On the distribution of suitable totients

celebrating P. Ribenboim's 90th birthday - IME USP

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Introduction

Euler's totient function:

$$\phi(n) := |\{1 \leq a \leq n \,;\, \mathrm{mdc}(a,n) = 1\}|$$

- the elements in the image of ϕ are called totients;
- $\phi(\mathbb{N}) \subsetneq \{1\} \cup \{\text{even numbers}\}$,

$$\mathcal{V} := \phi(\mathbb{N}) = \{1, 2, 4, 6, 8, 10, 12, 16, 18, 20, 22, 26, \ldots\}.$$

multiplicity:

$$\mathrm{A}(m) := |\phi^{-1}(m)|$$

Examples:

- $\phi^{-1}(1) = \{1, 2\}$, $\mathrm{A}(1) = 2$;
- $\phi^{-1}(14) = \emptyset = \phi^{-1}(90)$, $\mathrm{A}(14) = 0 = \mathrm{A}(90)$;
- $\mathrm{A}(4 \cdot 3^{66}) = 33$;

as usual:

given a subset $U \subseteq \mathbb{N}$, for any $1 < x \in \mathbb{R}$, set

$$U(x) := \{n \in U \mid n \leq x\} = U \cap [1, x]$$

Main classical questions:

- the order of $|\mathcal{V}(x)|$? (K. Ford theorem)
- What multiplicities are possible? (Sierpiński's Conjecture solved by K. Ford).
- Carmichael's conjecture: $\nexists m$ with $\mathrm{A}(m) = 1$;
- Set $\mathcal{V}_k := \{m \in \mathcal{V}\,;\, \mathrm{A}(m) = k\}$. What is the order of $|\mathcal{V}_k(x)|$?
- $\lim_{x \to \infty} \dfrac{|\mathcal{V}_k(x)|}{|\mathcal{V}(x)|} = C_k$, for some constant C_k?

Our own questions

totients are even numbers!

$$\forall\, n \,\exists\, k\ , \phi(n) \equiv 2^{k-1} \mod 2^k$$

simplest case: $k = 2$

Lemma A

Let r be an odd positive integer. It follows that $\mathrm{A}(2r) \in \{0, 2, 4\}$. If $\mathrm{A}(2r) = 2$, then $\phi^{-1}(2r) = \{p^n, 2p^n\}$, with p an odd prime, $n > 0$. If $\mathrm{A}(2r) = 4$, then $2r + 1$ is a prime number and $\phi^{-1}(2r) = \{2r + 1, q^m, 4r + 2, 2q^m\}$ with q a prime number and $m > 1$.

Lemma B

Given an integer number $t > 0$, let us consider the set

$$\mathcal{R}_t := \{k \in \mathbb{N}\,;\ p^i \in \phi^{-1}(k) \text{ with } i \geq t+1,\ p \in \mathcal{P},\ p > 2\} \subset \mathcal{V}\,.$$

We have

$$|\mathcal{R}_t(x)| = \mathrm{o}(\pi(\sqrt[t]{x}))\,.$$

Proof.

For every $k \in \mathcal{R}_t(x)$ we can see that $q \leq \sqrt[t]{x/2}$ and $t \leq \lceil \log x / \log 3 \rceil$. Now, let us take the set $\mathcal{U}(x) := \{q^j \leq x; q \text{ is prime}\}$. We can see that $|\mathcal{R}_t(x)| \leq |\mathcal{U}(x)|$. Hence

$$|\mathcal{U}(x)| \leq \pi(\sqrt[t]{x}) + \sum_{i=t+1}^{\lceil \log x / \log 3 \rceil} \pi(\sqrt[i]{x}).$$

Now, by the Prime Number Theorem follows

$$\frac{\pi(\sqrt[i]{x})}{\sqrt[t]{x}} = \mathrm{O}\left(\frac{t}{\sqrt[t(t+1)]{x} \log x} \right), \ \forall i > t \,. \tag{1}$$

Hence we can write

$$\sum_{i=t+1}^{\lceil \log x / \log 3 \rceil} \pi(\sqrt[i]{x}) = \mathrm{O}\left(\sum_{i=t+1}^{\lceil \log x / \log 3 \rceil} \frac{i}{\sqrt[t(t+1)]{x} \log x} \right) = \mathrm{O}\left(\frac{\log x}{\sqrt[t(t+1)]{x}} \right).$$

Since $\pi(\sqrt[t]{x}) = \mathrm{o}(\sqrt[t]{x})$, $|\mathcal{U}(x)| = \mathrm{o}(\sqrt[t]{x})$. Finally $\lim_{x \to \infty} \frac{|\mathcal{R}_t(x)|}{\sqrt[t]{x}} = 0 \,.$ $\square$

On the distribution

Let us start by taking the following useful sets

$$\mathcal{T}_k = \{2r\,;\ r \text{ odd and } \mathrm{A}(2r) = k\} \quad (k = 0, 2, 4).$$

Table 1: The number of totients 2 mod 4 $\leq x$ with a fixed multiplicity

x	$\pi(x)$	$\lvert\mathcal{T}_2(x)\rvert$	$\lvert\mathcal{T}_4(x)\rvert$	$\lvert\mathcal{T}_2(x)\rvert/\pi(x)$
10^3+2	168	87	5	$0.517857\ldots$
10^4+2	1229	625	8	$0.508543\ldots$
10^5+2	9592	4831	14	$0.503648\ldots$
10^6+2	78498	39400	20	$0.501923\ldots$
10^7+2	664579	332606	34	$0.500476\ldots$
10^8+2	5761455	2881495	78	$0.500133\ldots$

Corollary

$\lim_{x\to\infty} \dfrac{\lvert\mathcal{T}_4(x)\rvert}{\sqrt{x}} = 0.$

Proof.

