Steiner Ratio

Steiner Ratio definition and some examples

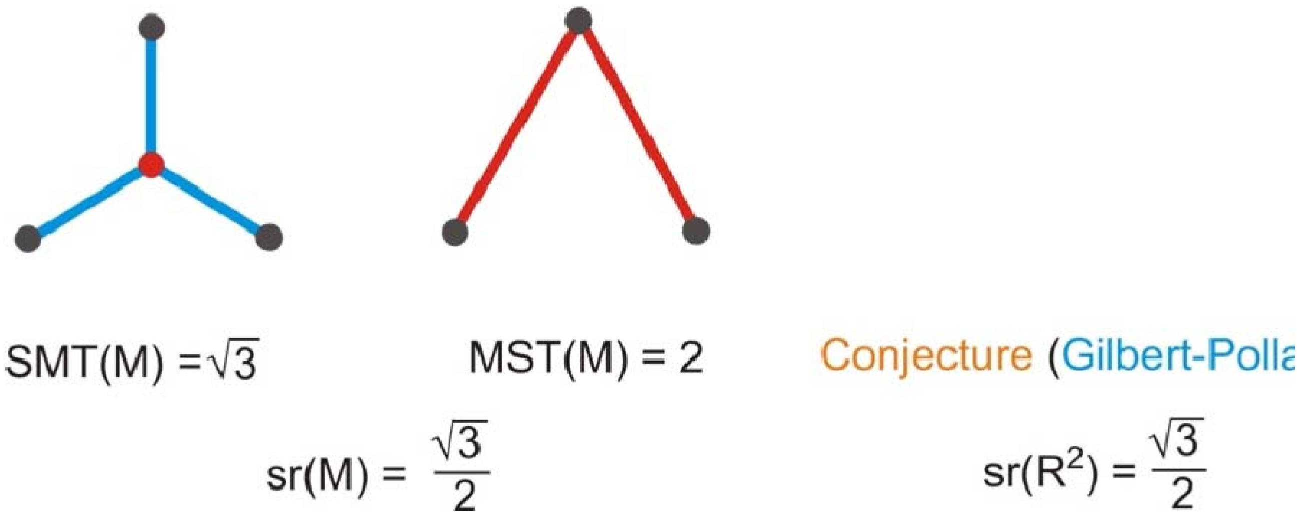
Let M be a finite subset of a metric space (X, ρ).

Recall that
smt(M) denotes the length of the shortest tree (of Steiner Minimal Tree) joining M,
mst(M) denotes the length of Minimal Spanning Tree (MST) for M.

Definition.
sr(M) = smt(M) / mst(M) is called the *Steiner ratio* for M
(it measures the precision of MST-approximation)

sr(X, ρ) = sr(X) = inf {sr(M) | M $\subset$ X, M is finite} is the Steiner Ratio for (X, ρ) (it measures the worst precision over all MST-approximations of SMTs for finite M $\subset$ X)

Example. Let M be a regular triangle in $\mathbb{R}^2$, whose sides are of the length 1, then



Steiner Ratio for n-points sets

sr_n(X, ρ) = inf { sr (M) | M $\subset$ X , #M $\leq$ n } is the *Steiner ratio of order* n.

Clearly, sr(X, ρ) = inf sr_n (X, ρ).

Besides that, sr_2(X, ρ) = 1 if X consists of at least 2 points.

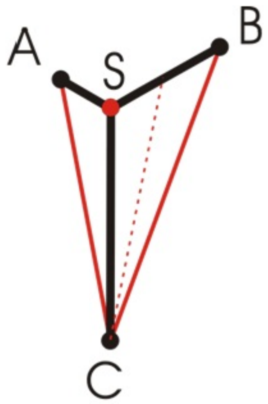
In 1990 Du and Hwang (Bell Labs., USA) announced a proof of Gilbert-Pollak Conjecture. However, it turns out that their proof has serious gaps.

Theorem (Gilbert, Pollak).The following equality holds :

$$sr_3\left(\mathbb{R}^2\right) = \frac{\sqrt{3}}{2}.$$

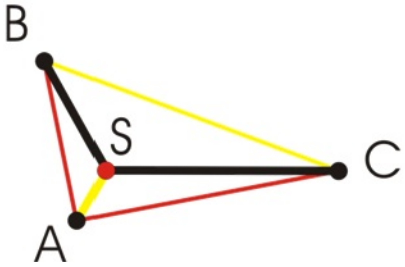
Proof.The Steiner ratio of a triangle with an angle of at most 120 ° equals 1 because in this case SMT = MST.

Consider a triangle whose angles are less than 120 °.



$$|AS| \leq |BS|$$
$$\Downarrow$$
$$|AC| \leq |BC|$$

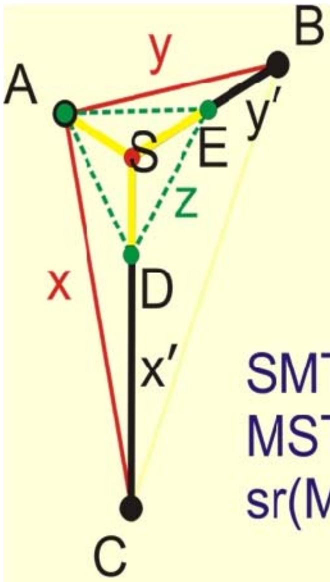
The shorter an edge of SMT,
the longer the opposite side
of the triangle



