Shortest Networks. Steiner Problem

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Definitions and some examples

Let (X, ρ) be a metric space, V be a finite subset of X, and G = (V, E) be a graph. We say that G is *a graph on the metric space* (X, ρ).

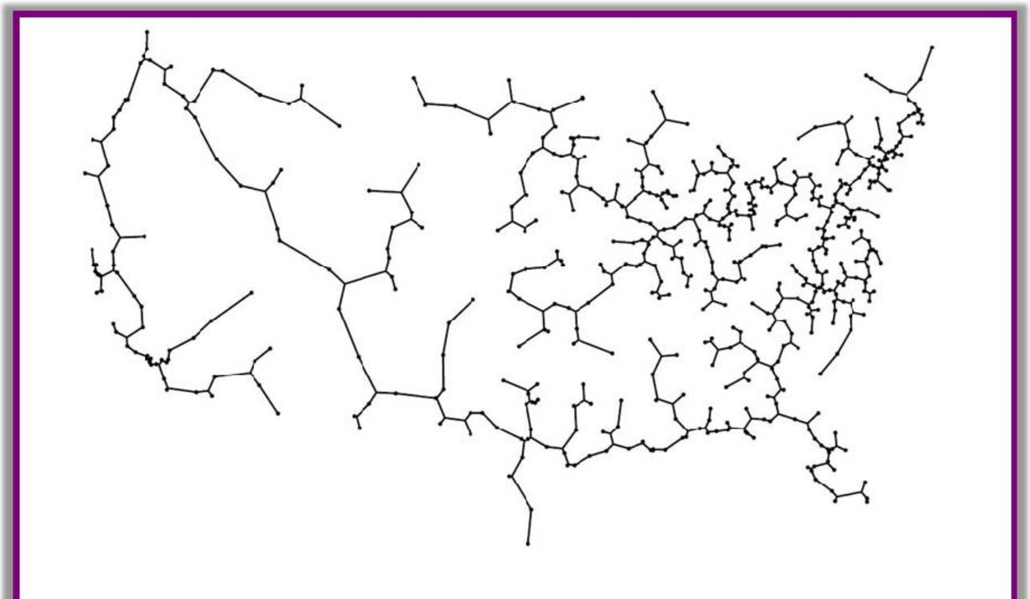
If e = { x, y} ∈ E, then the value ρ(e) = ρ(x,y) is called the *length of the edge* e. The sum of the lengths of all edges e ∈ E is called *the length of the graph* G.

Consider a finite M ⊂ X. If G = (V, E) is a graph on (X, ρ) such that M ⊂ V, then we say that G *joins* M

We put smt (M) = inf { ρ(G) | G is a tree joining M} and call this value by *the length of Steiner Minimal Tree* (SMT) for M. If (G) = smt(M), and G is a tree joining M, then G is called a *Steiner Minimal Tree* (or *shortest tree*) *with the boundary* M.

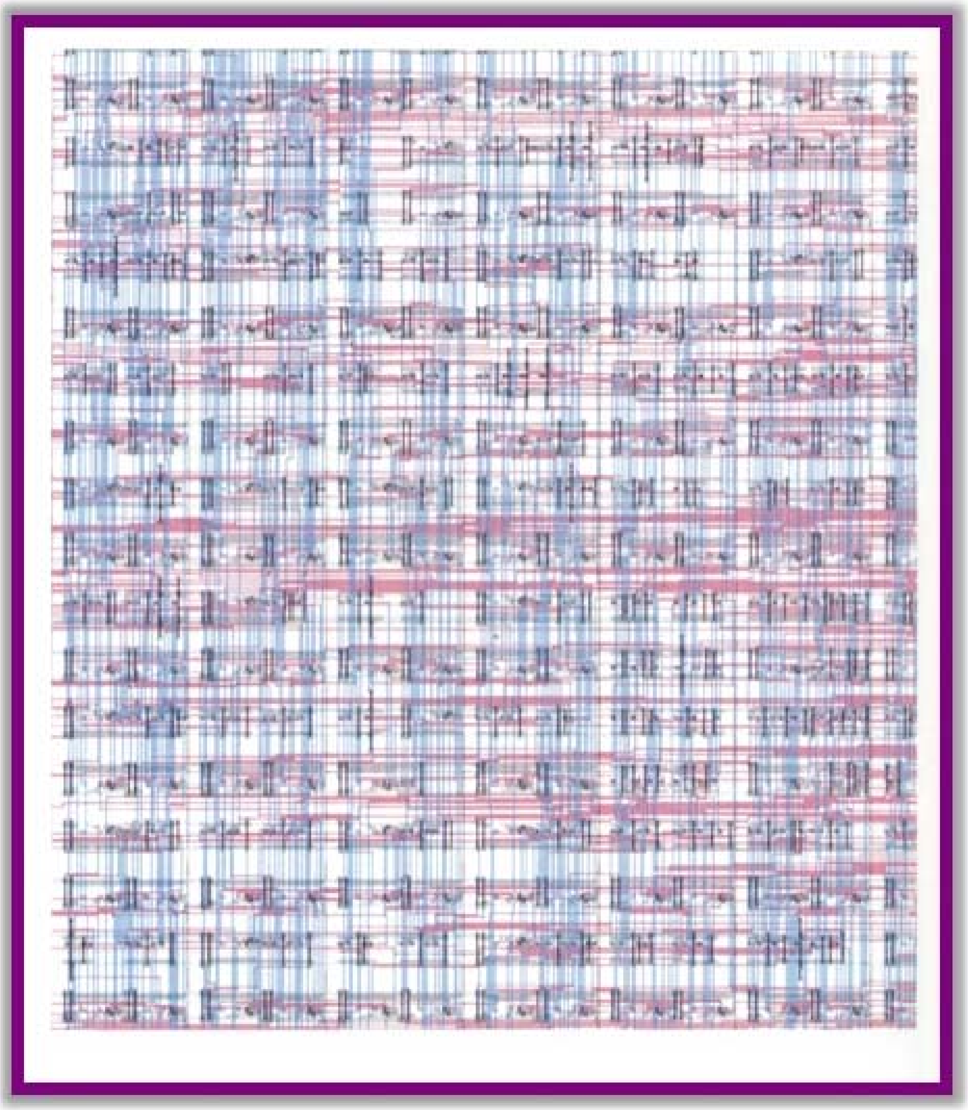
Applications.

1) Transportation problem. Here (X, ρ) is Euclidean plane.



532 cities in the United States

2) Chip design. Here (X, ρ) is Manhattan plane, i.e., if A = (x_1, y_1) and B = (x_2, y_2), then distance (A, B) = $|x_2 - x_1| + |y_2 - y_1|$.



3) Evolution (phylogenetic) tree.

Elementary editor operations on a word are

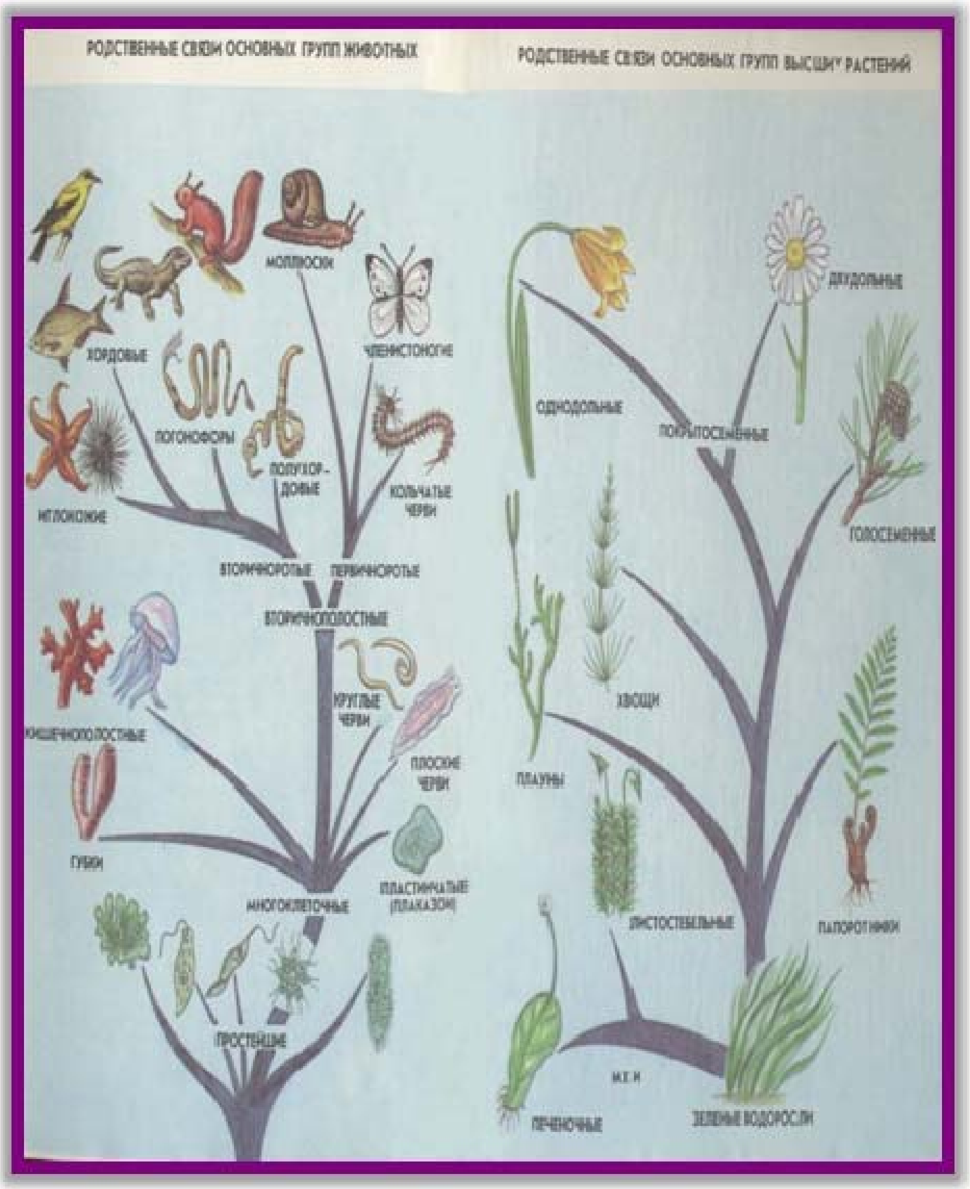
insertion deletion substitution

abed → abxed abxd → abd abxd → abyd

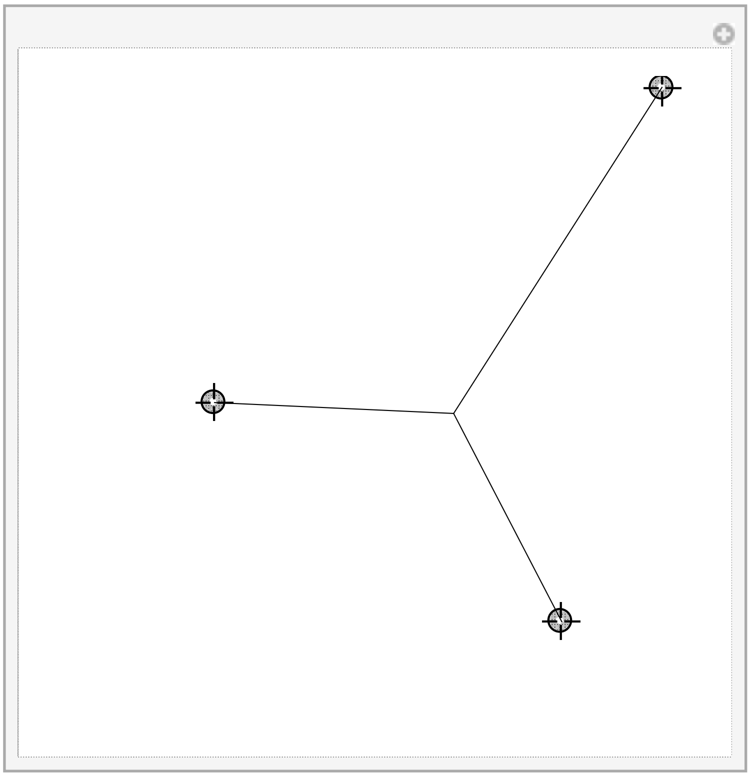
Hamming distance between two words w_1 and w_2 is the least number of elementary editor operations to pass from w_1 to w_2.

To measure the difference between two species one can code them by words, for example, 4 - letter DNA word, or 20 - letter protein word, or a word characterizing the presence of different phenotypic properties, etc., and to calculate the Hemming distance between these words.

Biological assumption : evolution was optimal in the sense of minimization of the changes number (for example, minimization of mutations number). Thus, the evolution tree has to be the shortest tree (in Hemming distance) joining the words corresponding to nowaday species. Thus enables to reconstruct the properties of predecessors.



Fermat case: M consists of three points in $\mathbb{R}^2$.

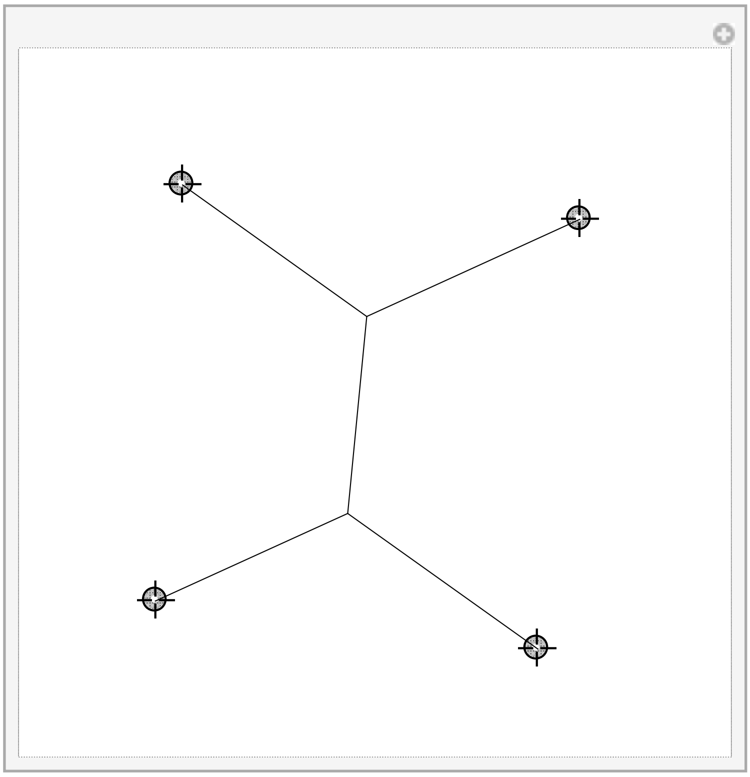


- **Observation**

If the three edges are nondegenerate segments, then the angles between the edges are equal to each other and, thus, to 120°. When one of the edges degenerates, then the angle between the remaining edges is at least 120°.