We just have to observe that $\mathcal{T}_4(x) \subset \mathcal{R}_2(x)$. $\square$

Proposition

Let $l > 1$ be a positive integer. Let us consider $\mathcal{V}_k^l := \mathcal{T}_t \cap \mathcal{V}_k$ be the set of totients with multiplicity k such that there is a power of a prime p^l in its inverse image by ϕ. With this

$$\lim_{x\to\infty} \frac{|\mathcal{V}_k^l(x)|}{|\mathcal{V}_k(x)|} = 0$$

Proof.

From Lemma A we get $|\mathcal{V}_k^l(x)| = \mathrm{o}(\sqrt[l]{x})$. By the Prime Number Theorem we have $\sqrt[l]{x} = \mathrm{o}(\pi(x))$. Now, a theorem of K. Ford implies that $\pi(x) = \mathrm{O}(|\mathcal{V}_k(x)|)$. Hence $|\mathcal{V}_k^l(x)| = \mathrm{o}(|\mathcal{V}_k(x)|)$. □

Theorem

$$|\mathcal{T}_2(x)| = \frac{\pi(x)}{2} + \mathrm{O}\left(\frac{1}{\log x}\right).$$

Proof.

$\mathcal{T}_2'(x) = \{2r \in \mathcal{T}_2(x)\,;\ 2r+1 \in \mathcal{P}\}$ and $\mathcal{T}_2^*(x) = \{2r \in \mathcal{T}_2(x);\ 2r+1 \notin \mathcal{P}\}$.

$$\mathcal{T}_2(x) = \mathcal{T}_2'(x) \cup \mathcal{T}_2^*(x) \text{ and } \frac{|\mathcal{T}_2(x)|}{\pi(x)} = \frac{|\mathcal{T}_2'(x)|}{\pi(x)} + \frac{|\mathcal{T}_2^*(x)|}{\pi(x)}.$$

$$\liminf_{x\to\infty} \frac{|\mathcal{T}_2(x)|}{\pi(x)} = \liminf_{x\to\infty} \frac{|\mathcal{T}_2'(x)|}{\pi(x)} \text{ and } \limsup_{x\to\infty} \frac{|\mathcal{T}_2(x)|}{\pi(x)} = \limsup_{x\to\infty} \frac{|\mathcal{T}_2'(x)|}{\pi(x)}.$$

$\{2r+1 \in \mathcal{P}(x+1; 4, 3)\} = \mathcal{T}_2'(x) \cup \{2r+1 \in \mathcal{P}(x+1; 4, 3); \mathrm{A}(2r) = 4\}$.

By the Prime Number Theorem in Arithmetic Progression

$$\frac{1}{2} = \lim_{x\to\infty} \frac{\pi(x; 4, 3)}{\pi(x)} = \liminf_{x\to\infty} \frac{\mathcal{T}_2'(x)}{\pi(x)} \le \limsup_{x\to\infty} \frac{\mathcal{T}_2'(x)}{\pi(x)} = \lim_{x\to\infty} \frac{\pi(x; 4, 3)}{\pi(x)}.$$

Thus

$$|\mathcal{T}_2(x)| = \frac{\pi(x)}{2} + \mathrm{O}\left(\frac{1}{\log x}\right).$$

$\square$

Corollary

$|\mathcal{T}_4(x)| = \mathrm{o}(|\mathcal{T}_2(x)|)$.

our preprint:

On the distribution of totients $2 \equiv \mod 4$,
`https://arxiv.org/pdf/1803.01396.pdf` (2018).

you should read:

- R. D. Carmichael, Note on Euler's ϕ-function, *Bull. Amer. Math. Soc.*, 28 (1922)109110.
- P. Erdös and R.R.Hall, On the values of Eulers-function, *Acta Arith.* 22 (1973)201-206.
- P. Erdös and C. Pomerance, On the normal number of prime factors of $\phi(n)$, *Rocky Mountain J. of Math.* 15 (1985) 343352.
- K. Ford,The distribution of totients, *The Ramanujan Journal*, 2 (1998)179.
- K. Ford, The number of solutions of $\phi(x) = m$, *Annals of math.*, 159 (1999) 129.

The next case: k=3. Working in progress

Theorem

Let r be an odd interger, we have

$$\limsup_{r} \mathrm{A}(4r) = \infty$$

Table 2: density of totients $\equiv 2^2 \bmod 2^3$

$x-4$	$\lvert\mathcal{T}_4(x)\rvert$	$\lvert\mathcal{V}(x)\rvert$	$\lvert\mathcal{T}_4(x)\rvert/\lvert\mathcal{V}(x)\rvert$
10^6	$54,172$	$180,184$	0.300648
$5 \cdot 10^6$	$250,824$	$840,178$	0.298536
10^7	$486,400$	$1,634,372$	0.297606
$25 \cdot 10^6$	$1,169,810$	$3,946,810$	0.296393
$125 \cdot 10^6$	$5,490,855$	$18,657,532$	0.294296
$3 \cdot 10^8$	$12,763,093$	$43,525,579$	0.293232
$5 \cdot 10^8$	$20,892,814$	$71,399,659$	0.292617

Corollary

For every $k \geq 2$ it follows

$$\limsup_r \mathrm{A}(2^k r) = \infty$$

Conjecture (Strong Sierpiński Conjecture)

Every multiplicity bigger than 1 is realized by a totient $\equiv 4 \mod 8$.