# Minimizing the object dimensions in circle and sphere packing problems

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**Abstract**

Given a fixed set of identical or different-sized circular items, the problem we deal with consists on finding the smallest object within which the items can be packed. Circular, triangular, squared, rectangular and also strip objects are considered. Moreover, 2D and 3D problems are treated. Twice-differentiable models for all these problems are presented. A strategy to reduce the complexity of evaluating the models is employed and, as a consequence, instances with a large number of items can be considered. Numerical experiments show the flexibility and reliability of the new unified approach.

**Key words:** Packing of circles and spheres, models, algorithms, nonlinear programming.

## 1 Introduction

Several papers deal with the packing problem of, given an object and a fixed set of items, minimize the object dimension (or equivalently maximize the items dimension) subject to placing the items within the object without overlapping. In particular, several papers focus on this problem using nonlinear optimization models and techniques.

In [26] the problem of packing identical circular items within a square is treated. A max-min model to maximize the minimum distance between the centers of the circular items is introduced with the aim of maximizing the items diameter. A classical nonlinear reformulation of the max-min problem is solved using MINOS [30] and the GAMS modeling language [17]. A multi-start strategy is used to enhance the probability of finding global solutions. Problems up to 30 items are solved attaining the best known solutions reported in the literature and yielding some new configurations. A different nonlinear programming formulation of the same problem is also introduced in [31]. Basically, the energy function $\sum_{i \neq j} (\lambda / d_{ij}^2)^m$, where $d_{ij}$ represents

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the Euclidean distance between the centers of items $i$ and $j$, $\lambda$ is a scaling factor and $m$ is a positive integer, is minimized subject to the items fitting inside the object. An unconstrained reformulation of that problem is solved using an hybrid line-search algorithm that uses gradient-type directions at the beginning and Newton-type directions near the solution. The solution obtained by the optimization algorithm is then used as a starting point to solve a system of nonlinear equations with the aim of improving the accuracy of the solution. Problems with up to 50 items are solved, finding some alternative solutions and improving some results presented in [18, 21]. In [25], the same problem is studied. The problem is modeled as a quadratic optimization problem. Two properties satisfied by at least an optimal solution are introduced and a clever and efficient branch-and-bound algorithm is developed. The algorithm is used to prove the $\epsilon$-optimality of solutions with up to 35 items, 38 and 39 items. Moreover, new solutions for 32 and 37 items are found. Finally, the authors of [25] highlight that, concerning the problem of maximizing the items diameter (instead of minimizing the object size), while most of the approaches are capable of obtaining a lower bound, their approach provides an upper bound on the solution.

The problem of packing equal circles within a circle is addressed in [29]. The nonlinear formulation is one of the most natural ones and maximizes the items radius while requires that the items do not overlap and fit inside the circular object. Two equivalent formulations, using cartesian and polar coordinates, are presented. A heuristic method called Reformulation Descent that iteratively alternates from one model to the other is presented and tested. The authors claim that these switches may reduce the probability of the method to stop at undesired stationary points. Numerical experiments show that the method is much faster than classical nonlinear programming methods and that, for instances with up to 100 items, the best known solution is found in 40% of the cases while in the other cases the error never exceeds 1%.

The problem of packing different-sized circles into a strip is considered in [35]. A natural nonlinear model is introduced and deeply studied. As pointed out by the authors, some peculiarities of the model allow to conclude that optimal solutions are attained at extreme points of the feasible region. A clever approach (based on Lagrange multipliers) to jump from a local minimizer to a better one, interchanging the positions of two non-identical circles, is devised. The numerical experiments in [35] show that this strategy, that can be viewed as a tunneling method for escaping from local minimizers, together with an efficient local solver [34] and a multi-start strategy, yields high-quality solutions. A comparison against a branch-and-bound algorithm using instances with up to 35 items show the advantages of the presented approach. Similar techniques are considered in [36], where the problem of packing various solid spheres into a parallelepiped with minimal height is addressed and numerical results with up to 60 spheres are given.

Nonlinear models have also been successfully used in the problem of, given an infinite number of identical items and an object, all with fixed dimensions, pack as many items as possible within the object. In [15] the authors deal with circular items within rectangular and circular containers. In [12, 11] the problems of packing orthogonal rectangles and free-rotated rectangles within arbitrary convex regions are considered, respectively. The Method of Sentinels introduced in [11, 28] can also be used for packing arbitrary non-identical polygons (with internal angles not smaller than $\pi/2$) within arbitrary convex regions. In all the cases the nonlinear models are solved using Algencan [2, 1, 7, 10], an Augmented Lagrangian method for the minimization of a smooth function with general constraints (freely available at the TANGO Project web page [6]).

All the nonlinear-model-based strategies described above have a point in common: the number of non-overlapping constraints between the items is $O(N^2)$, where $N$ is the number of items being packed. Based on efficient algorithms developed to reduce the asymptotic computational complexity of the $N$-body problem [22], we develop a methodology (and their corresponding data structures) to reduce the computational complexity of evaluating the nonlinear models. By using this strategy, packing problems with a large number of items are solved. See [27], where a similar strategy was successfully used for the generation of initial configurations for molecular dynamics.