$|AS|$ is the shortest edge

$\Downarrow$

the polygonal line CAB
is MST



$M=\{A, B, C\}$, $|AC|=x$, $|AB|=y$,

$|AD|=|DE|=|EA|=z$, $|DC|=x'$, $|EB|=y'$,

$|SA|=|SD|=|SE|=z/\sqrt{3}$,

$SMT(M)= \sqrt{3}z + x' + y'$,

$MST(M)= x + y$,

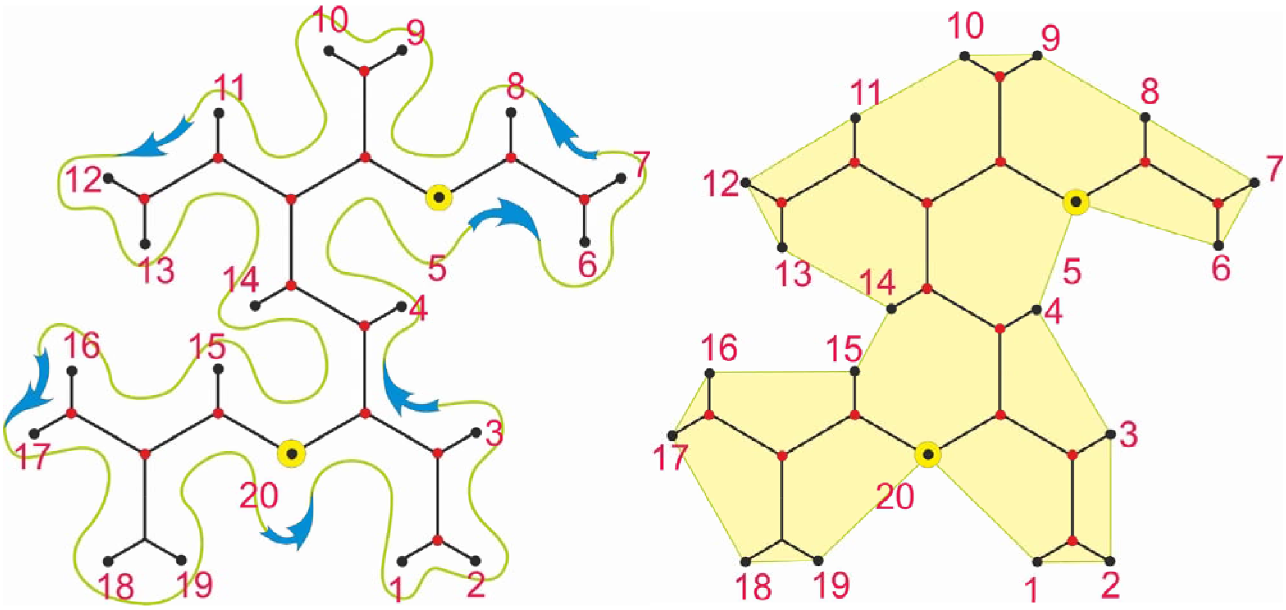
$sr(M) = SMT(M) \,/\, MST(M) = \dfrac{z\sqrt{3} + x' + y'}{x + y}$

$$\frac{\sqrt{3}z + x' + y'}{x + y} \geq \frac{\sqrt{3}z + x' + y'}{z+x'+z+ y'} = \frac{\sqrt{3}z + x' + y'}{2z + x' + y'} \geq \frac{\sqrt{3}}{2}$$

Basic Properties of Steiner ratio

Observation (E.F.Moore). For any metric space (X, ρ) we have $\frac{1}{2} \leq$ sr (X, ρ) ≤ 1.

Proof. Since smt(M) $\leq$ mst(M), then sr(X, ρ) = inf $\frac{\text{smt}(M)}{\text{mst}(M)} \leq 1$.
Now, let G be an arbitraty tree in X joining M. Let us draw G in the plane as a planar tree G', consider a tour around G', and corresponding "polygonal line" L.



We pass each edge twice, thus 2 ρ(G) $\geq$ ρ(L) $\geq$ mst(M), therefore $\frac{\rho(G)}{\text{mst}(M)} \geq \frac{1}{2}$. This holds for any G, and smt(M) = inf ρ(G), thus sr(M) $\geq \frac{1}{2}$.

Observation. For any real number $\frac{1}{2} \leq a \leq 1$ there exists a metric space (X, ρ) such that sr (X, ρ) = a. Moreover, for any $\frac{1}{2} <$ a ≤ 1 the set X can be chosen finite, but for a = $\frac{1}{2}$ can not.

Exercise. Prove the Observation.

Definition. A metric space (X, ρ) is called *ultrametric* if for any x, y, and z from X we have ρ (x, z) $\leq$ max {ρ (x, y), ρ (y, z)}.

Observation. Let (X, ρ) be an ultrametric space. Then sr (X, ρ) = 1.

A = $\{a_1, \ldots, a_n\}$ is a finite set of letters called *alphabet*, A*is the set of all possible words constructed in alphabet A

For x and y from A* let ρ (x, y) be the minimal number of editorial operations (deletions, insertions, and substitutions) necessary for passing from x to y.The function ρ is *Levenshtein distance*.

The metric space (A*, ρ) is called *phylogenetic space*.

Observation. For any phylogenetic space (X, ρ) we have sr (X, ρ) = $\frac{1}{2}$.

General properties of Steiner ratio

- Isometric spaces have equal Steiner ratios
- If (Y, ρ) is a subspace in (X, ρ), then sr (Y, ρ) $\geq$ sr (X, ρ)

Let ρ_1 and ρ_2 be two metrics on a set X, and suppose that there exist constants c_1 and c_2 such that for any x and y from X we have $c_1\ \rho_2$ (x, y) $\leq \rho_1$ (x, y) $\leq c_2\ \rho_2$ (x, y). Then

$$\frac{c_1}{c_2}\,\mathrm{sr}(X,\rho_2) \leq \mathrm{sr}(X,\rho_1) \leq \frac{c_2}{c_1}\,\mathrm{sr}(X,\rho_2).$$

