On Class Two p-Group G satisfying $G/Z(G) \simeq C_{p^2} \times C_{p^2}$

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Algebra: celebrating Paulo Ribenboim's ninetieth birthday

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- $G' \simeq C_{p^2}$ and $cl(G) = 2$
- The case $n = 1$ was considered by Leal and Milies

Leall Guilherme; Polcino Milies, C. Isomorphic group (and loop) algebras. J . Algebra 155 (1993), no. 1, 195–210

An upper bound of r(Z(G))

Upper Bound of $r(Z(G))$

Let G be a finite indecomposable $p-$group. If $G = \langle x, y, Z(G) : x^{p^2}, y^{p^2} \in Z(G) \rangle$ then $r(Z(G)) \leqslant 5$.

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- $r(Z(D_1)) \leqslant 3$

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- As $x^p = x_d a_1$, $a_1 \in A_1$, $x_d \in D_1$. Let's analyze two cases:

Case 1 : $\exists \widetilde{a} \in G$ such that $\widetilde{a}^p = a_1$.

Case 2 : $\nexists \widetilde{a} \in G$ such that $\widetilde{a}^p = a_1$.

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- First Possibility: $G' \nleqq D_1$. As $G' \simeq C_{p^2}, \exists C_{p^{\alpha_1}} \leqslant A_1$ such that $G' \leqslant C_{p^{\alpha_1}} \times Z(D_1)$. In this case

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- Second Possibility: $G' \leqslant D_1$. Then $r(Z(G)) \leqslant r(Z(D_1)) \leqslant 3 \leqslant 5$.

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- Define $\widetilde{A} = \{\widetilde{a_i} :< \widetilde{a_i} > \simeq C_{p^{\alpha_i}}, 1 \leqslant i \leqslant r\}$ and take $\prod\limits_{\widetilde{a_i} \in \widetilde{A}} \widetilde{a_i}$.

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- $< \prod_{\widetilde{a_i} \in \widetilde{A}} \widetilde{a_i} > \simeq C_{p^\beta}$, with $\beta = max\left\{\alpha_i :< \widetilde{a_i} > \simeq C_{p^{\alpha_i}}, \widetilde{a_i} \in \widetilde{A}\right\}$.

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- Then $\forall\, \widetilde{a}_i \in \widetilde{\widetilde{A}}$, $\exists \widetilde{\widetilde{a}}_i \in Z(G)$ such that $\left(\widetilde{\widetilde{a}}_i \right)^p = a^\star \in A_1$, falling in case 1.

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- $x^p = x_d \left(\prod_{\widetilde{a_i} \in \widetilde{A}} \widetilde{a_i}\right), < \prod_{\widetilde{a_i} \in \widetilde{A}} > \simeq C_{p^\beta} \Rightarrow x^p \in D_1 \times C_{p^\beta}, \beta \geqslant 1$

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- $< x^{p^2} >, < y^{p^2} >, G' \leqslant Z(D_1) \times C_{p^\beta} \times C_{p^\gamma}$

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- $< x^{p^2} >, < y^{p^2} >, G' \leqslant Z(D_1) \times C_{p^\beta} \times C_{p^\gamma}$
- $r(Z(G)) \leqslant r(Z(D_1)) + 1 + 1 \leqslant 3 + 1 + 1 = 5$

Example that upper limit is reached

$G =< x, y, Z(G) : x^{p^2}, y^{p^2} \in Z(G) >$, where

- $x^{p^4} = y^{p^4} = 1,\ a^{p^3} = b^p = e^{p^3} = 1.$
- $[x, y] = z,\ z = a^p b,\ x^{p^2} = c^p d,\ y^{p^2} = e^p d$
- $[x^p, y] = [x, y^p] = z^p$
- $\left[x^{p^2}, y\right] = \left[x, y^{p^2}\right] = [x^p, y^p] = 1$
- $[a, b] = [a, c]\,[a, d] = [a, e] = [b, c] = [b, d] = [b, e] = [c, d] = [c, e] = [d, e] = 1$
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$$
\begin{array}{ccccccccccc}
Z(G) & \simeq & C_{p^3} & \times & C_p & \times & C_{p^3} & \times & C_p & \times & C_{p^3} \\
 & & \| & & \| & & \| & & \| & & \| \\
 & & < a > & & < b > & & < c > & & < d > & & < e >
\end{array}
$$