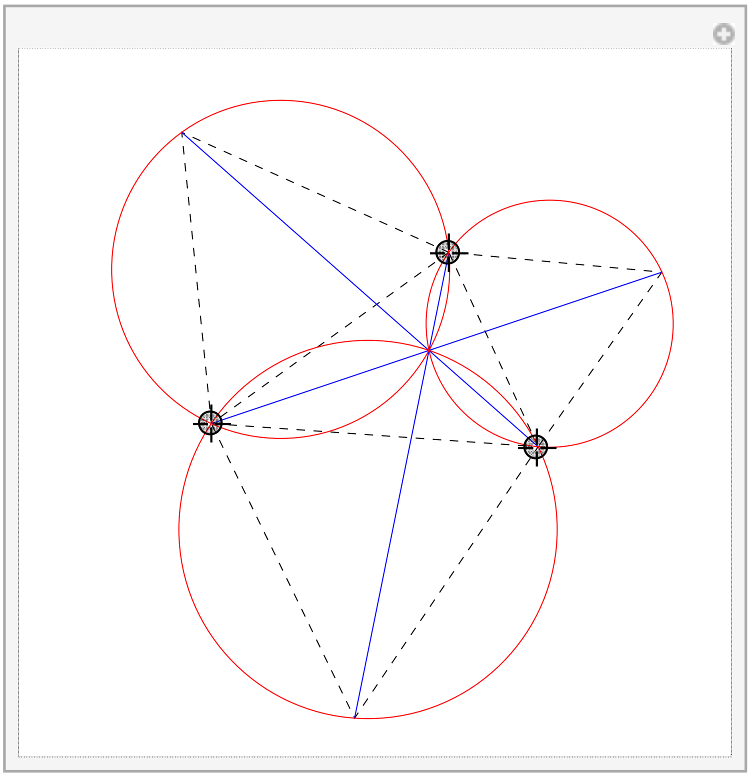
Remark. This property holds for general case.

Euclidean case : four points.



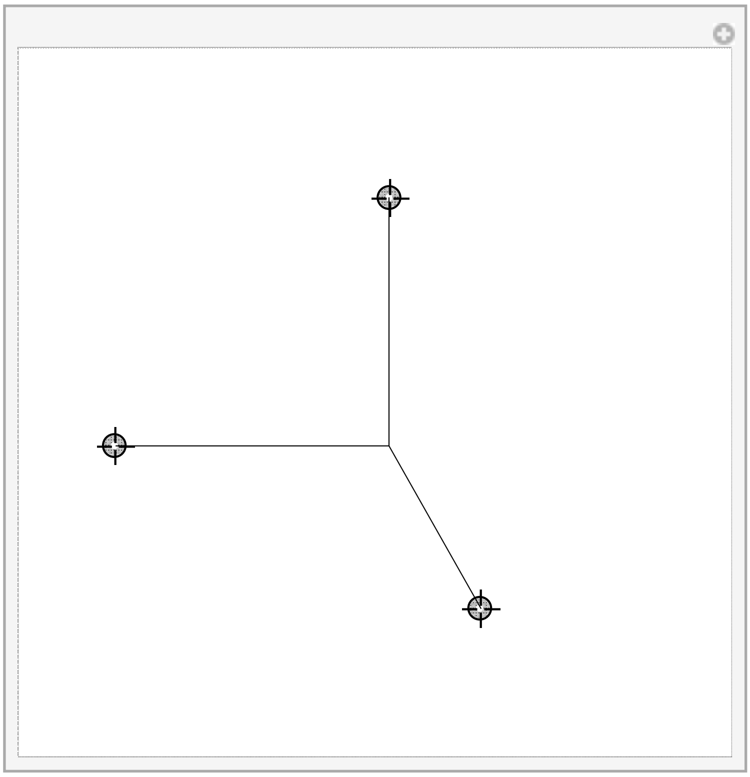
Notice bifurcations, i.e., sudden changes in the topological structure when a small continuous change made to the parameter values.

The 120°-property leads to the following construction (in non-degenerate case).

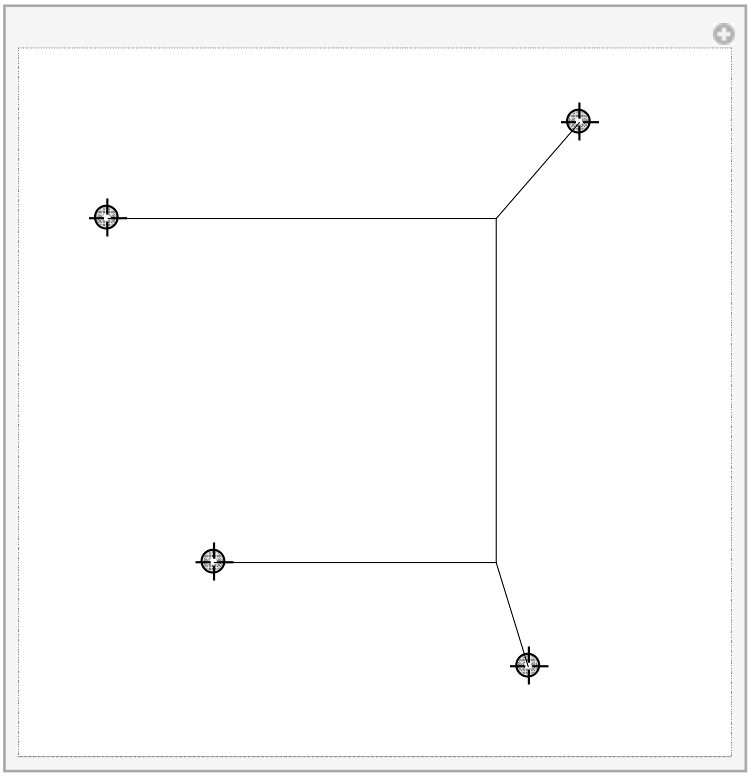


The intersecion of the three circles and three segments is called the Torricelli point of the triangle.

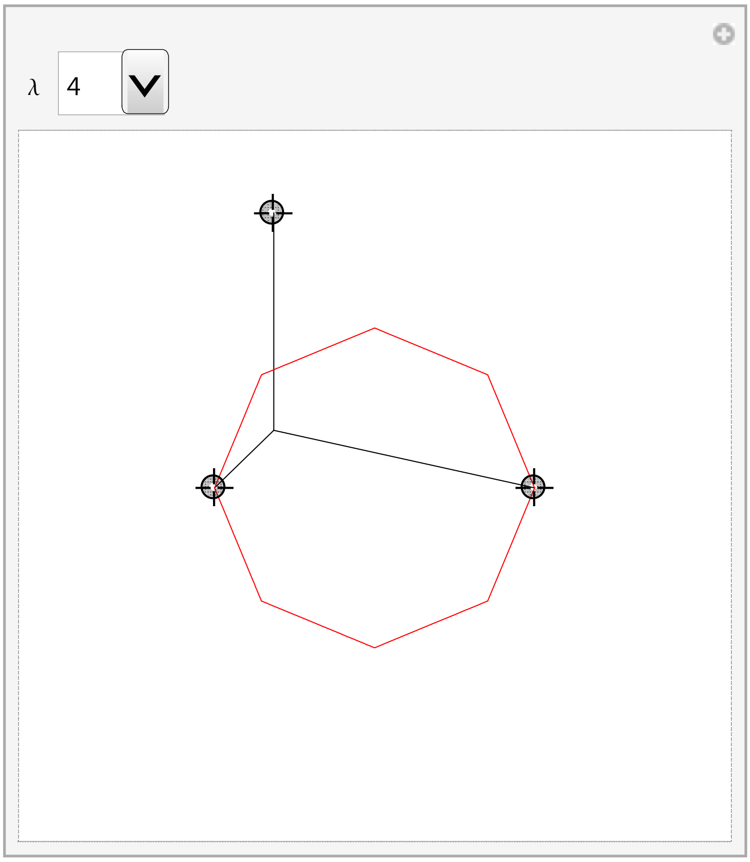
Manhattan case: M consists of three points in Manhattan plane.



Manhattan case: M consists of four points in Manhattan plane.

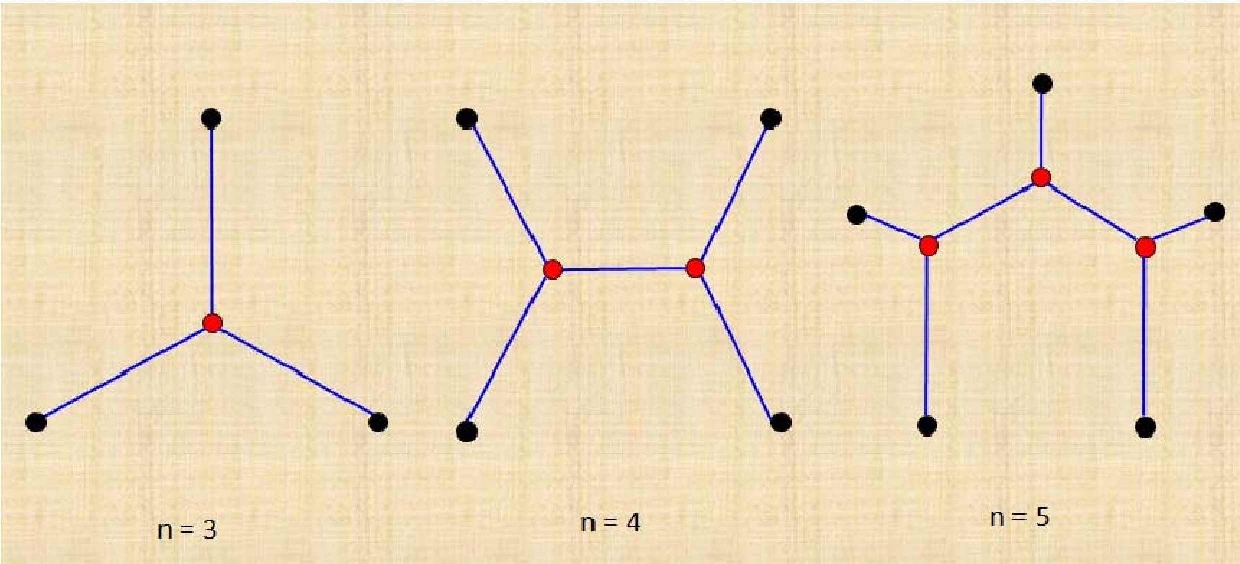


λ-geometry case: the distance function generated by a norn whose unit circle is a regular 2λ-gon centered at the origin.



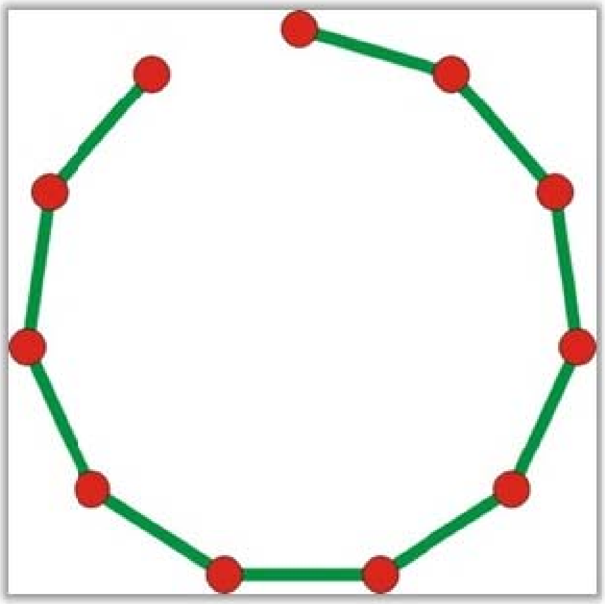
Regular case : M consists of vertices of a regular n-gon in the Euclidean plane.

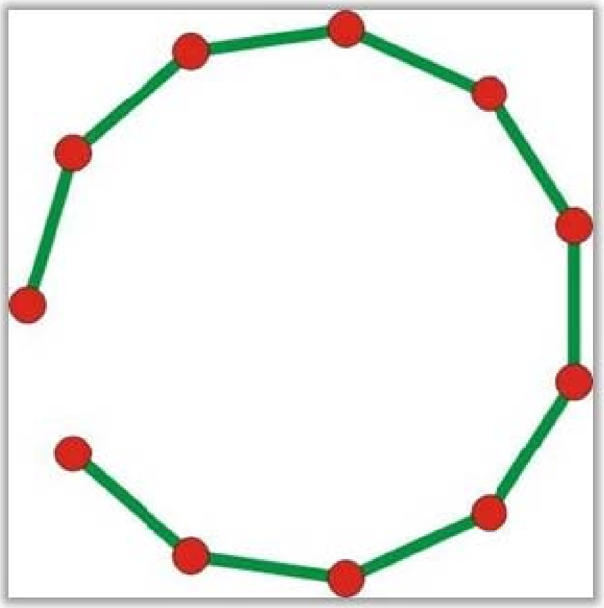
- Shortest trees for $n = 3, 4, 5$:



- Steiner minimal trees joining the vertices of a regular n - gon for $n \geq 6$:

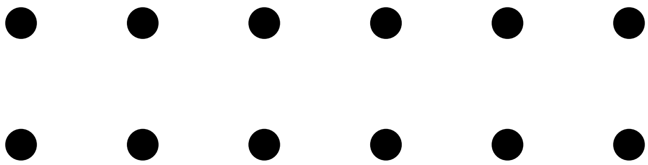
Theorem (**Jarnik, Kössler, Du, Hwang**). Given $n \geq 6$, each shortest tree joining vertices of a regular n - gon consists of all sides of the n - gon, except any one.



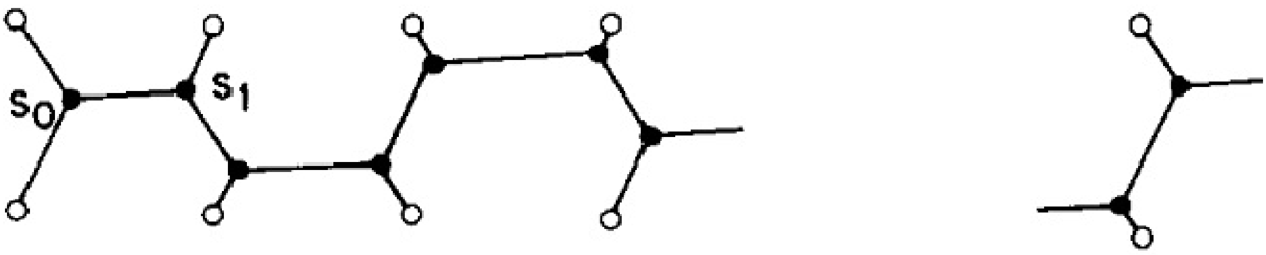


Case of ladders : M consists of vertices of 2n vertices in the Euclidean plane with coordinates {1,0}, {2,0}, ..., {n,0}, {1,1}, {2,1}, ..., {n,1}.

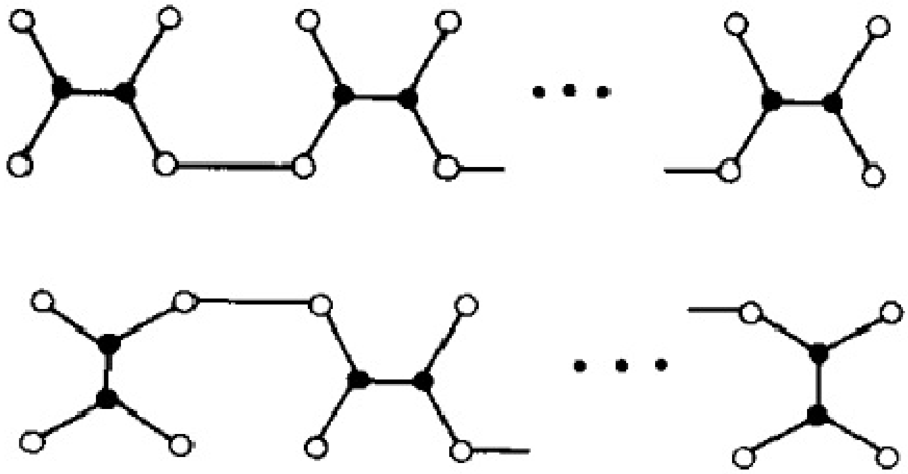
```
Module[{n = 6},
 Graphics[{PointSize[0.05], Flatten[Table[Point[{i, j}], {i, 1, n}, {j, 0, 1}]]}]
]
```



Theorem (**Chang, Graham**). For an odd n in looks like



and for even n like



Local Structure of the shortest trees.