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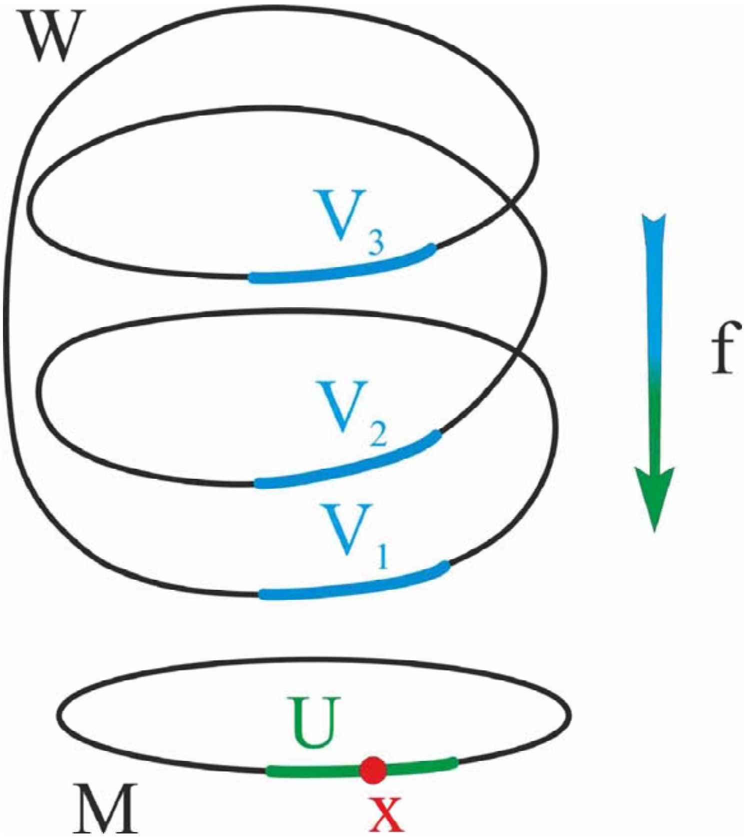
Steiner ratios of surfaces

Theorem (Cieslik, Ivanov, Tuzhilin). The Steiner ratio of any k-dimensional surface does not exceed the one for $\mathbb{R}^k$.

Corollary. The Steiner ratio of any 2-dimensional surface does not exceed $\frac{\sqrt{3}}{2}$.

Let f : W → M is a mapping between surfaces.

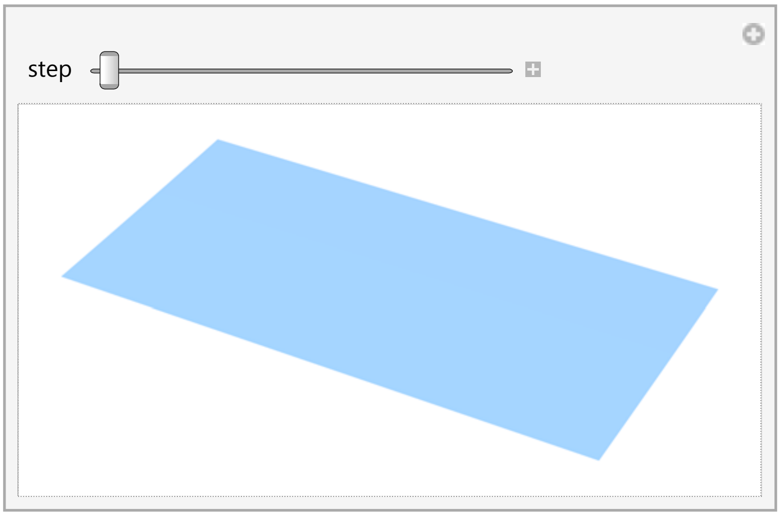
The mapping f is called a *k-leaved covering* if for any x ∈ M there exists a neighborhood U such that f^{-1} (U) is a union of k non-intersecting "copies" V_1, ... , V_k of U (each V_i is homeomorphic to U).
The surface M is called the *base*, and the surface W the *total space* of the covering f.

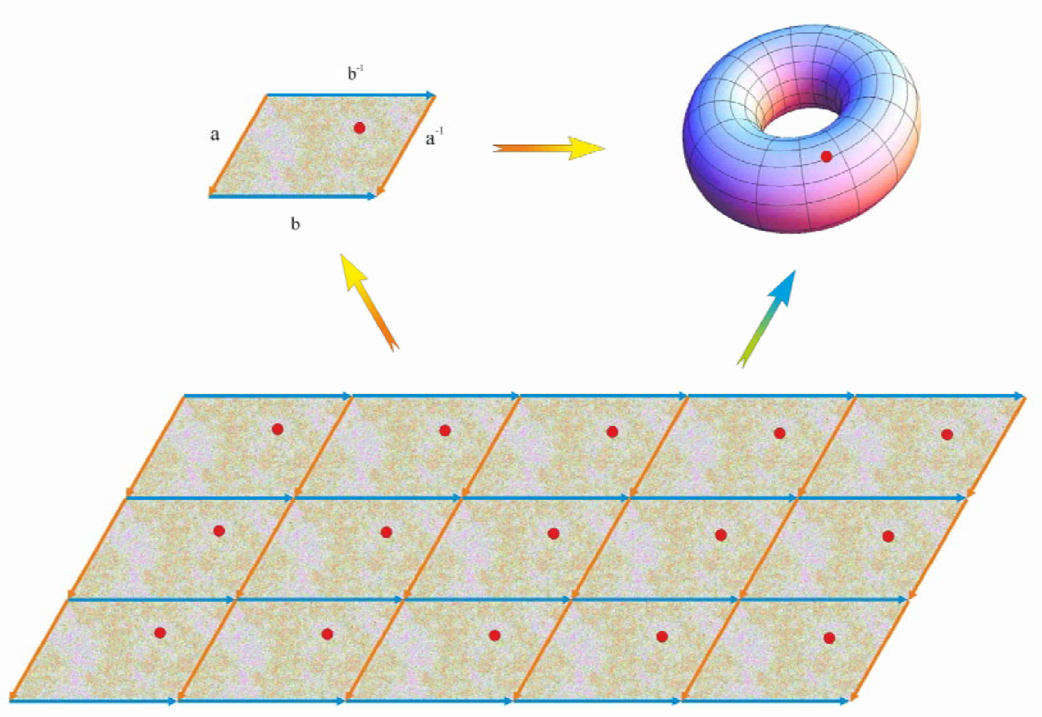


A covering f is called *locally isometric* if all the mappings f : V_i → U preserves the distances between points.