- $< a^p b >= G'$, $< x^{p^2} >=< c^p d >$ and $< y^{p^2} >=< e^p d >$

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- $< a^p b >= G'$, $< x^{p^2} >=< c^p d >$ and $< y^{p^2} >=< e^p d >$
- $\mid G \mid = p^{15}$, $\mid Z(G) \mid = p^{11}$

The Simple Components of $\mathbb{Q}G$

Degree of Simple Components

Let G a finite p-group. If $G = \langle x_1, x_2, Z(G) : x_1^{p^2}, x_2^{p^2} \in Z(G) \rangle$ then $\{1, p, p^2\}$ are exactly the set of the degrees of the absolutely irreducible representation coming from the primitive central indempotents of $QG_i = Q(G/\langle z_i \rangle), i = 1, 2,$ where $C_{p^2} \simeq G' = \langle z_2 \rangle \geqslant \langle z_1 \rangle \geqslant \langle z_0 \rangle = \{1\}$, $\langle z_i \rangle \simeq C_{p^i}$.

The Simple Components of $\mathbf{QG}$

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- (1) $QG_i((1 - \widehat{\overline{z}_{i+1}}) \simeq \bigoplus_{s=1}^{b} Q(G_i/K_s)(1 - \widehat{\overline{z}_{i+1}}) = \sum_{s=1}^{b} M_{t_s \times t_s}(F_s)$

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- $G_i' \leqslant C_{p^{\beta_i}}, G_i \cap \overline{K_s} = \{1\}$
- Using (1), (2) and calculating the dimension
- The degrees of the absolutely irreducible representations are $1, p, p^2$

The Simple Components of $\mathbb{Q}G$

Center of Simple Components

The center of the irreducible components of $QG(1-\hat{G}')$ *are isomorphic to* $QZ(G/K)$ *where* K *is the kernel of the irreducible representation* $G \to Ge$ *and* e *is a primitive central idempotent of* $QG(1-\hat{G}')$*.*

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- and $\widetilde{Z}/K = C_{G/K}(Z_2(G/K)) = Z(G/K)$.
- The primitive central idempotent of $QG(1 - \widehat{G'})$
- $e = \widehat{K}(1 - \widehat{z})$, where $o(\overline{z}) = p$, $< \bar{z} > \leq \tilde{Z}/K$ and $< \bar{z} > \leq (G'K)/K \simeq G'/(G' \cap K)$.

The Simple Components of $\mathbb{Q}G$

- $Z(\mathbb{Q}G(1-\widehat{G'})) = Z(\bigoplus_{i=1}^{s} \mathbb{Q}G\widehat{K_i}(1-\widehat{\overline{z_i}})) \simeq \bigoplus_{i=1}^{s} Z(\mathbb{Q}(G/K_i)(1-\widehat{\overline{z_i}})) \simeq \bigoplus_{i=1}^{s} Z(M_{t_i \times t_i}(\mathbb{Q}(\rho^{\beta}))$

The Simple Components of **QG**

- $Z(QG(1-\widehat{G'})) = Z(\bigoplus_{i=1}^{s} QG\widehat{K_i}(1-\widehat{\overline{z_i}})) \simeq \bigoplus_{i=1}^{s} Z(Q(G/K_i)(1-\widehat{\overline{z_i}})) \simeq \bigoplus_{i=1}^{s} Z(M_{t_i \times t_i}(Q(\rho^{\beta}))$
- $Q(\rho^{\beta}) \simeq QZ(G/K_i)$

Necessary and Sufficient conditions to $\mathbf{QG} \simeq \mathbf{QH}$

Necessary Conditions

If $QG = QH$ then $G' = H' \simeq C_{p^2}$, $|Z(G)| = |Z(H)|$ and $G/Z(G) \simeq H/Z(H) \simeq C_{p^2} \times C_{p^2}$, in particular $cl(H) = 2$.
Moreover we have:
$(i) exp(Z(G)) = exp(Z(H))$,
$(ii) exp(Z(G/ < z_1 >)) = exp(Z(H/ < h_1 >))$, *where* $< z_1 >= O_p(G')$ *and* $< h_1 >= O_p(H')$.