The case of in Euclidean space.

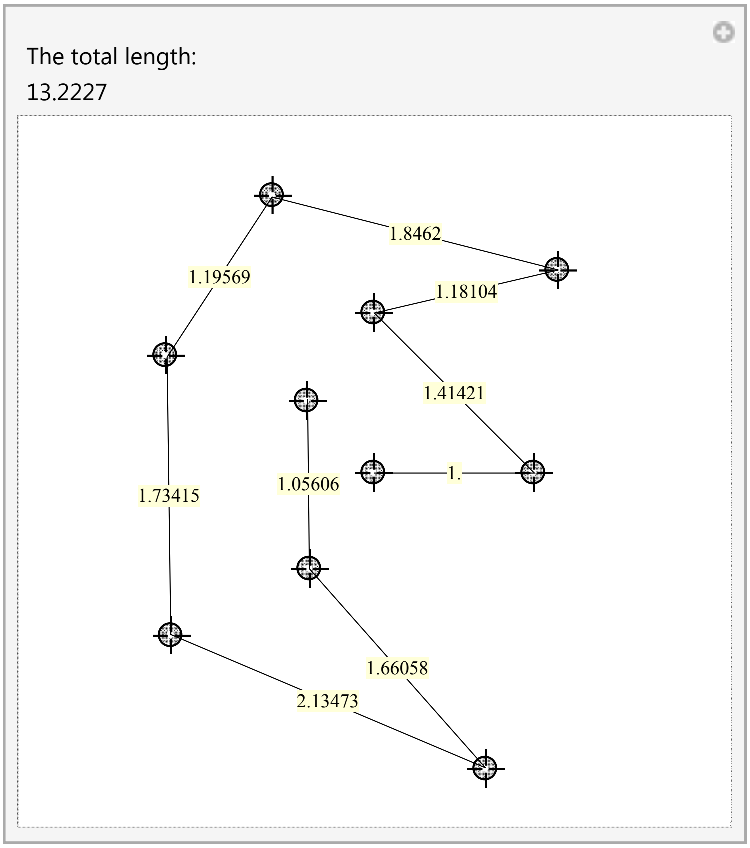
Theorem.
(1) Each shortest tree in Euclidean consists of straight segments meeting by the angles of at least 120°. In particular, the degree of any vertex of such a tree does not exceed 3.
(2) All degree 1 vertices belong to the boundary.
(3) If a vertex of degree 2 does not belong to the boundary, then the angle between two edges incident to it equals 180°.

Remark.
Vertices which do not belong to the boundary are called *Steiner points* or *movable vertices*.
Movable vertices of degree 2 can be as removed from, so as added to a shortest tree, without violating the minimality property of the tree. Such vertices are called *false*, **and one usually assumes that a shortest tree does not contain them**.

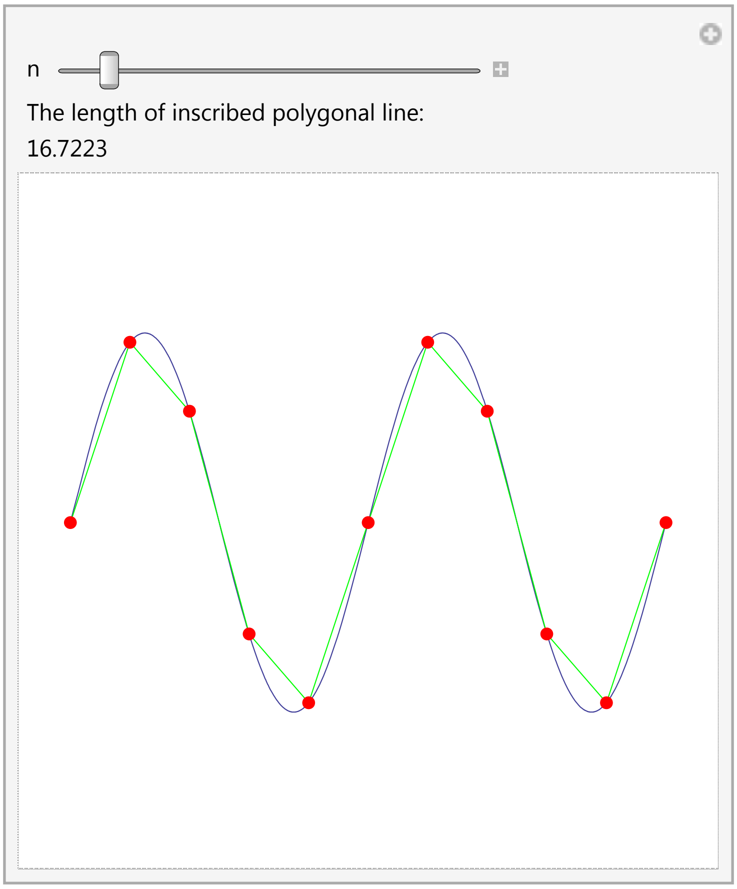
The case of surfaces.

- Shortest curves on surfaces. Geodesics.

For a polygonal line in $\mathbb{R}^n$, its length is the sum of length of its links.



To calculate the length of continuous curve in $\mathbb{R}^n$, we approximate it by polygonal lines and take the supremum of their lengths.



If the curve $\gamma[t] = \{x_1[t], x_2[t], \ldots, x_n[t]\}$, $t \in [a, b]$, is smooth, i.e., its coordinate functions $x_i[t]$ are continuously-differentiable, then one can calculate the length in a simpler way:

1) calculate the velocity $\gamma'[t] = \{x_1'[t], x_2'[t], \ldots, x_n'[t]\}$ of $\gamma[t]$,

2) calculate the length $|\gamma'[t]| = \sqrt{\sum_{i=1}^{n} (x_i'[t])^2}$ of the velocity,

3) integrate the later function: $\int_a^b |\gamma'[t]| \, dt$. The result is the length of γ.

In our case

```
γ[t] := {t, 2 Sin[2 t]};
D[γ[t], t]
√(D[γ[t], t].D[γ[t], t])
N[∫_{-π}^{π} √(D[γ[t], t].D[γ[t], t]) ⅆt]
ClearAll[γ]
```

```
{1, 4 Cos[2 t]}
```

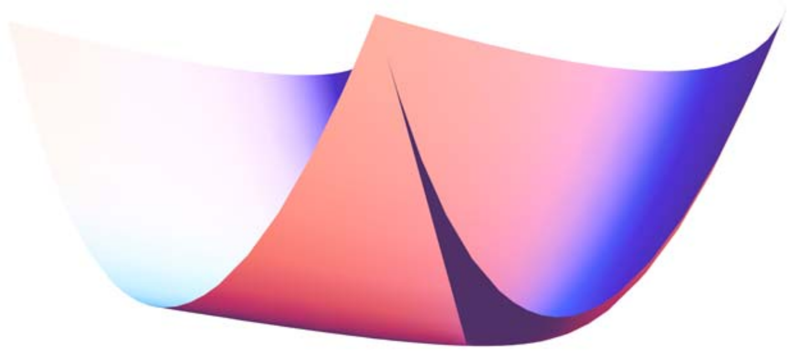
$\sqrt{1 + 16 \operatorname{Cos}[2 t]^2}$

```
17.6286
```

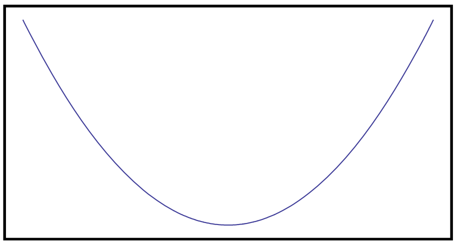
Curves on surfaces

A *k-dimensional surface* M *in* $\mathbb{R}^n$ is a map r: $\Omega \to \mathbb{R}^n$, where Ω is a domain in $\mathbb{R}^k$. If $u_1, \ldots, u_k$ are coordinates in Ω, and $x_1, \ldots, x_n$ are coordinates in $\mathbb{R}^n$, then the surface r is given by n functions $x_1[u_1, \ldots, u_k], \ldots, x_n[u_1, \ldots, u_k]$. We always suppose that the functions are smooth.

```
Module[{r},
 r[u_, v_] := {u + v, u^2 - v^2, u^2 + v^2};
 ParametricPlot3D[r[u, v], {u, -1, 1},
  {v, -1, 1}, Mesh → None, Boxed → False, Axes → None, Ticks → None]
]
```



A *curve* γ *on* M is a map $\gamma : [a, b] \to \Omega$, i.e., the set of k functions $u_1[t], \ \ldots, \ u_m[t]$, where $t \in [a, b]$. The outer representation of the curve γ is the map $r[\gamma[t]]$.



The velocity vector of the outer representation is given by the formula $\sum_{i=1}^{k} r_{u_i} u_i'$, i.e., all velocity vectors of curves passing through P $\in$ M are linear combinations of vectors $r_{u_1}, \ \ldots \ , \ r_{u_k}$.

$r[u,v] = \{u+v,\ u^2 - v^2,\ u^2 + v^2\}$

$r_u[u,v] = \{1,\ 2u,\ 2u\},\ r_v[u,v] = \{1,\ -2v,\ 2v\}$

$\gamma[t] = \{t,\ t^2\}$

$\gamma'[t] = \{1,\ 2t\}$

at $\gamma[t]$ we have

$r[u[t],v[t]] = \{t + t^2,\ t^2 - t^4,\ t^2 + t^4\}$

$r_u[u[t],v[t]] = \{1,\ 2t,\ 2t\},\ r_v[u[t],v[t]] = \{1,\ -2t^2,\ 2t^2\}$

$r[u[t],v[t]]' = \{1 + 2t,\ 2t - 4t^3,\ 2t + 4t^3\}$

$= \gamma'[t].\{r_u[u[t],v[t]],\ r_v[u[t],v[t]]\} = \{1 + 2t,\ 2t - 4t^3,\ 2t + 4t^3\}$

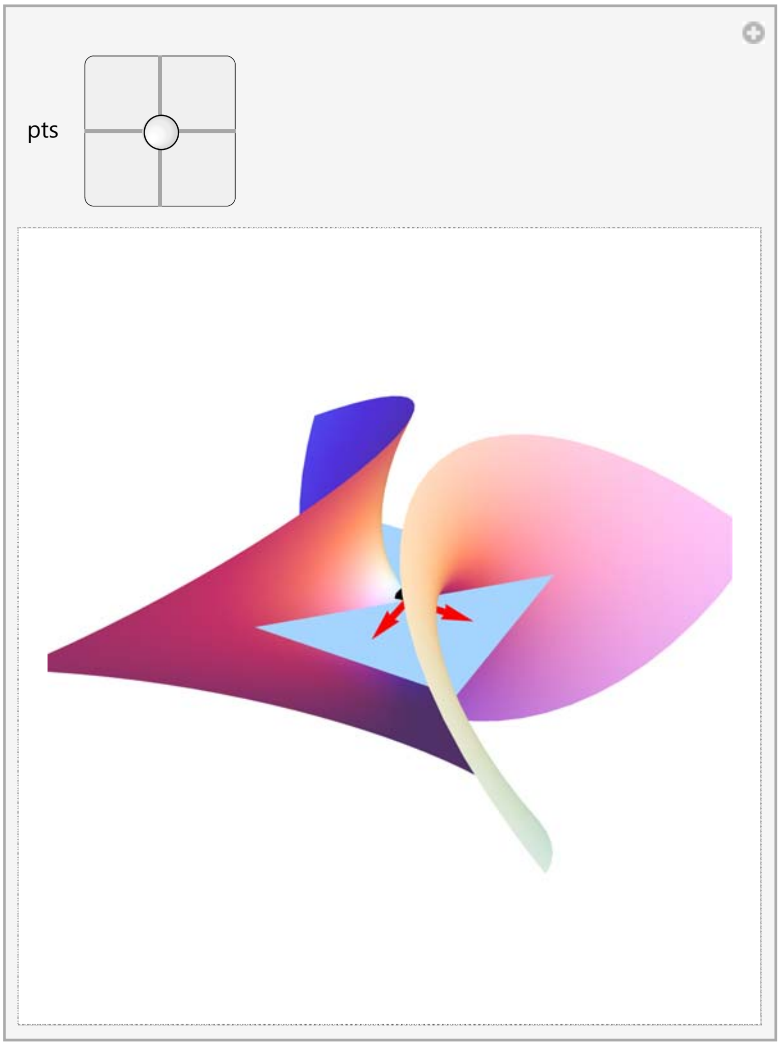
Surfaces r is regular if the vectors r_{u_1}, ... , r_{u_k} are linear independant and each point of Ω, i.e., the rank of the matrix { r_{u_1}, ... , r_{u_k}} equals k.

$r_u[0,0] = \{1,\ 0,\ 0\}$

$r_v[0,0] = \{1,\ 0,\ 0\}$

are linear dependant => M is not regular

We shall consider only regular surfaces. For them { r_{u_1}, ... , r_{u_k}} is called the *cannonical basis* of the *tangent space* T_P *M at P*.

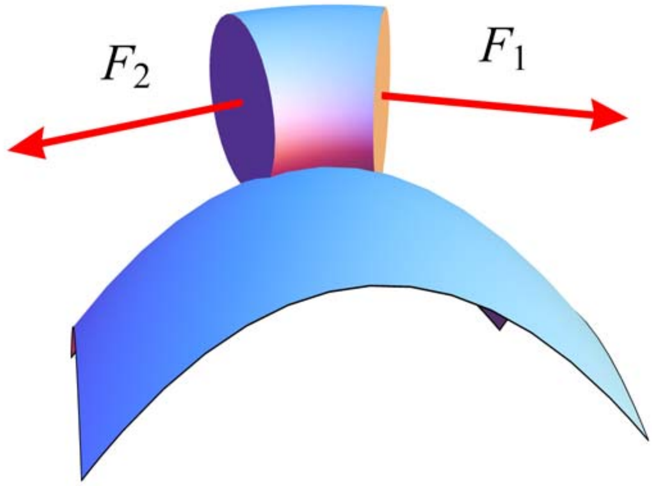


Given two points A and B on a surface M, find a shortest curve on M joining A and B.

Theorem. If γ[t] is a shortest curve on a surface M given by the mapping r[u,v], then the acceleration of the outer representation r[γ[t]] of the curve is perpendicular to the surface M. In particular, the length of the velocity vector of r[γ[t]] is constant.