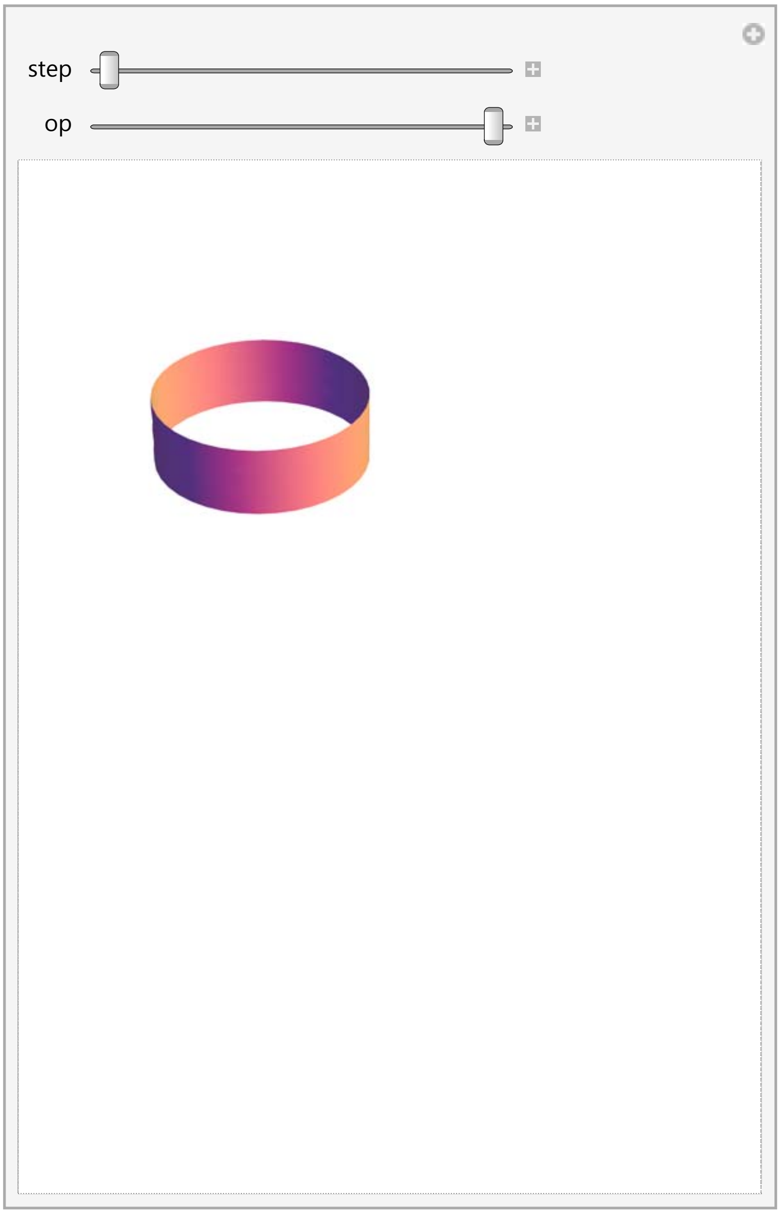
Examples.

- Torus





- Klein Bottle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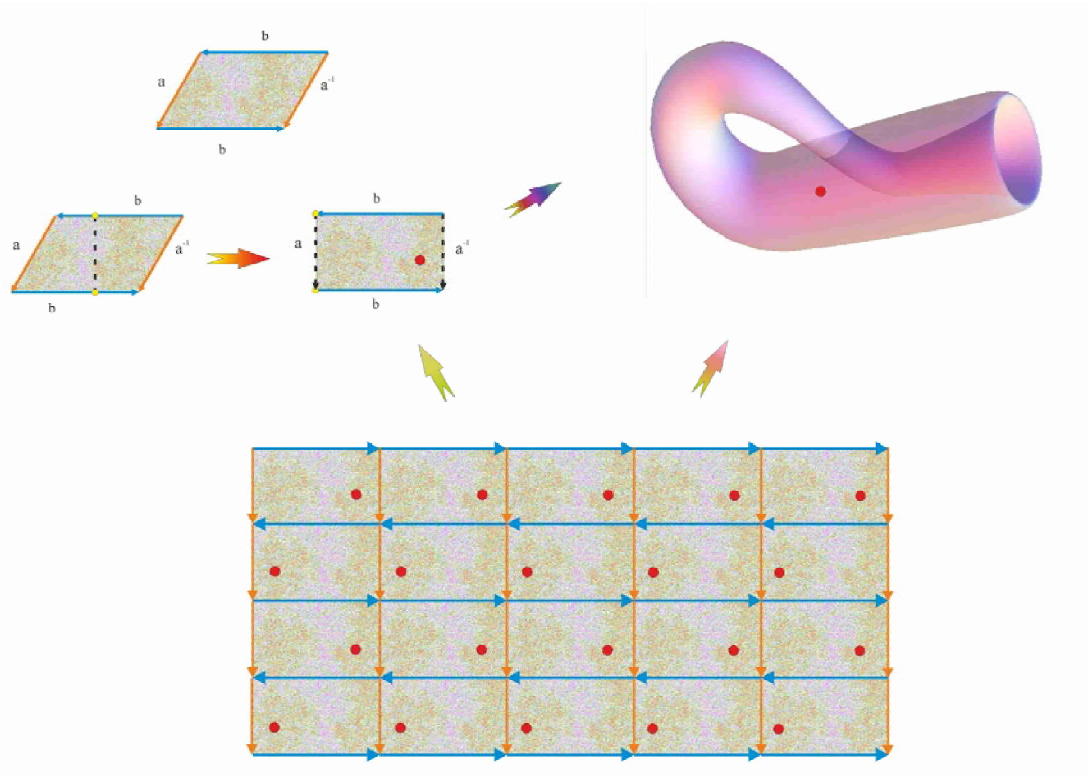


Theorem (Cieslik, Ivanov, Tuzhilin). Let $f : W \to M$ be a locally isometric covering. Then $\mathrm{sr}(W) \leq \mathrm{sr}(M)$.