Necessary and Sufficient conditions to $\mathbf{QG} \simeq \mathbf{QH}$

- Split the demonstration in two cases:

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Theorem 1

Let G and H be p-groups of nilpotent class 2. If $QG = QH$, then:

1. $G' = H'$;
2. $|Z(G)| = |Z(H)|$;
3. *The biggest central cyclic components of $Z(G)$ and $Z(H)$ are isomorphic.*

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  - $\overline{H} = H/Z(H)$ is a capable class 2 p-group having a cyclic commutator subgroup of order p. By Theorem A of Yadav

Theorem A

Let G be a finite capable p-group of nilpotency class 2 with cyclic comutator subgroup G'. Then $G/Z(G)$ is generated by 2 elements and $|G/Z(G)| = |G'|^2$.

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- $H/Z_2(H) \simeq \frac{\overline{H}}{Z(\overline{H})} \simeq C_p \times C_p$, giving $\frac{H}{Z_2(H)} \simeq C_p \times C_p$

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- $\widehat{z_1} = \widehat{h_1}$, and $Q\left(G/ < z_1 >\right) \simeq QG\widehat{z_1} = QH\widehat{h_1} \simeq Q\left(H/ < h_1 >\right)$

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- $\widehat{z_1} = \widehat{h_1}$, and $Q\left(G/ < z_1 >\right) \simeq QG\widehat{z_1} = QH\widehat{h_1} \simeq Q\left(H/ < h_1 >\right)$
- $< z_1 > = < h_1 >$ and $H' \simeq G'$.
- force $cl(H) = 2$, contradiction

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Sufficient Conditions

Let p be an odd prime and suppose G and H are two $p-$groups of same order satisfying the following conditions:

$(i) G/Z(G) \simeq C_{p^2} \times C_{p^2} \simeq H/Z(H)$
$(ii) G/G' \simeq H/H'$
$(iii) exp(Z(G)) = exp(Z(H))$
$(iv) exp(Z(G/<z_1>)) = exp(Z(H/<h_1>))$

Then $QG \simeq QH$.

- **Remark** In the case $p = 2$, The above theorem remains true, if we add in condition (*iii*) the hypotesis $exp(Z(G)) \geq 2^3$

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- two models of $2 - groups$ that satisfy the hypotheses of previous theorem with $exp(Z(G)) = exp(Z(H)) = 2^2$ but $QG \not\simeq QH$

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- two models of $2 - groups$ that satisfy the hypotheses of previous theorem with $exp(Z(G)) = exp(Z(H)) = 2^2$ but $QG \not\simeq QH$
- In the first appears $M_{2\times 2}(Q)$ and in the second the Quaternion $\mathbb{H}(Q)$.

- $\mathbf{G_1} = < \mathbf{x}, \mathbf{y}, \mathbf{Z}(\mathbf{G_1}) : \mathbf{x^4}, \mathbf{y^4} \in \mathbf{Z}(\mathbf{G_1}) >$, where $x^8 = y^8 = 1$, $[x, y] = z$, $[x, z] = [y, z] = 1$, $z^4 = 1$, $x^4 = a$, $y^4 = b$, $[a, x] = [a, y] = [b, x] = [b, y] = [a, z] = [b, z] = 1$.

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- $Z(G_1) = < z > \times < a > \times < b > \simeq C_4 \times C_2 \times C_2$

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- $\mathbf{G_2} = < \mathbf{x}, \mathbf{y}, \mathbf{Z}(\mathbf{G_2}) : \mathbf{x^4}, \mathbf{y^4} \in \mathbf{Z}(\mathbf{G_2}) >$, where $x^8 = y^8 = 1$, $[x, y] = z$, $[x, z] = [y, z] = 1$, $z^4 = 1$, $x^4 = az^2$, $y^4 = bz^2$, $[a, x] = [a, y] = [b, x] = [b, y] = [a, z] = [b, z] = 1$.

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- $Z(G_2) = < z > \times < a > \times < b > \simeq C_4 \times C_2 \times C_2$

Thank You