\mathcal{A} : \mathbb{Q}] = 4$.
4. $k = \mathbb{Q}$ *and* $\mathcal{A} = M_2(\mathbb{Q})$.
5. $k = \mathbb{Q}(\sqrt{-d})$ *and* $\mathcal{A}$ *is a quaternion division algebra* $(\mathbb{Q}(\sqrt{-d}), \alpha, \beta)$ *with* $d \in \mathbb{N}^*$.
6. $k = \mathbb{Q}(\sqrt{d})$ *and* $\mathcal{A}$ *is a totally definite quaternion algebra* $(\mathbb{Q}(\sqrt{d}), -\alpha, -\beta)$, $d \in \mathbb{N}^*$.

Theorem (Continuation)

7. *$k = \mathbb{Q}$, maximal subfields of $\mathcal{A}$ have signature $[0, 2]$ and $\mathcal{A}$ is a central division crossed product $(\mathbb{L}, \mathbb{Z}_2 \times \mathbb{Z}_2, \kappa)$, $[\mathbb{L} : \mathbb{Q}] = 4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 = Gal(\mathbb{L}/\mathbb{Q})$, the Klein 4-group. Equivalently, $\mathcal{A}$ is a central division $\mathbb{Q}$-algebra of degree 4 such that $[\mathcal{A}] \notin Br(\mathbb{Q}_0/\mathbb{Q})$, where $\mathbb{Q}_0$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{R}$. In particular this is the case if $[\mathcal{A}] \notin Br(\mathbb{R}/\mathbb{Q})$.*
8. *The upper-triangular 2×2 matrix algebra*
$$T_2(\mathbb{Q}) = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}.$$

In particular, if B is a maximal abelian finitely generated R-subalgebra of $\mathcal{A}$ then $[B : \mathbb{Q}] \leq 4$. Moreover, $\mathcal{U}(B)$ is a finite group if and only if $\mathcal{A} \in \{\mathbb{Q}, \mathbb{Q}(\sqrt{-d}), (\mathbb{Q}, -\alpha, -\beta)\}$.

Note that in case 7 $\mathcal{A}$ is also known to be a cyclic algebra (see [1, Theorem $IX.32$]). In particular, $\mathbb{L} = \mathbb{Q}(\alpha)$ has an element t, say, with $m_t(x) = x^4 - \gamma \in \mathbb{Q}[x]$. Since $\mathbb{Q}(t)$ is a maximal subfield of $\mathcal{A}$ its signature must be $[0, 2]$ and hence $\gamma < 0$.

If $\mathbb{L}$ is also Galois with $\mathbb{Z}_2 \times \mathbb{Z}_2$ as Galois group then, looking at the Lagrange resolvent of $m_t(x)$, it is easily seen that $\gamma = \alpha^2$ with $\alpha \in \mathbb{Q}$, a non-square.

In the semi-simple case we have that $[\mathcal{A} : k] \in \{1, 2, 3, 4\}$ and the degree of $\mathcal{A}$ also is in the set $\{1, 2, 3, 4\}$. We know (see [6, Theorem 10.1.2] or [15]) that for any order Λ of $(\mathbb{Q}(\sqrt{-d}), -1, -1)$, with $d \in \mathbb{N}^*$ square free and $d \equiv 1 \pmod 8$, $\mathcal{U}(\Lambda)$ is a hyperbolic group.

The other quaternion algebras appearing in the Main Associative Theorem (MAT) are totally definite and hence their unit groups are virtually cyclic. The unit group of an order of the algebras of degree 4 are, as far as we know, not yet determined. Since we have all the building blocks of an algebra with the hyperbolic property we can now give the Wedderburn decomposition for these algebras.

Theorem

A k-algebra $\mathcal{A}$ has the hyperbolic property if and only if one of the following holds:

1. $J = 0$ *and* $\mathcal{A}$ *is a direct sum of algebras appearing in (MAT) and at most one of them has an order whose unit group is not finite.*
2. $k = \mathbb{Q}$, J *is a one dimensional central nilpotent ideal and* $\mathcal{A}$ *is a direct sum of* J *and algebras appearing in (MAT) all which have an order whose unit group is finite.*
3. $k = \mathbb{Q}$, J *is a one dimensional non-central nilpotent ideal and* $\mathcal{A}$ *is a direct sum of* $T_2(\mathbb{Q})$ *and algebras appearing in (MAT) all which have an order whose unit group is finite.*

A natural question is the following: which of the algebras appearing in (MAT) are isomorphic to a Wedderburn component of the group algebra $\mathbb{Q}G$ of a finite group G? We address this question in the following result. In particular, we shall prove that the division algebra $(\mathbb{Q}(\sqrt{-d}), \alpha, \beta)$, with $d \in \mathbb{N}^*$ square free, is not a Wedderburn component of $\mathbb{Q}G$ for any finite group G.

Corollary

Let $\mathcal{A}$ be one of the noncommutative algebras of (MAT). There exists a finite group G such that $\mathcal{A}$ is a Wedderburn component of $\mathbb{Q}G$ if and only if $\mathcal{A}$ is isomorphic to $M_2(\mathbb{Q})$ or a quaternion division algebra $(\mathbb{Q}(\sqrt{d}), -1, -1)$, with $d \in \mathbb{N}^*$ square free.
In particular, if G is a finite group then the quaternion algebra $(\mathbb{Q}(\sqrt{-d}), -1, -1)$, with $d \in \mathbb{N}^*$ square free, is not a Wedderburn component of $\mathbb{Q}G$.