Corollary. If n is the standard Euclidean n-dimensional space, and $f : \mathbb{R}^n \to M$ is a locally isometric covering, then sr (M) = sr (n).
In particular, sr (flat 2-dim cylinder) = sr (flat 2-dim torus) = sr (flat Klein bottle) = $\mathrm{sr}\left(\mathbb{R}^2\right)$.

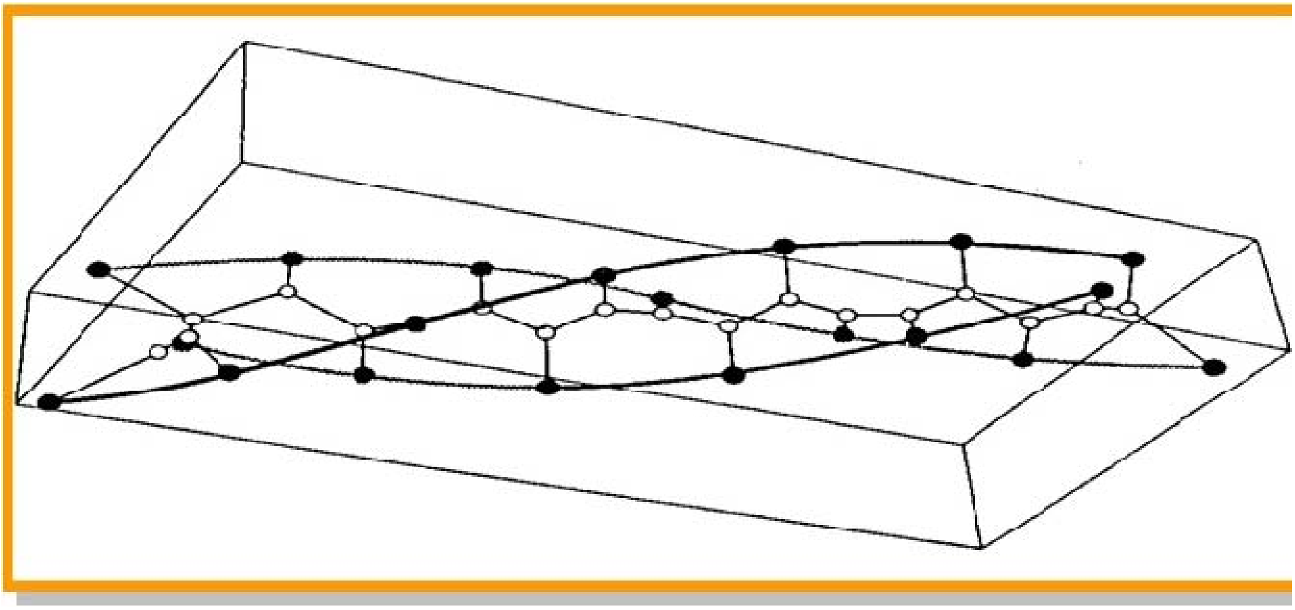
■ Steiner ratio of Euclidean $\mathbb{R}^n$

If Gilbert-Pollak conjecture is true, then the Steiner Ratio of $\mathbb{R}^2$ is attained on vertices of regular triangle.

However, for any $n \geq 3$, if $M \subset \mathbb{R}^n$ is the vertices set of a regular simplex, then $sr(M) > sr(\mathbb{R}^n)$.

Also, the best known estimation of the Steiner Ratio for $\mathbb{R}^3$ is attained at infinite set, namely,

Conjecture (W.D.Smith & J.M.Smith). The Steiner ratio for $\mathbb{R}^3$ is attained at the “sausage” infinite points boundary:



If so, the Steiner ratio of $\mathbb{R}^3$ equals

$$\sqrt{\frac{283}{700} - \frac{3\sqrt{21}}{700} + \frac{9\sqrt{11-\sqrt{21}}\,\sqrt{2}}{140}} = 0.78419\ldots$$

Theorem (Graham, Hwang). For any $n \geq 2$ we have $sr(\mathbb{R}^n) \geq \frac{1}{\sqrt{3}}$.

Theorem (Cieslik). For any $n \geq 2$ we have $sr(\mathbb{R}^n) \leq \sqrt{\frac{1}{2} + \frac{1}{2n}}$.

D.Z.Du and W.D.Smith (1996) proved that if the Steiner ratio is attained on a finite subset $M \subset \mathbb{R}^n$, then the number of points in M can not be less than the value of a rapidly increasing function f(n).

For example, f (50) = 53, f (200) = 3 481 911, etc.

This also motivates the interest to generalize SMT theory to infinite boundary sets (see paragraph "Fine sets").

Steiner ratio of normed spaces

Definition of a norm. A *norm* on a vector space V is a function which takes each vector $v \in V$ to a non-negative real number $\| v \|$ and satisfies the following properties

(1) $\| v \| = 0 \Leftrightarrow v = 0$;
(2) for any real number λ we have $\| \lambda v \| = |\lambda| \; \| v \|$;
(3) $\| v + w \| \leq \| v \| + \| w \|$.

Exercise. Prove thet $\rho(v, w) = \| w - v \|$ is a metric on V.

The set $B = \{ v \in V \mid \| v \| \leq 1 \}$ is called the *unit ball* of the norm. It's easy to see that each norm is uniquely defined by its unit ball B. Namely, to reconstruct the norm by its ball B, one can put

$\| v \|_B = \inf \{ \lambda \in \mathbb{R} \mid v / \lambda \in B \}$.