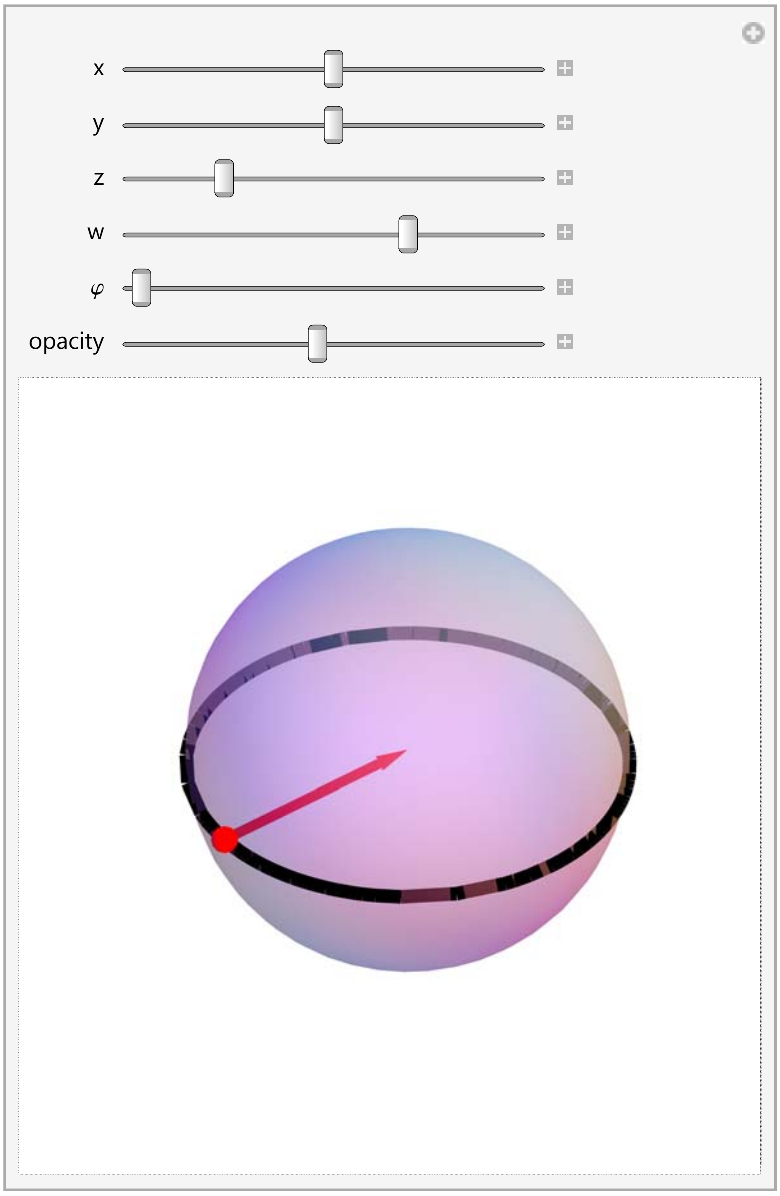
Proof. Suppose that we move along the shortest curve with constant speed and represent this curve by an elastic. Since γ[t] is shortest, it can not shrink itself. Thus the elastic force acting at each sufficiently small piece of the elastic γ[t] is perpendicular to M. This is equivalent to that the acceleration of the small piece is perpendicular to M and the forces distributed uniformly, i.e., they are proportional to velocity with some coefficient k, thus

$F_1 = - k\,\gamma'[t]$, $F_2 = k\,\gamma'[t + \Delta t]$. The mass m of the piece is $\sim \rho\,\Delta t$, thus the acceleration of the piece is $(F_1 + F_2 + N)/m =$ $\frac{k}{\rho} \frac{\gamma'[t+\Delta t] - \gamma'[t]}{\Delta t} + \frac{N}{m} \sim \gamma''[t] + \frac{N}{m}$, where N is orthogonal to M. Thus, $\gamma''[t]$ is orthogonal M as well.



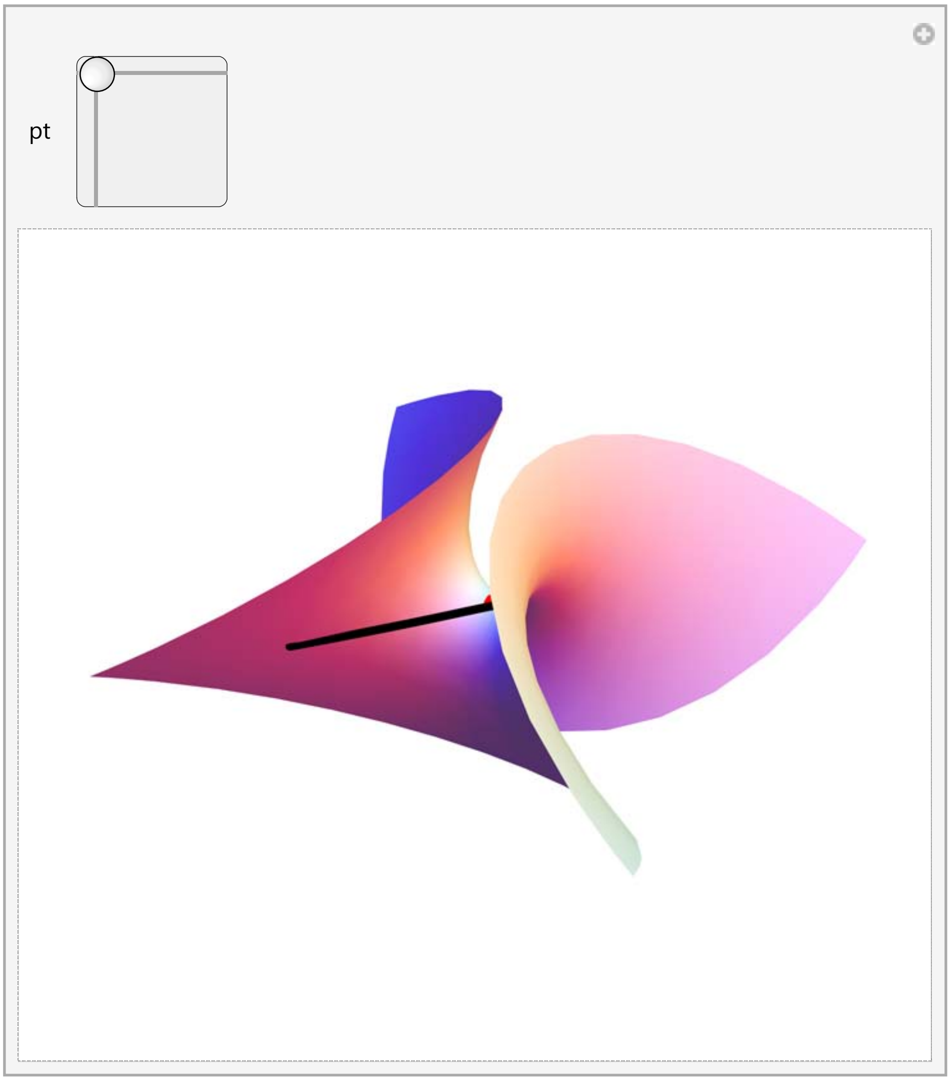
Definition. A curve on a surface whose outer representation acceleration perpendicular to the surface, is called *geodesics*.

Example. Big circles on a sphere parameterized proportionally to the angle are geodesics.

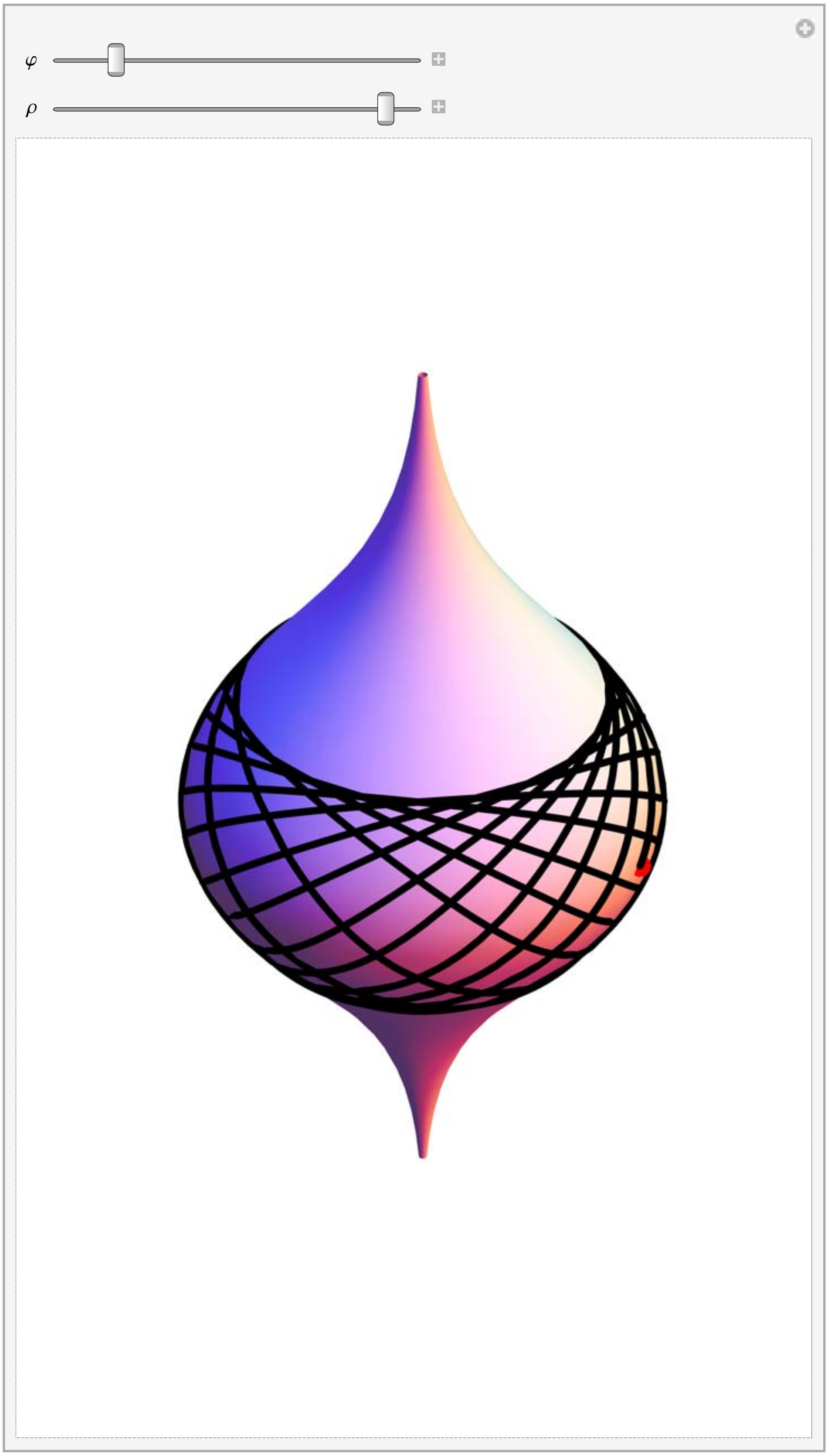


To find a geodesic, one needs to solve the corresponding system of differential equations $\{r_u[u[t], v[t]].r[u[t], v[t]]'' = 0,\ r_v[u[t], v[t]].r[u[t], v[t]]'' = 0\}$. The solution depends on initial conditions. To join points A and B, we use the conditions $\gamma[-1]$=A, $\gamma[1]$ = B.

Another kind of initial conditions is $\gamma[-1] = A$, $\gamma'[-1] = v$.



Theorem (Clairaut). Let M be a revolution surface, $\gamma[t]$ be a geodesic on M. Denote by $\rho[t]$ the distance from $\gamma[t]$ to the rotation axis, and by $\varphi[t]$ the angle between γ and parallel at $\gamma[t]$. Then $\rho[t]$ Cos[$\varphi[t]$] is constant.



Exercise. Prove Clairaut theorem.

The local structure of shortest networks on surfaces.

Theorem.
(1) Each shortest tree on a surface in Euclidean space consists of shortest geodesics meeting by the angles of at least 120°. In particular, the degree of any vertex of such a tree does not exceed 3.
(2) All degree 1 vertices belong to the boundary.
(3) If a vertex of degree 2 does not belong to the boundary, then the angle between two edges incident to it equals 180°.

Remark. As above, we **usually assume that a shortest tree on a surface does not contain false vertices (movable vertices of degree 2)**.

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Local minimal networks

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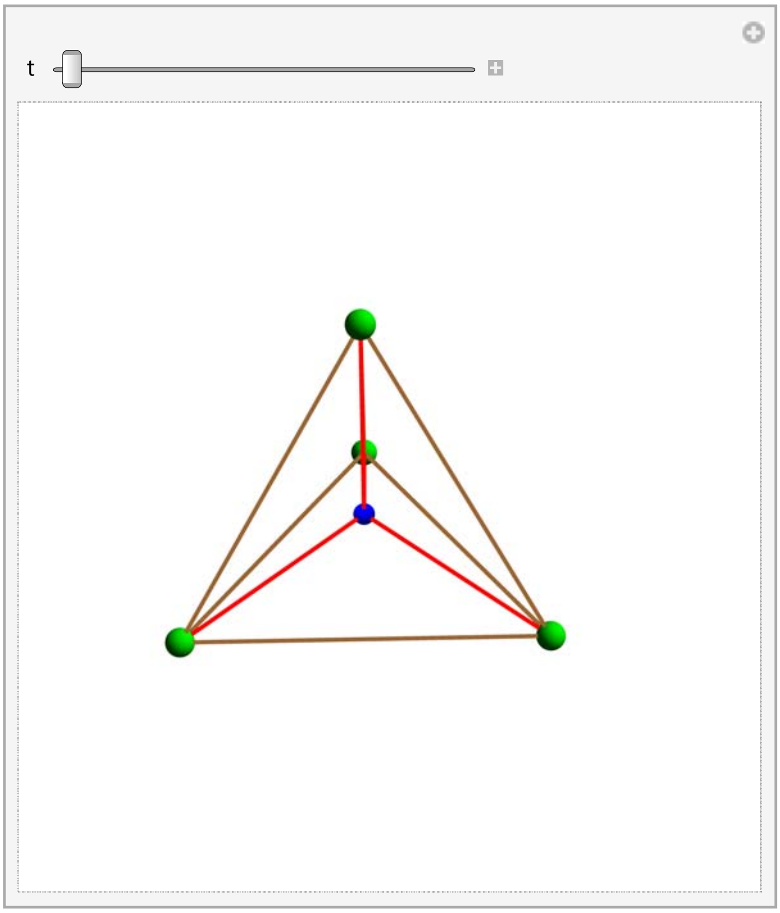
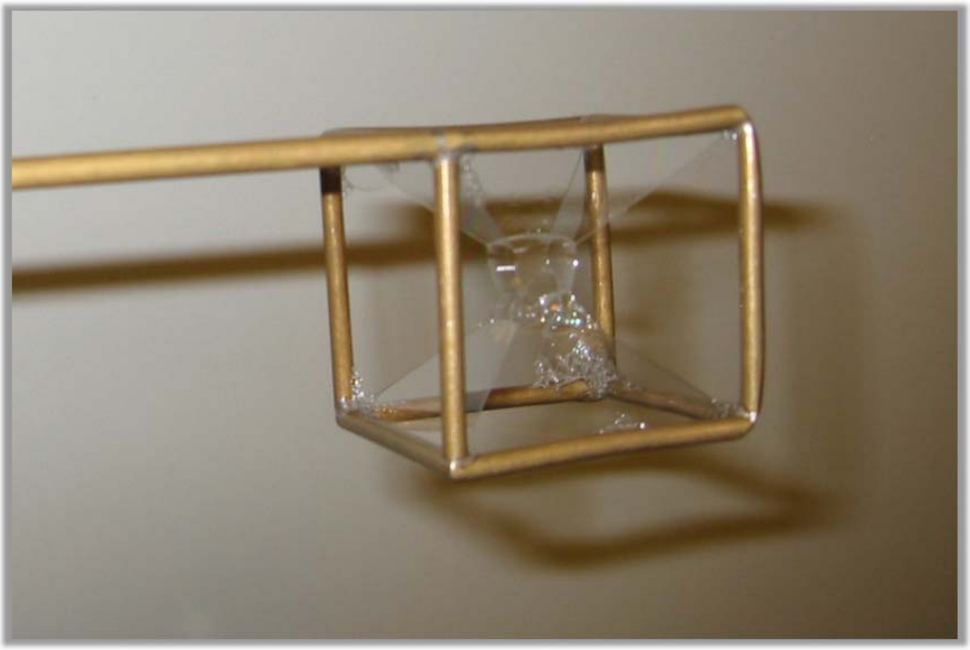
Definitions and some examples

A connected graph in Euclidean space or on a surface in Euclidean space is called *local minimal network* if its local structure is the same as the one of Steiner minimal trees described in the previous theorems.
(Indeed, sufficiently small fragments of a local minimal network are Steiner minimal trees).
Remark. According to agreement on the false vertices, the boundary of local minimal network contains all its vertices of degree 1 and 2.