We proceed now on the Nonassociative Algebras where the Zorn matrix algebra does not have the hyperbolic property and give a complete classification to the semi-simple an non-semi-simple alternative algebras.

subject

In this section we consider nonassociative alternative algebras with the hyperbolic property. Most results and notations we use here are form the books of Goodaire-Jespers-Milies [7, Section I] and Schafer [16].

The basic building blocks for these simple algebras are the 2×2 matrix algebras $M_2(\mathbb{Q})$ and the quaternion division ring $\mathbf{H}(\mathbb{Q}) = (\mathbb{Q}, -1, -1)$. This should not be so surprising since both, quaternion algebras and nonassociative simple alternative algebras are strongly linked to Geometry.

Definition (Totally Definite Cayley-Dickson Algebra)

Let k be a totally real field and consider the Cayley-Dickson algebra $\mathcal{A} = (k, -\alpha, -\beta, -\gamma)$ where $\alpha, \beta, \gamma \in k$ are totally positive elements. Then $\mathcal{A}$ is called a *totally definite Cayley-Dickson algebra*.

Theorem

Let $\mathcal{A}$ be an indecomposable nonassociative alternative central k-algebra. Then $\mathcal{A}$ has the hyperbolic property if and only if $\mathcal{A}$ is of the following types:

1. $k = \mathbb{Q}(\sqrt{-d})$ *and $\mathcal{A}$ is a Cayley Dickson division algebra* $(\mathbb{Q}(\sqrt{-d}), \alpha, \beta, \gamma)$ *with* $d \in \mathbb{N}^*$.
2. $k = \mathbb{Q}(\sqrt{d})$ *and $\mathcal{A}$ is a totally definite Cayley Dickson algebra* $(\mathbb{Q}(\sqrt{d}), -\alpha, -\beta, -\gamma)$, *with* $d \in \mathbb{N}^*$.

Corollary

Let $\mathcal{A}$ be a nonassociative alternative k-algebra with the hyperbolic property and non-central radical J. Then $k = \mathbb{Q}$ and $\mathcal{A} = \mathcal{A}_1 \bigoplus \mathcal{A}_2$ such that

1. *$\mathcal{A}_1$ and $\mathcal{A}_2$ are ideals of $\mathcal{A}$.*
2. *$\mathcal{A}_1$ is a direct sum of totally definite Cayley Dickson algebras $(\mathbb{Q}, -\alpha, -\beta, -\gamma)$.*
3. *$\mathcal{A}_2$ is an associative subalgebra containing J, $T_2(\mathbb{Q})$ is a direct summand of $\mathcal{A}_2$ and all other direct summands of $\mathcal{A}_2$ are isomorphic to a subalgebra of a totally definite quaternion algebra $(\mathbb{Q}, -\alpha, -\beta)$.*

In particular we have that $[J, \mathcal{A}, \mathcal{A}] = [\mathcal{A}, J, \mathcal{A}] = [\mathcal{A}, \mathcal{A}, J] = (0)$, i.e., J associates with all elements of $\mathcal{A}$.

Note however that if $\mathcal{A}$ is any nonassociative k-algebra satisfying that its radical J is 2-nilpotent and one dimensional it must have the same decomposition. The fact that $k = \mathbb{Q}$ is only used to ensure $T_2(\mathbb{Q})$ is a direct summand. So in general one should get $T_2(k)$ as a direct summand.

Proposition

Let $\mathcal{A}$ be a finite-dimensional algebra over $\mathbb{Q}$ such that $\mathcal{A} \cong S \oplus J$ with $J \neq 0$ being the radical of $\mathcal{A}$. If $\mathcal{A}$ has the hyperbolic property then J is nilpotent of index 2. Furthermore, there exists $j_0 \in J$ such that and $J \cong \langle j_0 \rangle_{\mathbb{Q}}$ is the $\mathbb{Q}$-linear span of j_0 over $\mathbb{Q}$. In particular, the center of $\mathcal{A}$ equals $\mathbb{Q}$.

Theorem

Let k be a number field, $I_k = \mathfrak{o}_k$ its ring of algebraic integers and $\mathcal{A}$ a Cayley-Dickson k-algebra. Let $\mathfrak{O}$ be a maximal order in $\mathcal{A}$ with $\mathcal{U}(\mathfrak{O})$ its loop of units. The following are equivalent.

1. *$SL_1(\mathfrak{O})$, the loop of units in $\mathfrak{O}$ having reduced norm 1, is finite.*
2. $[\mathcal{U}(\mathfrak{O}) : \mathcal{U}(\mathfrak{o}_k)] < \infty$.
3. *$\mathcal{A}$ is a totally definite Cayley-Dickson algebra.*

Corollary

Let $\mathcal{A} = (k, -\alpha, -\beta, -\gamma)$ be a totally definite Cayley-Dickson algebra over a number field k. Then $\mathcal{A}$ has the hyperbolic property if and only if $\mathcal{A} = (\mathbb{Q}(\sqrt{d}), -\alpha, -\beta, -\gamma)$, with $d \in \mathbb{N}^*$.

We finish the paper giving a full classification of those RA-loops L such that $\mathcal{U}(\mathbb{Z}L)$ has the hyperbolic property.

Theorem

Let L be a finitely generated RA-loop. The loop $\mathcal{U}(\mathbb{Z}L)$ has the hyperbolic property if and only if the following conditions hold.

1. *The torsion subloop $T(L)$ is a Hamiltonian 2-loop or an Abelian group of exponent dividing 4 or 6 or a Hamiltonian 2-group.*
2. *All subloops of $T(L)$ are normal in L.*
3. *$h(\mathcal{Z}(L))$, the Hirsch lenght of the center of $\mathcal{Z}(L)$, is at most 1.*

In particular $\mathcal{U}_1(\mathbb{Z}L) = L$.

Thank you

subject

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