Exercise. Prove that the function $\| \bullet \|_B$ conicides with the initial norm $\| \bullet \|$.

The later motivates that the Steiner ratio of a normed space with the unit ball B is usually denoted by sr (B).We refer to B as *Banach-Minkowski ball* or *BM-ball*.

Observation.
1) If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, and B is a BM-ball, then sr (B) = sr (f (B)).
2) If $L \subset \mathbb{R}^n$ is a linear subspace, B is a BM-ball in $\mathbb{R}^n$, and $B' = B \cap L$ is the corresponding BM-ball in L, then sr (B) ≤ sr (B').
3) For any BM-balls B and B' and any positive numbers c and c' such that $\frac{B}{c} \subset B' \subset \frac{B}{c'}$ we have

$$\frac{c}{c'} \operatorname{sr}(B) \geq \operatorname{sr}(B') \geq \frac{c'}{c} \operatorname{sr}(B).$$

For a BM-ball $B \subset \mathbb{R}^n$ we put $[B] = \{ T(B) \mid T \in GL(n, \mathbb{R}) \}$.

Define
  $d([B], [B']) = \inf \{h \geq 1 \mid \text{there exists } T \in GL(n, \mathbb{R}) : B \subset T(B') \subset h B\}$.
Notice that log d is a metric which is called the *Banach - Mazur metric*.

Theorem (Cieslik). For any BM-balls B and B' in $\mathbb{R}^n$ we have
  $d([B], [B']) \; \operatorname{sr}(B) \geq \operatorname{sr}(B') \geq \frac{\operatorname{sr}(B)}{d([B],[B'])}$.

Corollary (Du, Gao, Graham, Liu, Wan).
1) Since, by John, $d([B], [D^n]) \leq \sqrt{n}$ for any BM-ball $B \subset \mathbb{R}^n$, where D^n is the standard Euclidean ball in $\mathbb{R}^n$, we have $\operatorname{sr}(B) \geq \frac{\operatorname{sr}(D^n)}{\sqrt{n}}$.

2) For any BM-ball B we have $\frac{2}{3} \leq \operatorname{sr}(B) \leq \frac{\sqrt{13} - 1}{3}$.

The case of l_p - spaces

Let us put

$$B^n_p = \{ (x_1, \ldots, x_n) \mid |x_1|^p + \ldots + |x_n|^p \leq 1\}, \; p \geq 1.$$

Corollary (Cieslik). Since, by Gurarii, Kadec, Macaev,

$$d\left([B^n_p], [B^n_q]\right) = n^{\frac{1}{p} - \frac{1}{q}}$$

for any 1 ≤ p ≤ q ≤ 2, or 2 ≤ p ≤ q ≤ , we have

$$n^{\frac{1}{p}-\frac{1}{q}}\,\mathrm{sr}\left(\left[B_p^n\right]\right) \geq \mathrm{sr}\left(\left[B_q^n\right]\right) \geq \frac{1}{n^{\frac{1}{p}-\frac{1}{q}}}\,\mathrm{sr}\left(\left[B_p^n\right]\right).$$

Theorem (Hwang).The following equalities hold :

$$\mathrm{sr}\left(\left[B_1^2\right]\right) = \mathrm{sr}\left(\left[B_\infty^2\right]\right) = \frac{2}{3}.$$

So, the lower bound on the Steiner ratios for l_p -spaces is achieved.

Theorem (Du, Liu). For p, q ≥ 1 such that $\frac{1}{p}+\frac{1}{q}=1$ we have

$$\mathrm{sr}\left(\left[B_q^2\right]\right) \leq \frac{(2^p-1)^{\frac{1}{p}}+(2^q-1)^{\frac{1}{q}}}{4}.$$

Thus, the Steiner ratio for l_p -spaces does not exceed $\frac{\sqrt{3}}{2}$, and the equality holds just for p = 2 (modulo Hilbert - Pollak Conjecture).

- **λ - geometries**

For positive integer λ we denote by $B^{(\lambda)}$ a BM-ball on the plane which is a regular 2λ-gon. The obtained normed plane is said to be λ-geometry.

Theorem. For any integer λ ≥ 2 we have

$$\mathrm{sr}\left(\mathbb{R}^2\right)\frac{1}{\cos\frac{\pi}{2\lambda}} \geq \mathrm{sr}\left(B^{(\lambda)}\right) \geq \mathrm{sr}\left(\mathbb{R}^2\right)\cos\frac{\pi}{2\lambda}.$$

Theorem (Lee, Shen).
1) For λ ≡ 3 (mod 6) we have $\mathrm{sr}\left(B^{(\lambda)}\right) = \mathrm{sr}\left(\mathbb{R}^2\right)\cos\frac{\pi}{2\lambda}$.

2) For λ ≡ 0 (mod 6) we have $\mathrm{sr}\left(B^{(\lambda)}\right) = \mathrm{sr}\left(\mathbb{R}^2\right)$.

Steiner ratio and Yung Number

Let ($\mathbb{R}^n$, $\|.\|$) be a normed space with the unit ball B, and X $\subset \mathbb{R}^n$ be a bounded set.

The *diameter* of X is $D_B(X) = \sup \{ \|v - w\| : v, w \in X\}$.

The *radius* of X is $R_B(X) = \inf \{r > 0 : X \subset v + r B \text{ for some } v \in \mathbb{R}^n\}$.

The Yung number is $J_n(B) = \sup \{ \frac{R_B(X)}{D_B(X)} \mid X \subset \mathbb{R}^n, \ X \text{ is bounded} \}$.

Observation (Leichtweiss). If B $\subset \mathbb{R}^n$ is a BM-ball, then

$$\frac{1}{2} \leq J_n(B) \geq \frac{n}{n+1}.$$

Theorem (Cieslik).
1) For any B $\subset \mathbb{R}^2$ we have sr (B) $\leq \frac{3}{2} J_2(B)$.