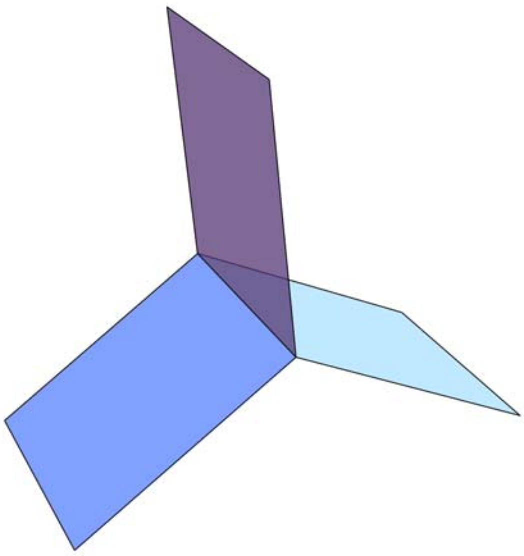
Example 1. Tetrahedron.

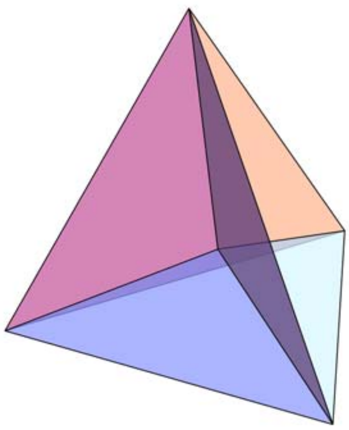


Example 2. Closed (without boundary) local minimal networks on standard sphere and soap films singularities.

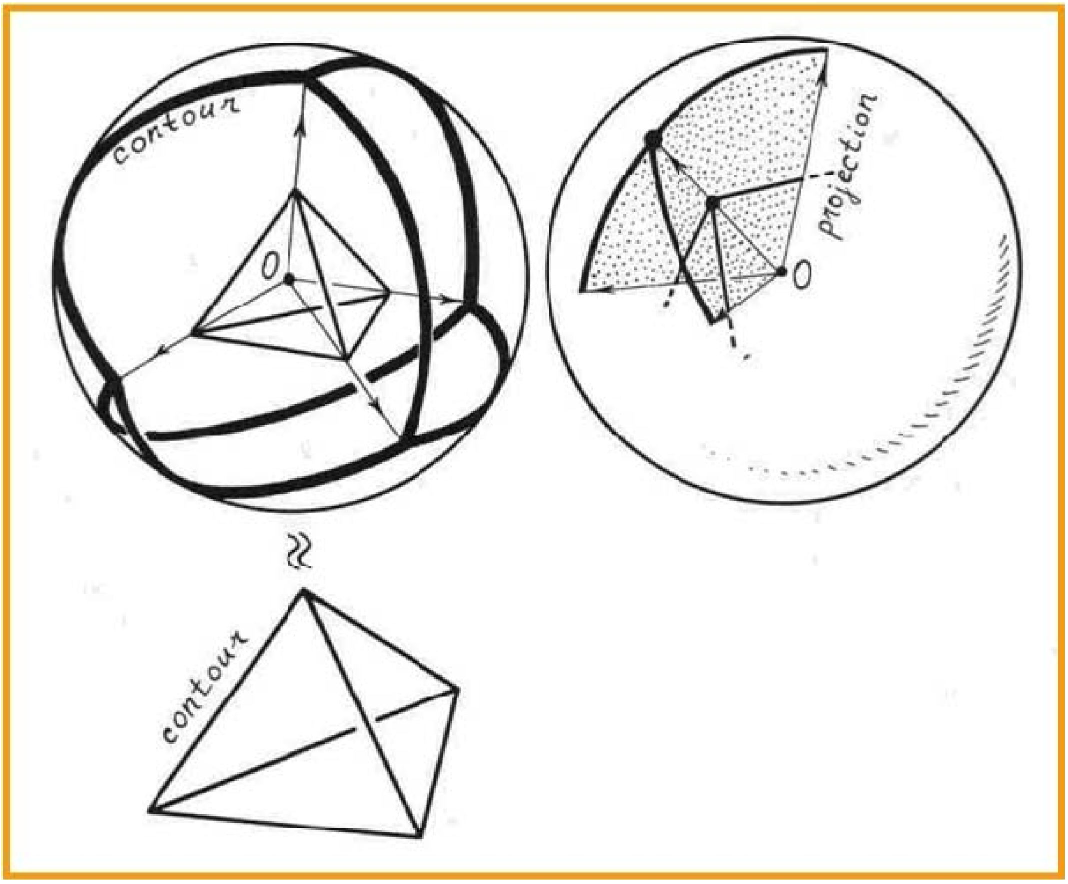


Plateau (1801-1883) formulated four principles, which describe possible singularities on soap films (stable minimal surfaces).



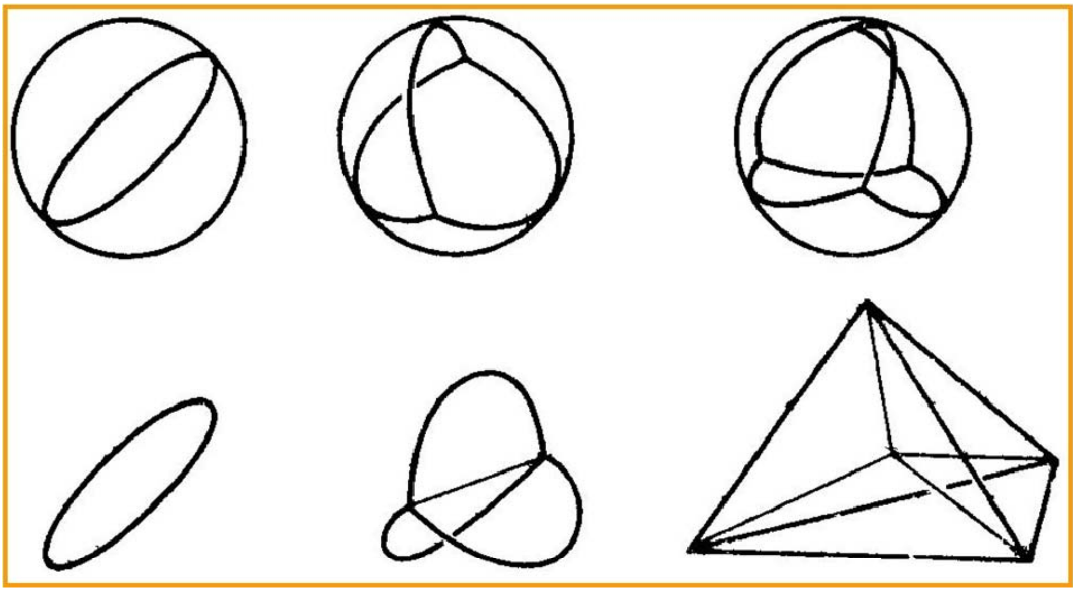


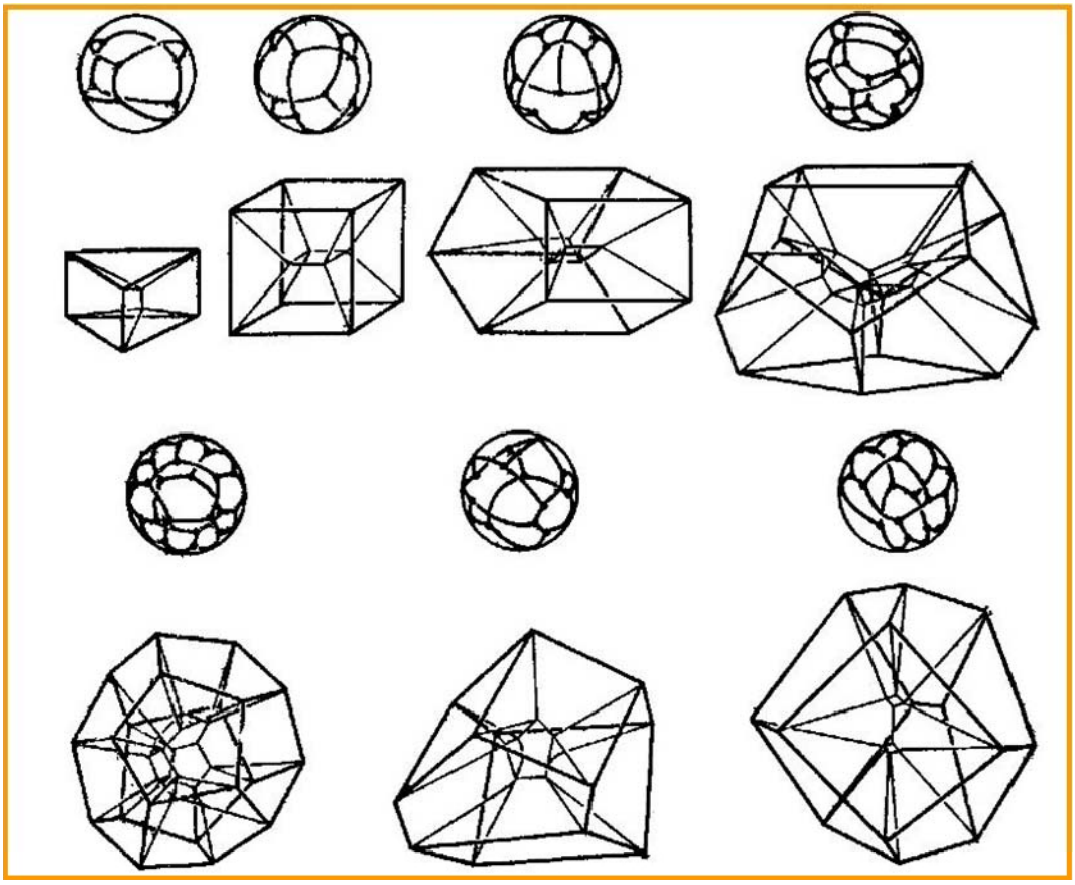
How can one prove that?



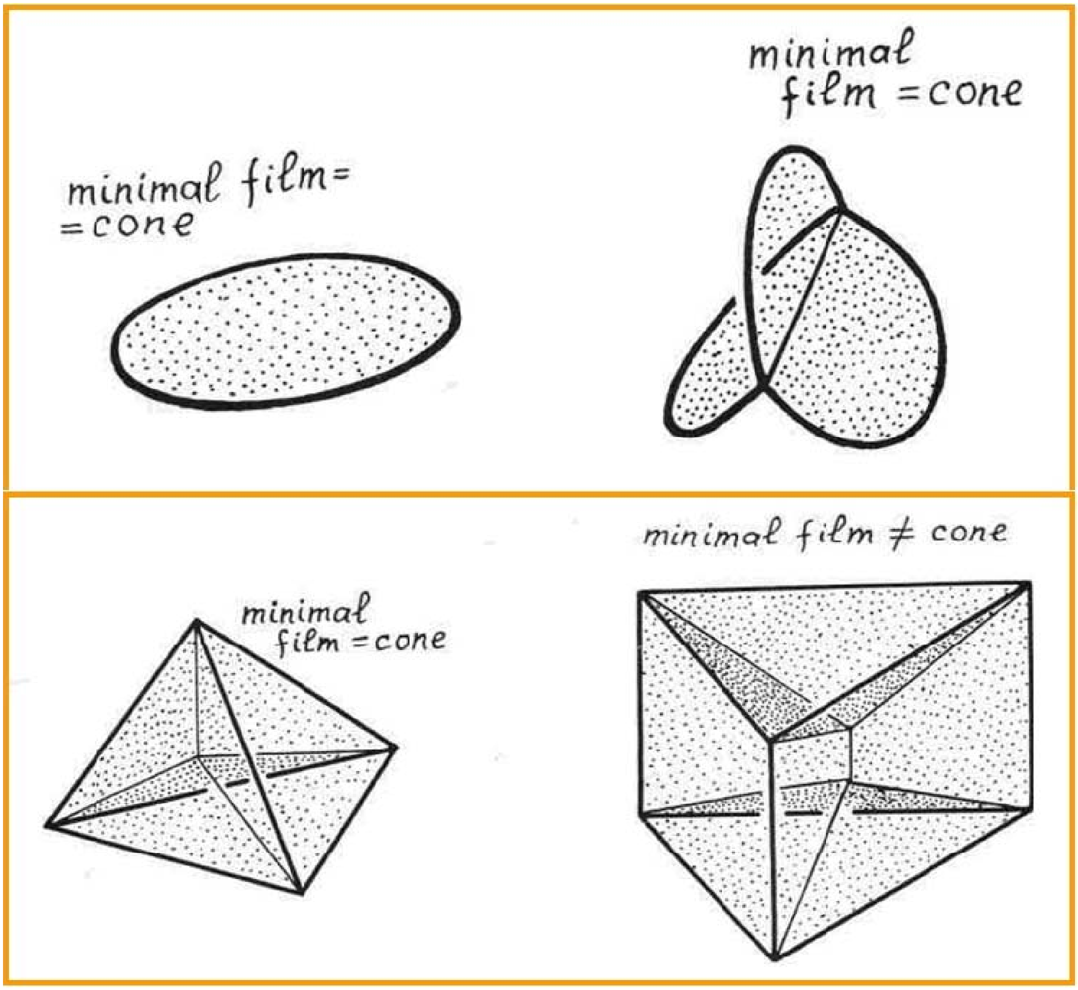
The limiting network minimizes the length locally (each its sufficiently small part is shortest). Thus such networks are local minimal.