2) For any B $\subset \mathbb{R}^3$ we have sr (B) $\leq \frac{4}{3} J_3(B)$.

(V.V.Makeev 2008: http://www.mathnet.ru/php/archive.phtml?wshow=paper&jrnid=znsl&paperid=1637&option_lang=rus)

3) If there exists a regular (n + 1)-simplex in ($\mathbb{R}^n$, $\|.\|$), then sr (B) $\leq \frac{n+1}{n} J_n(B)$.

Dekster : for any BM-ball sufficiently close to Euclidean ball in the sense of Banach-Mazur distance there do exists a regular (n + 1)-simplex.

Packing and covering

Packing is a collection of convex bodies with mutually disjoint interiors.
Covering is a family of subsets of $\mathbb{R}^n$ whose union is $\mathbb{R}^n$.

For a convex compact body K and some $V = \{ v_i \} \subset \mathbb{R}^n$, we put

$M(V, K) = \{ v_i + K \}$,
$M_r(V, K) = \{ v_i + (1 + r) K \}$,
$\frac{1}{c(V,K)} = \inf \{ r \mid M_r(V, K) \text{ is a covering} \}$,
$V(K) = \{ V \mid M(V, K) \text{ is a packing} \}$,
$C(K) = \sup \{ c(V, K) \mid V \in V(K) \}$ is called *closeness.*

Theorem (Cieslik). For any BM-ball $B \subset \mathbb{R}^n$ we have

$$\mathrm{sr}(B) \leq \frac{3}{4} (1 + 1 / C(K)).$$

Tammes' Problem

Let (X, ρ) be a metric space and $M \subset X$. Assuming that $\sup\{\emptyset\} = 0$ and $\inf\{\emptyset\} = \infty$, we put

$$s(M) = \inf\{\rho(x, y) \mid x, y \in M,\ x \neq y\},$$

$$\epsilon_k(M) = \sup\{s(A) \mid A \subset M,\ \#A = k\}.$$

Let $B \subset \mathbb{R}^n$ be a BM-ball and $\Sigma = \partial B$ be the boundary of B.

We put $r_k^n(B) = \epsilon_k(\Sigma)$, and if B is Euclidean ball, the value $r_k^n(B)$ we denote by r_k^n.

Tammes' Problem : To calculate r_k^n. The solution is known only for $k \leq 13$ and $k = 24$ ($k = 13$ - Musin and Tarasov, 2010).

Theorem (Cieslik). For $k \geq 3$ and any BM-ball $B \subset \mathbb{R}^n$ we have

$$sr(B) \leq \frac{k}{(k-1)\, r_k^n(B)}.$$

Some problems

1) Fill the gap in Du-Hwang approach to investigation of Steiner ratio.

2) For which spaces the Steiner ratio is achieved at finite subsets?

3) For any BM-ball $B \subset \mathbb{R}^n$ define the dual ball $B' \subset \mathbb{R}^n$ as $B' = \{x \in \mathbb{R}^n \mid < x, y > \leq 1$ for any $y \in B\}$, where $< , >$ is the standard scalar product in $\mathbb{R}^n$.For example, the dual ball for B_p^2 is B_q^2, where $\frac{1}{p} + \frac{1}{q} = 1$. Describe the relation between the Steiner ratios of dual BM-balls.

Fine sets

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Defenition of fine sets

Definition.A set M of a metric space X is called *fine* if it can be joined by a finite length tree.

Remark. Any fine set is at most countable.

Constructions

Recall definition of the function $\epsilon_n(M)$

Let M be a subset of a metric space (X, ρ). Assuming that $\sup\{\emptyset\} = 0$ and $\inf\{\emptyset\} = \infty$, we put

$$s(M) = \inf\{\rho(x, y) \mid x, y \in M,\ x \neq y\},$$

$$\epsilon_n(M) = \sup\{s(A) \mid A \subset M,\ \#A = n\}.$$

Some properties of $\epsilon_n(M)$ and some examples.

(1) $\epsilon_n(M) = \sup\{s(A) \mid A \subset M,\ \# A \geq n\}$,

(2) $\epsilon_n(M)$ does not increase when n increases.

Examples.

(1) Let $M = \{1/a, \frac{1}{a^2}, \frac{1}{a^3}, \ldots\}$, $a \geq 2$. Then $\epsilon_n(M) = \frac{1}{a^{n-2}}$.

(2) Let $M = \mathbb{Q} \cap [0, 1]$. Then $\epsilon_n(M) = \frac{1}{n-1}$.

(3) Let $M = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$.Then $\epsilon_2(M) = 1$, $\epsilon_3(M) = \frac{1}{2}$, $\epsilon_4(M) = \frac{1}{4}$, $\epsilon_5(M) = \frac{1}{6}$, $\epsilon_6(M) = \frac{1}{10}$, …

Exercise. In the previous example, obtain a formula for general $\epsilon_n(M)$.

Observation. Let M be a fine subset of a metric space. Then $\epsilon_n(M)\, n \to 0$ as $n \to \infty$.

Example. The set $\mathbb{Q} \cap [0, 1]$ is not fine because

$$\epsilon_n(M)\, n = \frac{n}{n-1} \to 1 \text{ as } n \to \infty.$$

Example. Let $0 < \alpha < \beta$, and

$$M = \left\{\left(\frac{1}{p}, \frac{1}{q}\right) \,\middle|\, p, q \in \mathbb{N},\ \alpha \leq \frac{p}{q} \leq \beta\right\},$$

then $\epsilon_n(M)\, n$ does not tend to 0, thus M is not fine.