Ten possible local shortest networks on standard sphere and corresponding soap films (A.Heppes, 1964)

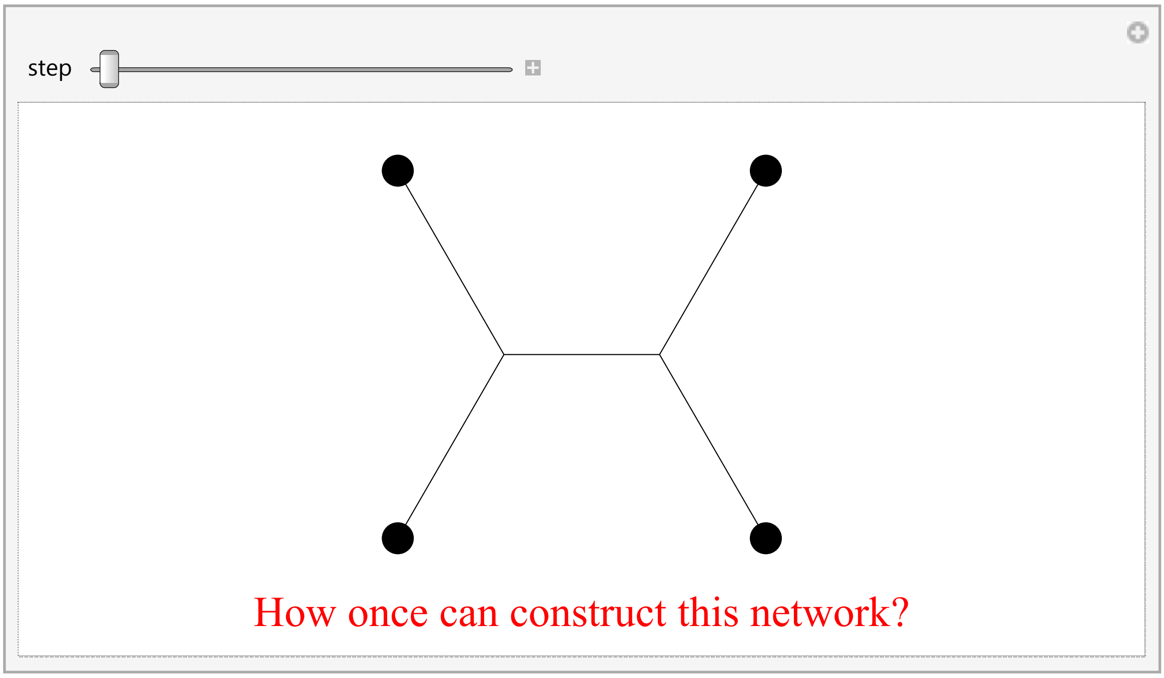




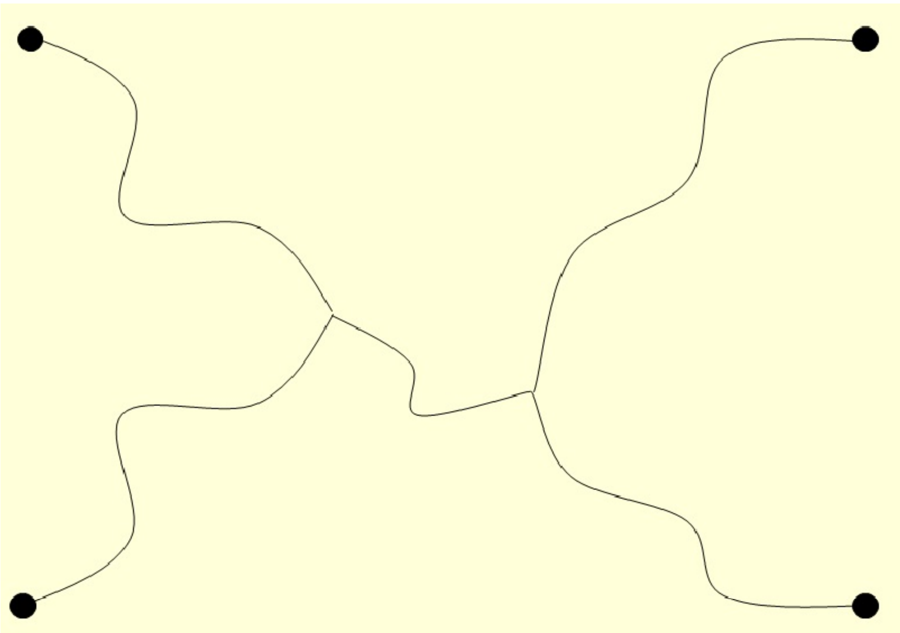
Only three of the films are cones.


minimal film=
=cone
minimal
film =cone
minimal
film =cone
minimal film ≠ cone

- **How one can construct a local minimal binary tree of a given combinatorial structure in the plane? Melzak algorithm.**



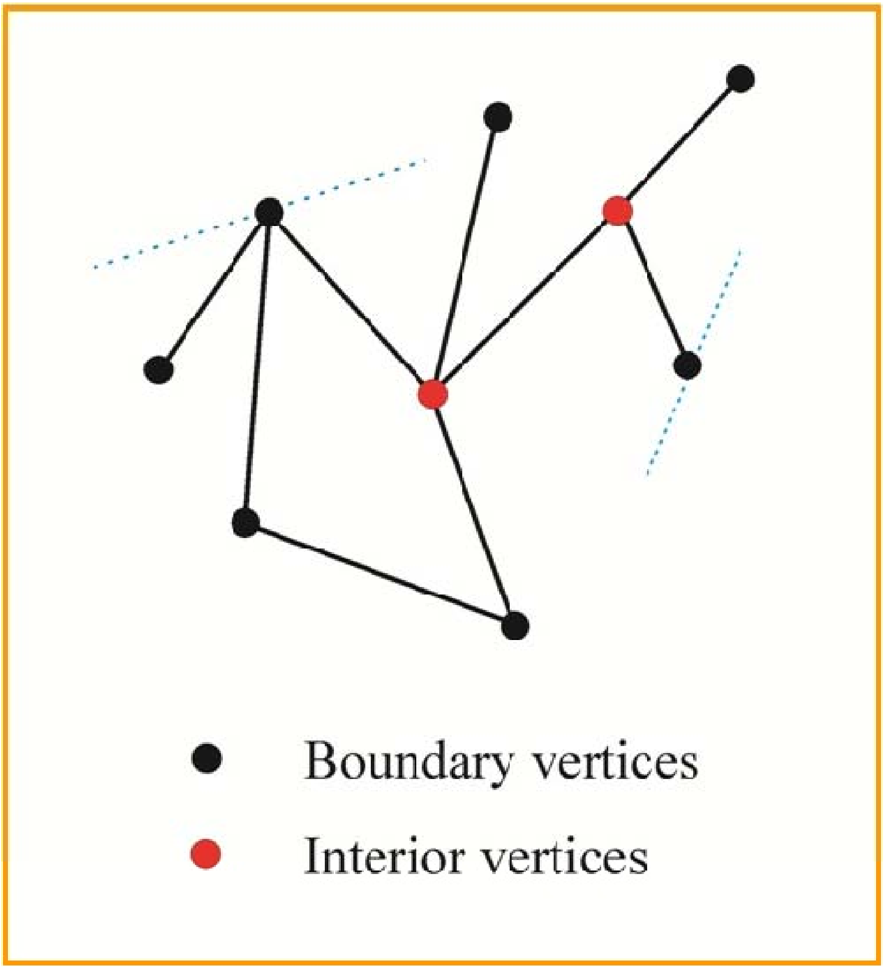
These ideas lead to general algorithm of constriction of the local minimal binary tree with given combinatorial structure joining a given set of points in the plane.



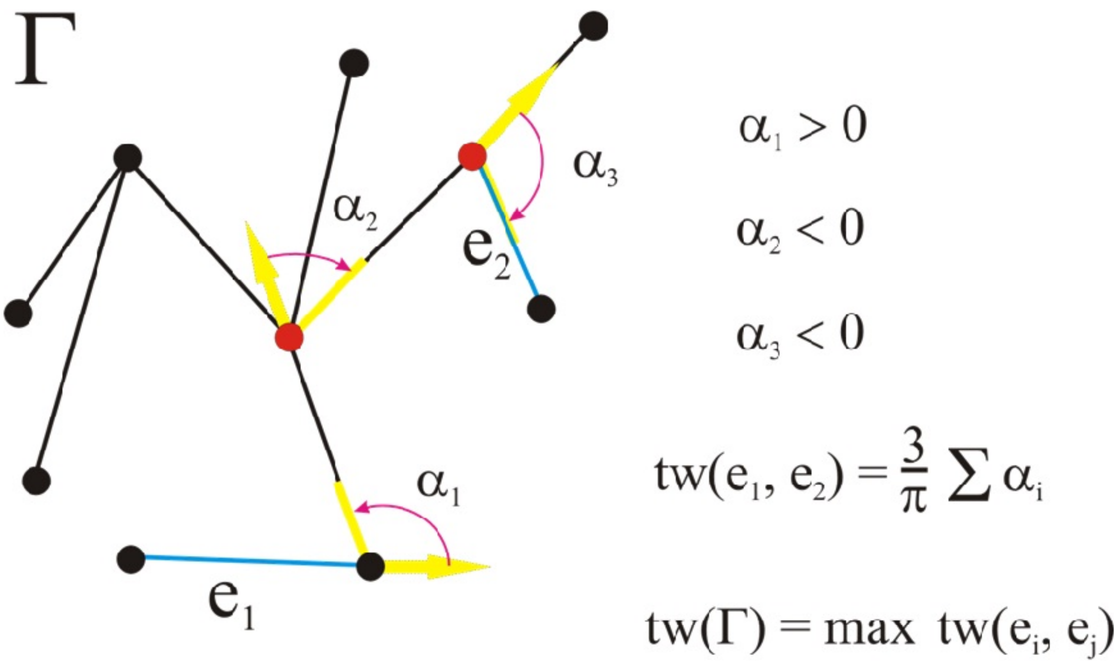
However, this construction is not always possible (for example, it is impossible for triangle with an angle of more than 120°).

Onion peelings and local minimal binary trees.

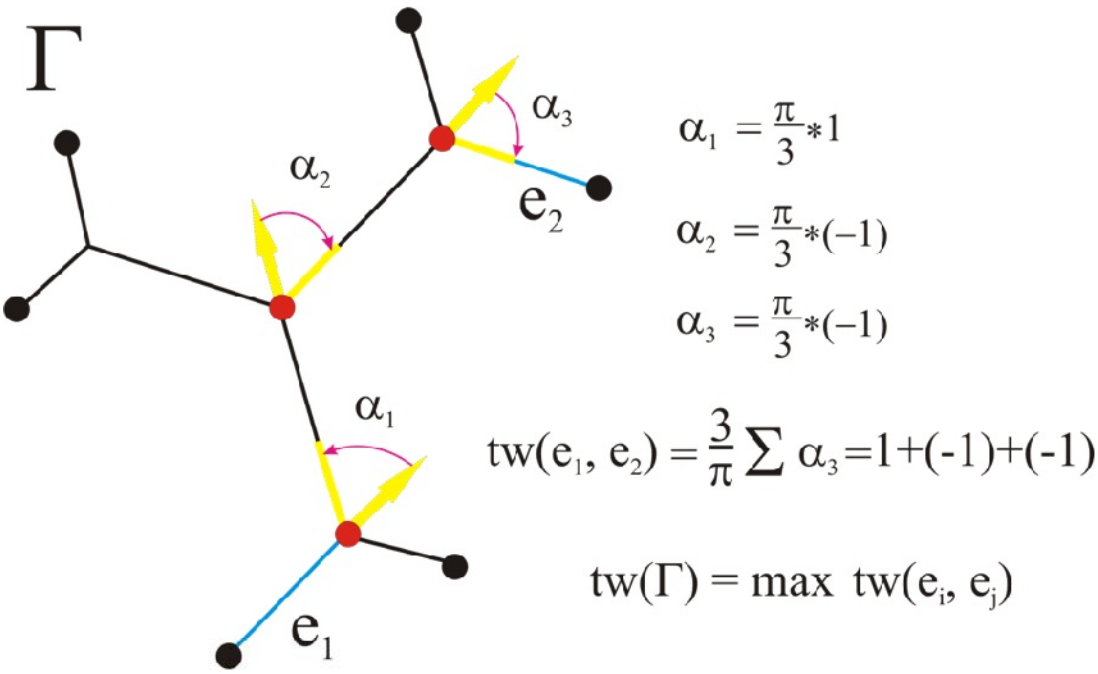
- The boundary of a plane linear tree



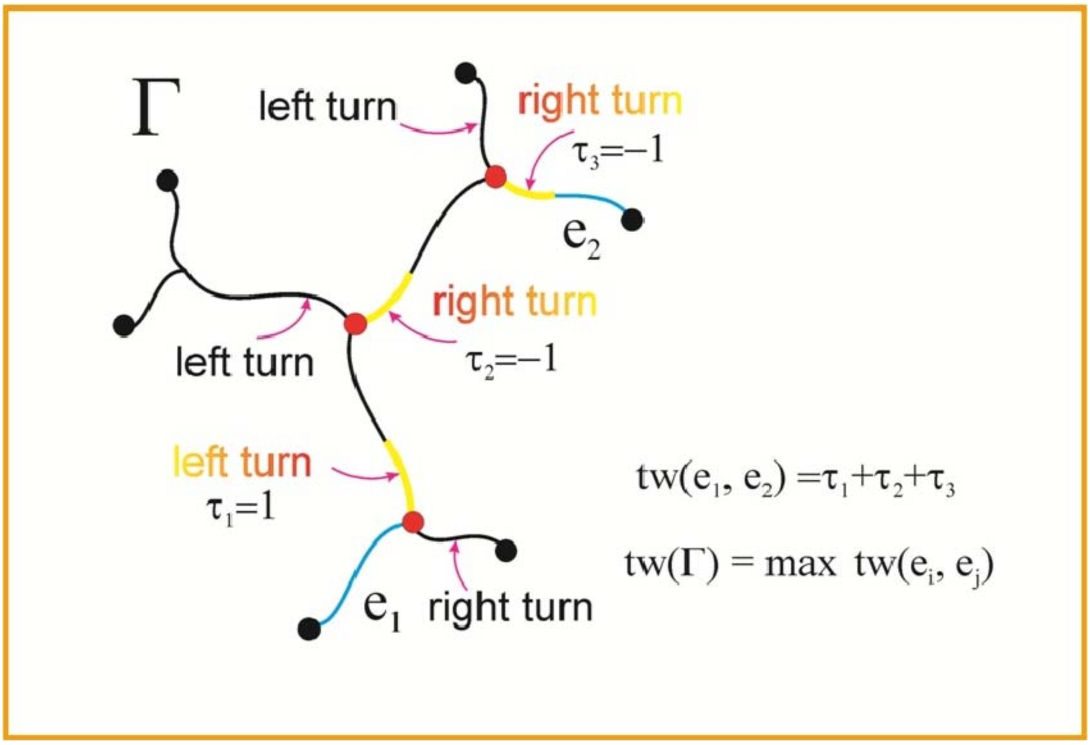
- The twisting number of plane linear trees



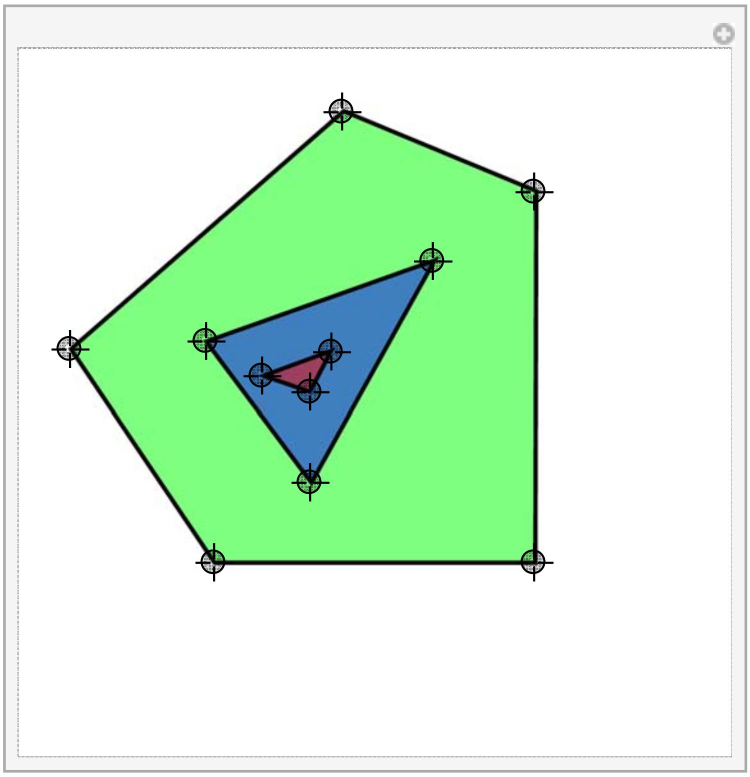
- The twisting number of a local minimal binary tree



- The twisting number of a plane binary tree



Recall that the number of convexity levels of a set $M \subset \mathbb{R}^n$ is called the height of M.



Theorem (Ivanov, Tuzhilin). Let Γ be a linear plane tree and n the height of its boundary.Then tw (Γ) ≤ 12 (n - 1) + 6.

Corollary. Let Γ be a local minimal plane binary tree and n the height of its boundary. Then tw(Γ) ≤ 12(n - 1) + 5.

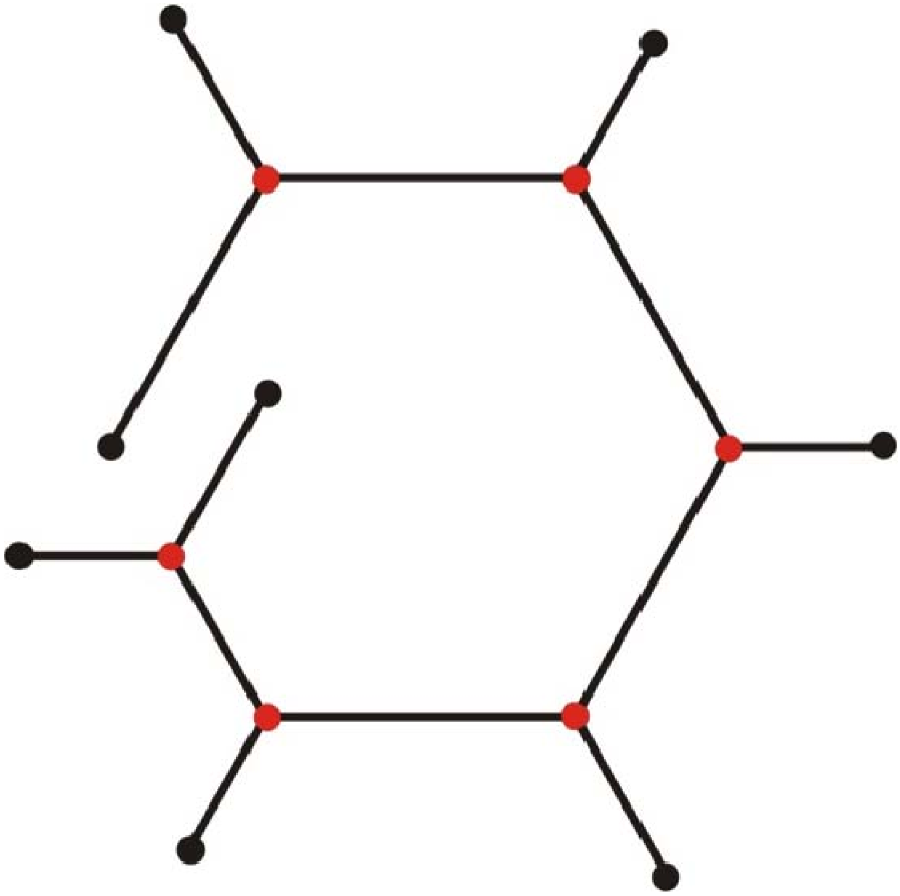
The boundary set consisting of just one convexity level is called *convex*.

We say that a plane tree Γ *can be realized as a local minimal tree* if there exists a local minimal tree Γ ' such that Γ ' and Γ are planar equaivalent.

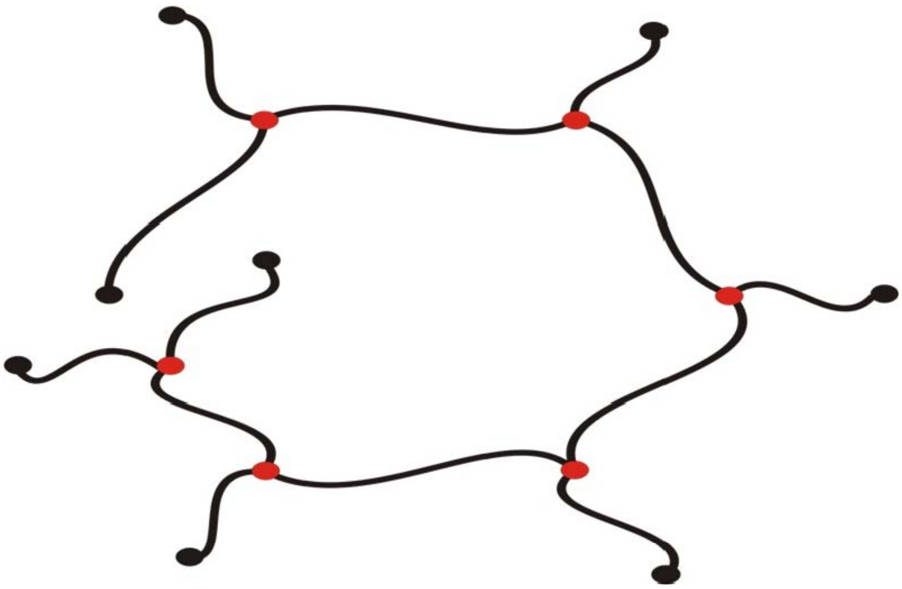
Theorem (Ivanov, Tuzhilin). Let Γ be a plane binary tree.Then Γ can be realized as a local minimal binary tree with a convex boundary if and only if
tw (Γ) ≤ 5.

- **Illustration**

It is impossible to deform this local minimal tree by changing its edges lengths to obtain the one with a convex boundary and without self-intersections.



There does not exist a local minimal binary tree with a convex boundary, such that it is planar equivalent to the tree depicted below.



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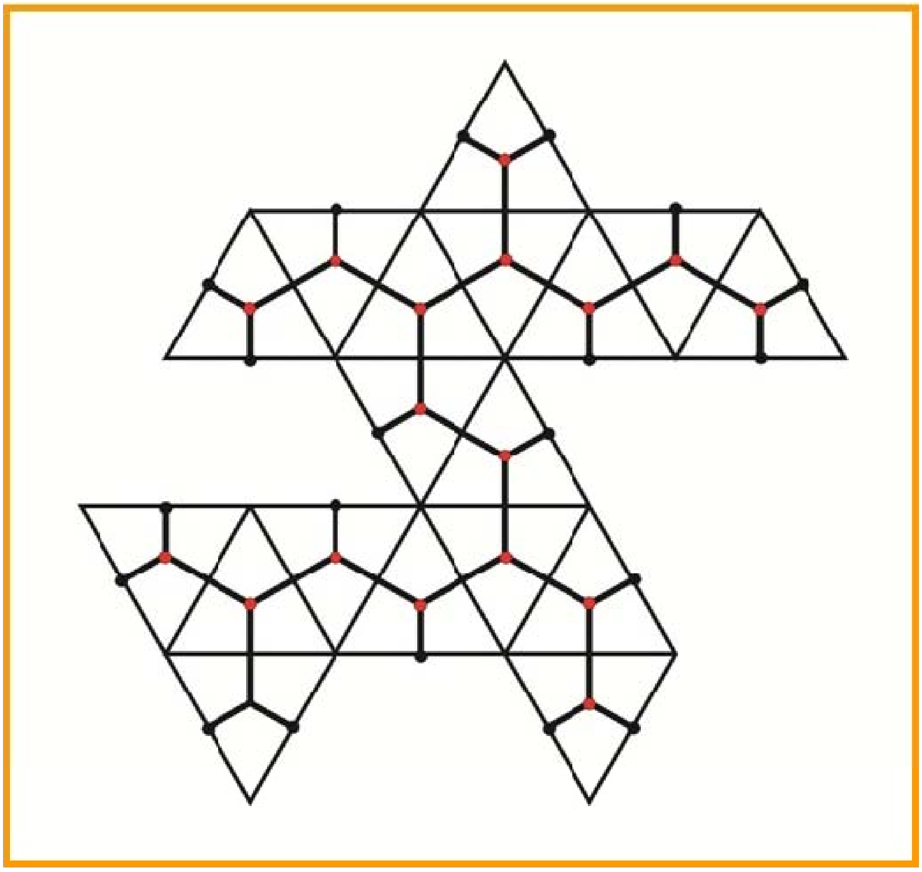
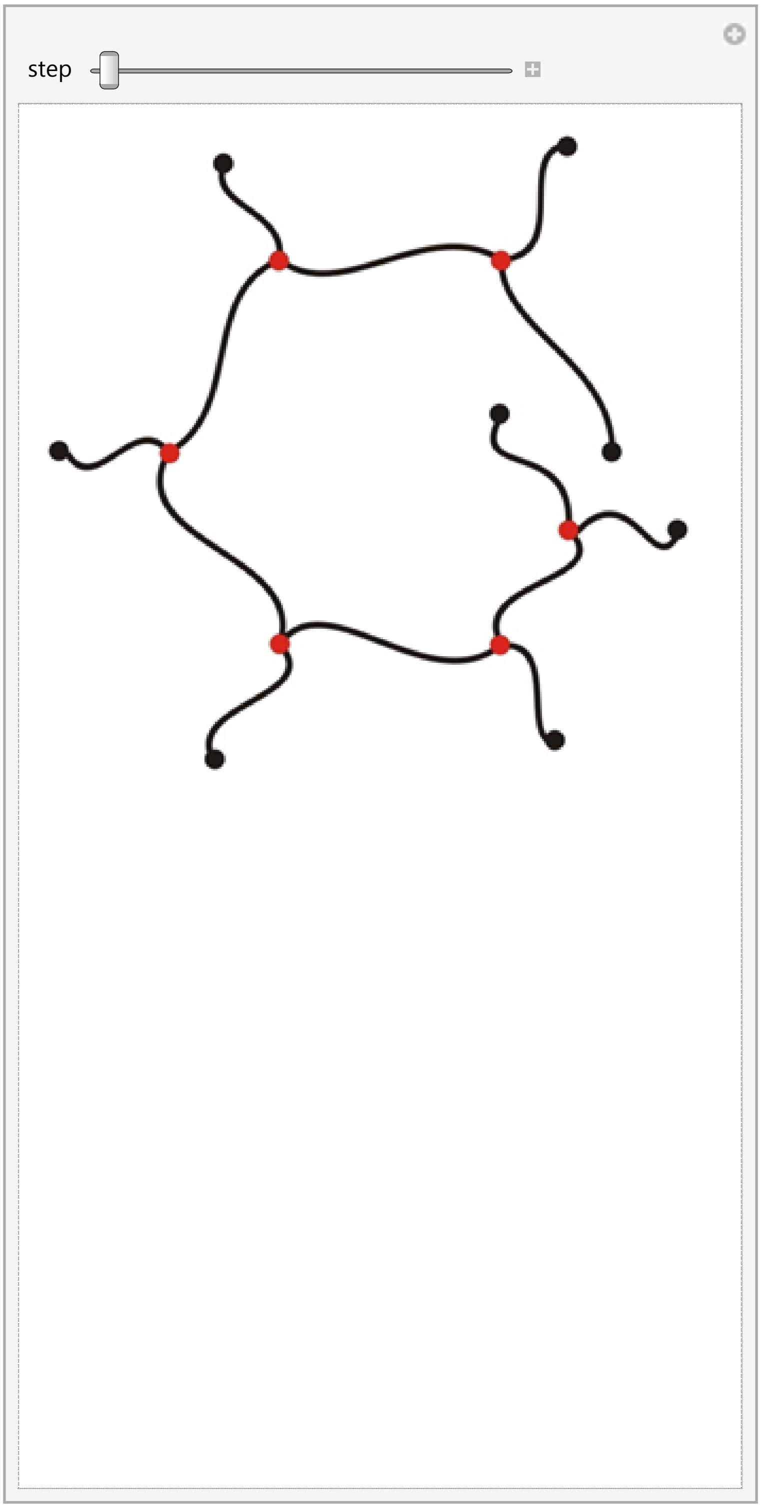
Local minimal trees joining the vertices of convex polygons.

- Dual language of plane binary trees - the language of *tilings*



Which binary trees are dual graphs of tilings?

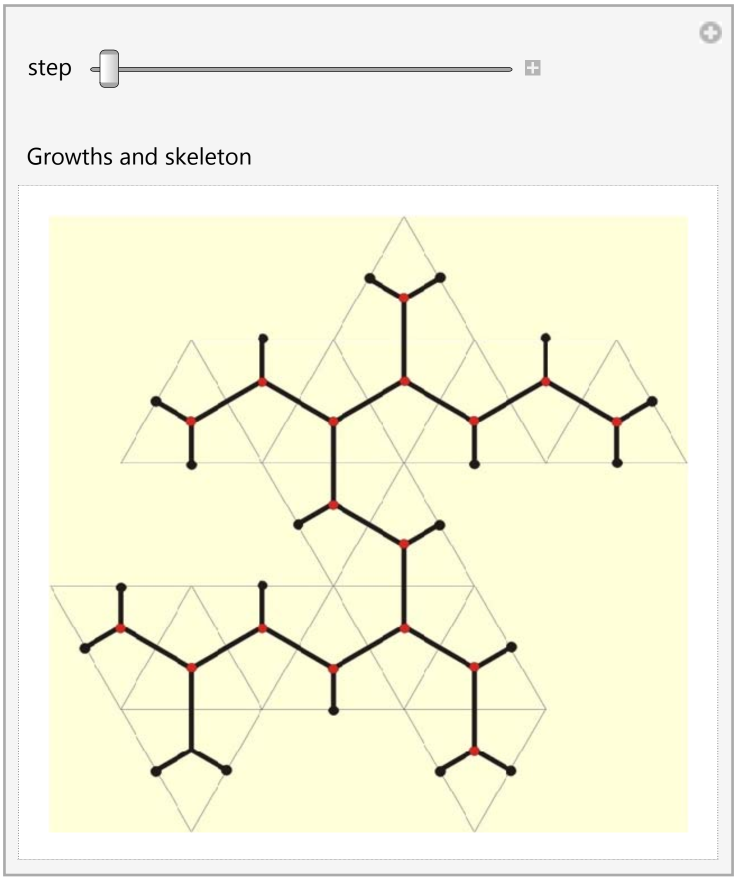


Tiling realization

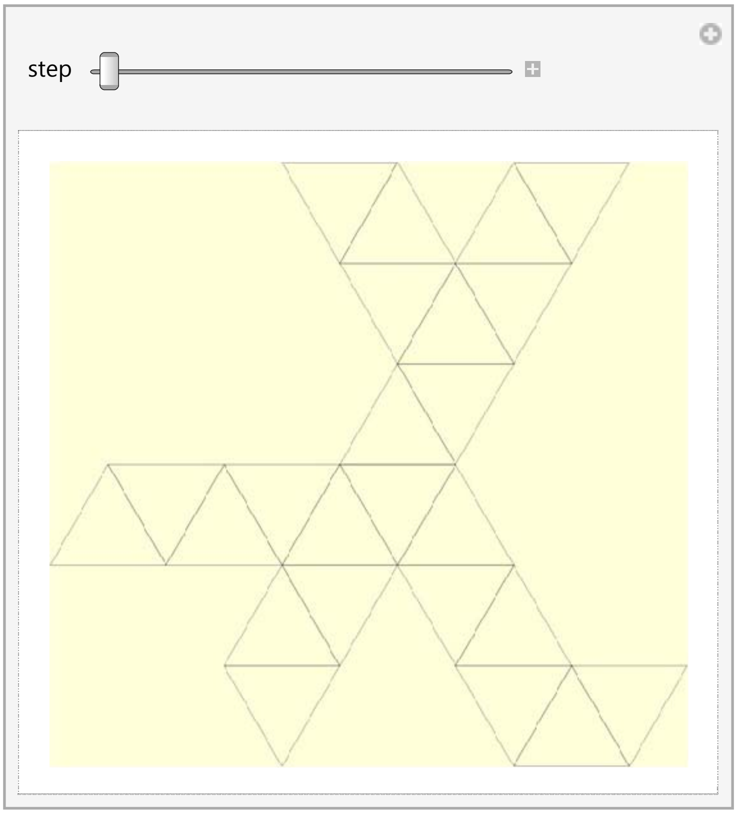
Theorem (Ivanov, Tuzhilin). If the twisting number of a plane binary tree does not exceed 5, then it can always be realized as the dual graph of a tiling.