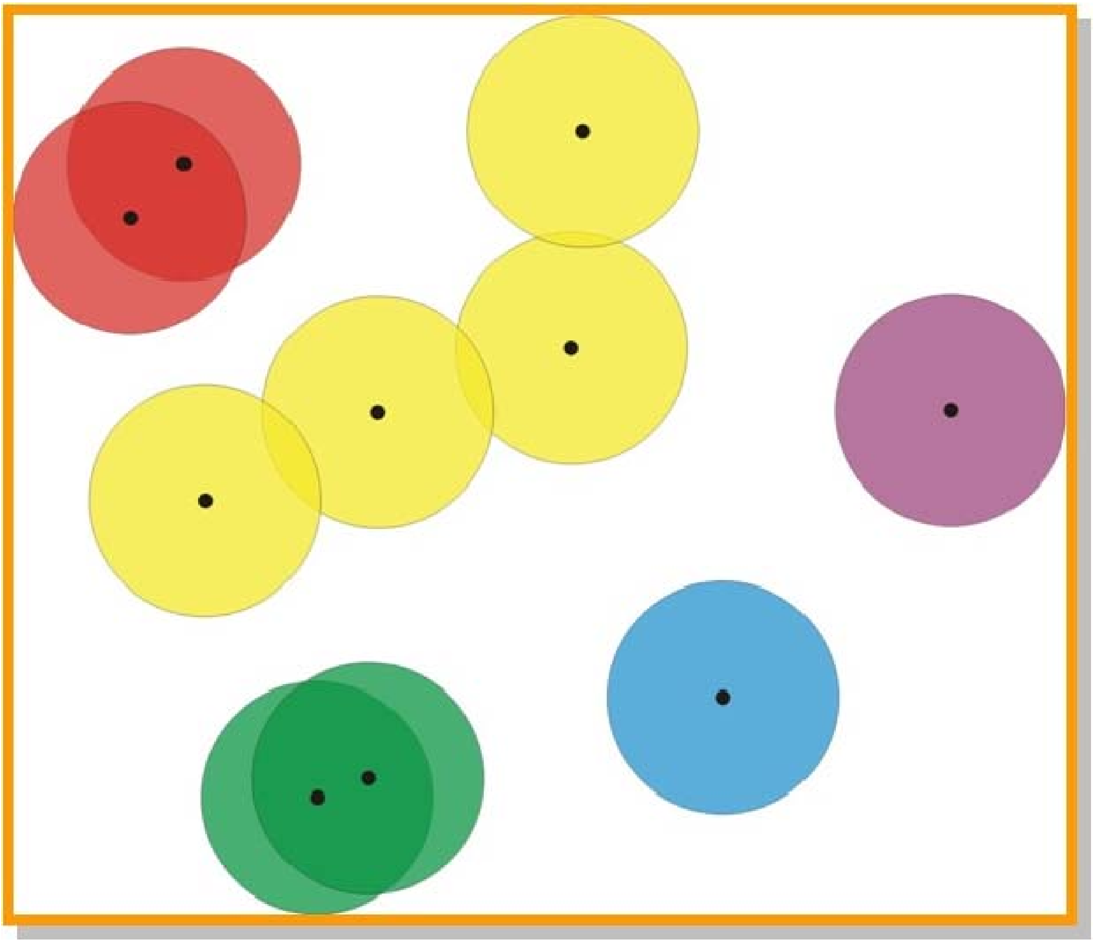
Let (X, ρ) be a metric space, $M \subset X$, $\lambda \geq 0$.

Let $U(\lambda, M)$ be the open λ-neighborhood of the set M, i.e., $U(\lambda, M)$ is the union of all open balls of radius centered at points from M.

Let $\{U_\alpha(\lambda, M)\}$ be the family of connected components of $U(\lambda, M)$.

We put $M_\alpha = M \cap U_\alpha(\lambda, M)$.

So $\{M_\alpha\}$ is a partition of M which will be denoted by $P_\lambda(M)$.



For any subset N of X we put diam(N) = sup $\{\rho(x, y) \mid x, y \in \mathbb{N}\}$.

For any subset M of X and any $\lambda \geq 0$ we put $\mathrm{Diam}_\lambda(M) = \Sigma$ diam(c) over all $c \in P_\lambda(M)$.

Example. Let $M = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ and $\frac{1}{12} < \lambda \leq \frac{1}{6}$.

Then $P_\lambda(M) = \{ \{1\}, \{\frac{1}{2}\}, M \setminus \{1, \frac{1}{2}\} \}$, thus $\mathrm{Diam}(M) = \frac{1}{2}$.

Fine sets in ℝ

Recall that the outer Jordan measure $\mu(M)$ of a set $M \in \mathbb{R}$ is defined as follows :

$$\mu\,(M) = \inf\left\{\sum_{k=1}^{N} (b_k - a_k) \;\middle|\; M \subset \bigcup_{k=1}^{N} (a_k,\ b_k)\right\}.$$

Observation. Let $M \subset \mathbb{R}$ be bounded and countable.Then
(M is fine) $\Leftrightarrow$
($\epsilon_n(M)\, n \to 0$ as $n \to \infty$) $\Leftrightarrow$
($\mathrm{Diam}_\lambda(M) \to 0$ as $\lambda \to 0$) $\Leftrightarrow$
($\mu(M) = 0$).
Moreover, for a fine set $M \subset \mathbb{R}$ we have mst (M) = diam (M).

Fine sets criterion

Main Theorem (A.Ivanov, I.Nikonov, A.Tuzhilin). Let M be a bounded countable subset of a metric space, and we put $\pi_\lambda(M) = \# P_\lambda(M)$. Then M is fine iff

$$(1) \int_0^{\operatorname{diam}(M)} \pi_\lambda(M)\, d\lambda < \infty, \text{ and}$$

$$(2) \operatorname{Diam}_\lambda(M) \to 0 \text{ as } \lambda \to 0.$$

Moreover, for a fine set M we can calculate the length mst(M) of Minimal Spanning Tree on M as follows:

$$\operatorname{mst}(M) = \int_0^{\operatorname{diam}(M)} \pi_\lambda(M)\, d\lambda - \operatorname{diam}(M).$$