- Deconpositions

Decomposition of a tiling into skeleton and growths



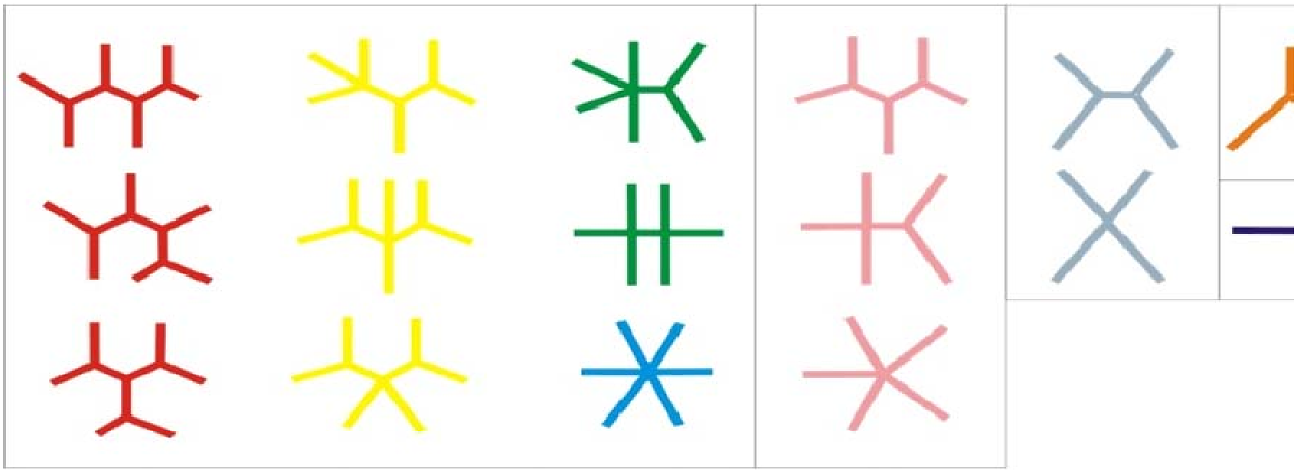
Decomposition of a skeleton into linear parts and branching cells. The code of a skeleton.



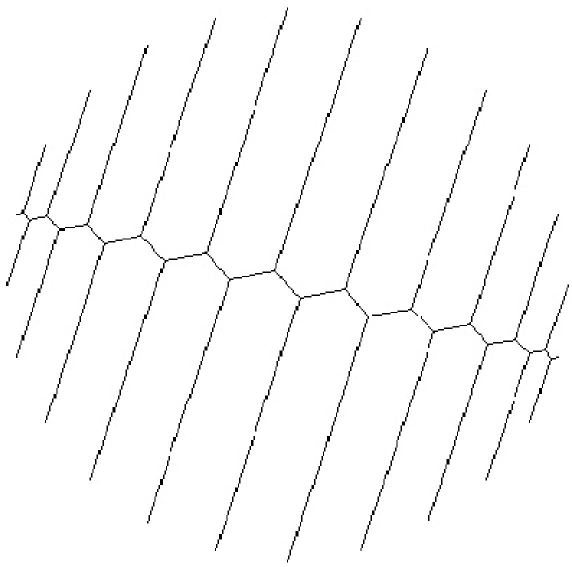
- Classification of skeletons codes

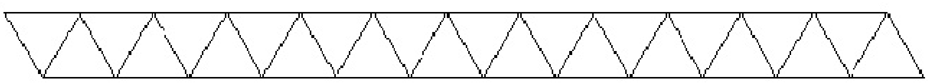
Theorem (Ivanov, Tuzhilin). Consider all skeletons whose dual graphs twisting numbers are at most 5 and for each of these skeletons construct its code. Then, up to planar equivalence, we obtain all plane graphs with at most 6 vertices of degree 1 and without vertices of degree 2. In particular, every such skeleton contains at most 4 branching points and at most 9 linear parts.

All possible codes of the skeletons

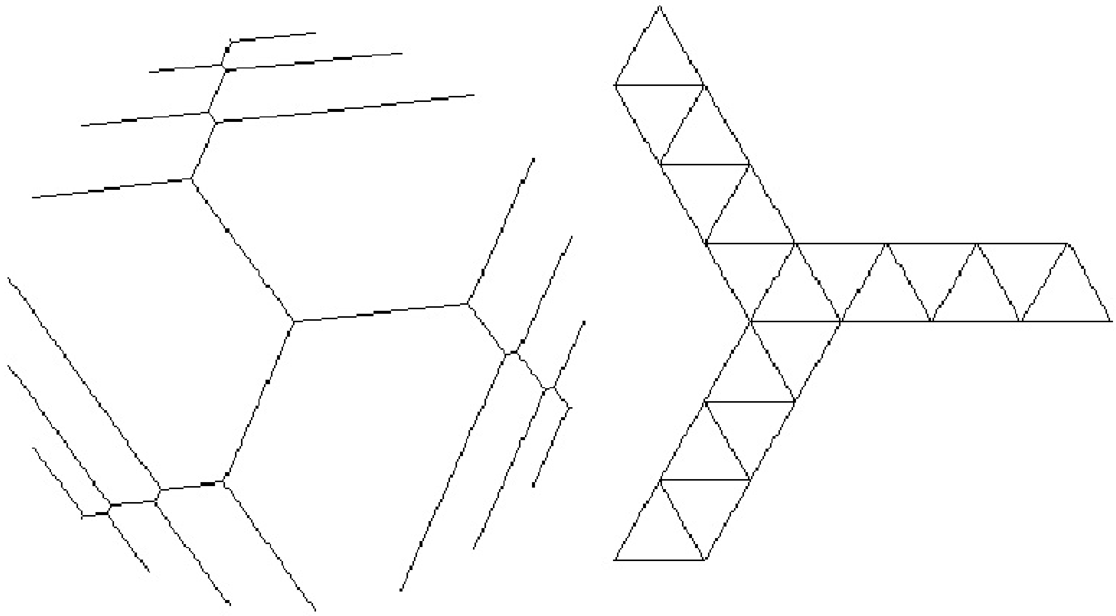


- **Application: complete classification of local minimal binary trees of skeleton type joining the vertices of regular n-gons**
  - The tree of the *snake type* exists for any n.

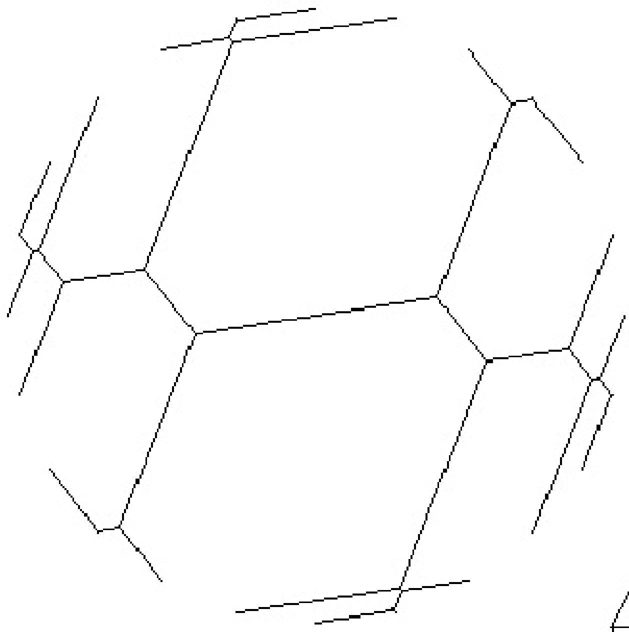


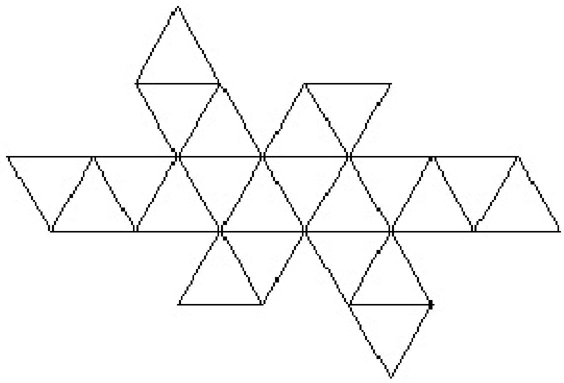


  - The tree of the *T-joint typ*e exists just for n = 6 k + 3

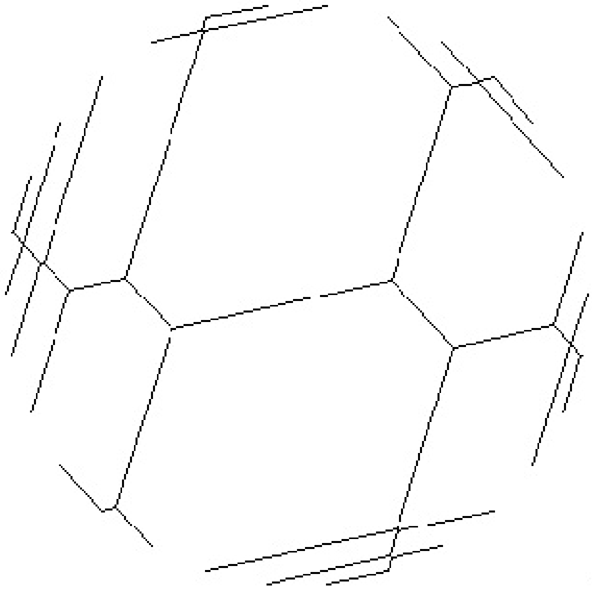


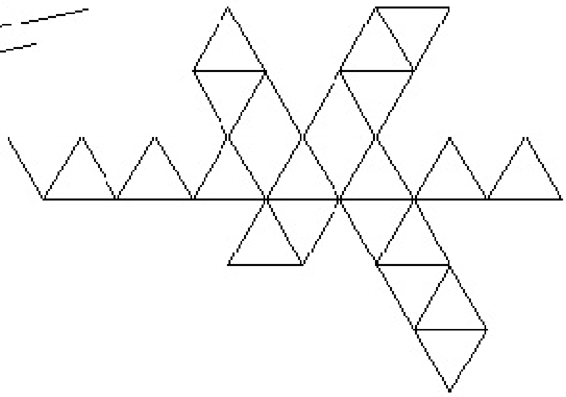
- The tree of the *6-fold type* exists just for four values of n: 24, 30, 36, 42.
  - n = 24



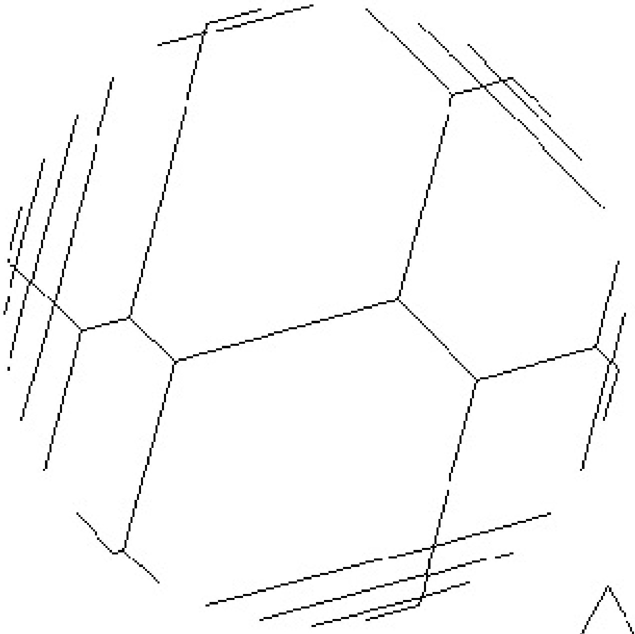


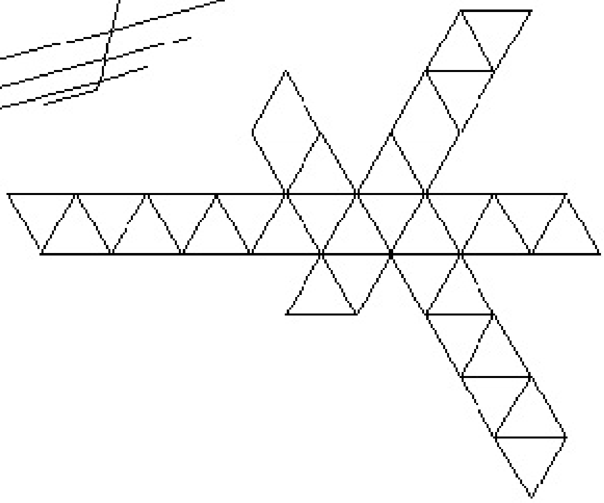
- n = 30



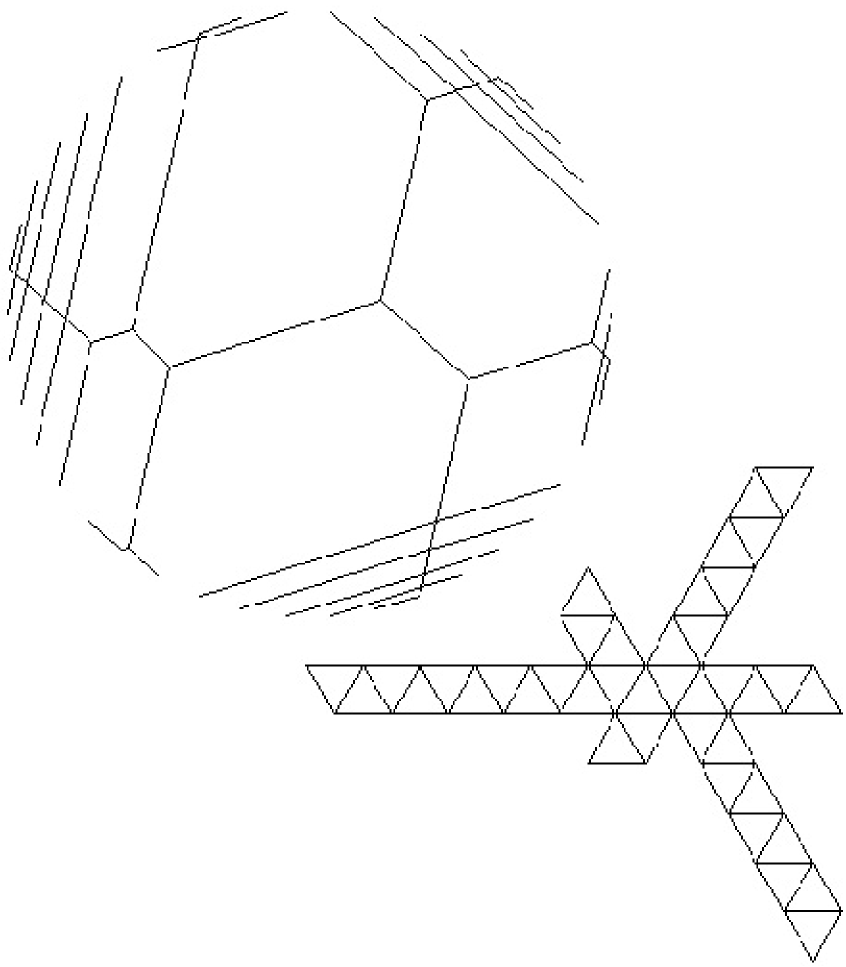


- n = 36





- n = 42



Remark. The general classification is not completed. A few examples.