This paper is organized as follows. In Section 2 we introduce a variety of nonlinear models for packing circles and spheres in several kinds of objects. The strategy to reduce the complexity of the non-overlapping constraints is described and analyzed. Numerical experiments are shown in Section 3. In Section 4 we state some conclusions and directions for future research.

# 2 Nonlinear models

The nonlinear models considered in this paper have the following structure (see [33]):

$$\begin{array}{ll} \text{Minimize} & \text{the object dimension} \\ \text{subject to} & \text{fitting the items inside the object,} \\ & \text{non-overlapping of items.} \end{array} \tag{1}$$

Problem (1) has three main ingredients: (i) the object dimension to be minimized; (ii) the constraints of the items being placed within the object; and (iii) the non-overlapping constraint between the items. Clearly, the first one depends just on the object and the last one depends just on the items, while the other one depends on both, the object and the items. In the present section we will show how to deal with each model ingredient in the cases of circular or spherical items and a wide range of objects.

## 2.1 Non-overlapping constraints

In this subsection we describe the non-overlapping constraints. Moreover, we describe a methodology, based on strategies developed for the $N$-body problem [22], to reduce the complexity of computing the non-overlapping constraints.

Let $c_i, i = 1, \ldots, N$, be the centers of $N$ circular or spherical items with radii $r_i, i = 1, \ldots, N$. The non-overlapping between the items can be modeled as

$$d(c_i, c_j)^2 \geq (r_i + r_j)^2, \ \forall\, i < j, \tag{2}$$

or

$$\sum_{i<j} \max\{0, (r_i + r_j)^2 - d(c_i, c_j)^2\}^2 = 0, \tag{3}$$

where $d(\cdot, \cdot)$ is the Euclidean distance (see Figure 1). Both (2) and (3) are related to the distances among the centers of the $N(N-1)/2$ pairs of items. Squares in (2) and (3) are used to make

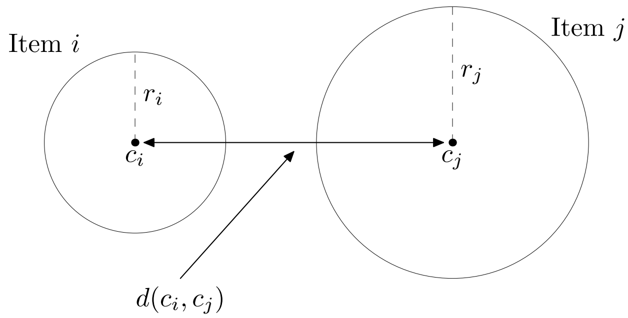


Figure 1: Two items do not overlap if and only if the distance among their centers is greater than or equal to the sum of their radii.

them differentiable. Clearly, (2) and (3) are equivalent and any set of items that satisfy (2) or (3) has no overlapping.

However, (3) has a great advantage over (2) in practice: not all the $O(N^2)$ distances need to be computed to evaluate the left hand side of the constraint at a given point. Basically, if two circular items $k_1$ and $k_2$ are far one from the other, the contribution of the term $\max\{0, (r_{k_1} + r_{k_2})^2 - d(c_{k_1}, c_{k_2})^2\}^2$ to the sum in (3) is null. In fact, if the items are more or less well distributed, the number of pairs that contribute to the sum in (3) is $O(N)$. Moreover, these pairs can also be identified in $O(N)$ operations. This strategy has been successfully used in molecular conformation problems in which the number of items can be very large (see [27]).

The $N$-body problem consists on computing the gravitational force between $N$ particles in the 3D space, where each particle exerts a force on all the other particles, implying pairwise interactions. This kind of physical system occurs in several fields, like celestial mechanics, plasma physics, fluid mechanics and semiconductor device simulations [22]. Several efficient algorithms were proposed to reduce the asymptotic computational complexity from $O(N^2)$, in the naive approach, to $O(NlogN)$ (see [3, 5]) and to $O(N)$ (see [19] for particles in two dimensions and [20, 38] for particles in three dimensions). In [24] a study of the performance of these methods in parallel computers is analyzed. Based on these ideas, we develop a strategy to reduce the computational complexity of evaluating (3).

It is hard to realize a procedure to detect which pairs *will contribute* to the sum in (3) without computing all the $O(N^2)$ distances. However, it is easy to detect which pairs *will not contribute* and just to compute the distances for the other pairs (which may contribute or not). Consider a partition of the object in regions in such a way that circular items whose centers are in non-adjacent regions can't overlap. For the case of identical circular items with radius $r$ it is easy to see that a partition in squared regions of side $\delta = 2r$ has such property (see Figure 2). For the case of circular items with radii $r_i$, $i = 1, \dots, N$, the size of the squared regions should be $\delta = 2 \max_i\{r_i\}$. Now, given an item, it is just necessary to compute the distances from its center to the centers of the items which are in the same region or in an adjacent region.

Considering a squared object, the total number of squared regions needed to cover it is $N_{reg}^2$, where $N_{reg} = \lceil L_{ub}/\delta \rceil$ and $L_{ub}$ is an upper bound on the squared object side. The squared regions are then surrounded by unbounded regions just to make sure that any point in the space belongs to a region, i.e., we are considering a partition of the whole space (see Figure 3). Given

$x \in I\!R^n$ ($n = 2, 3$ are the natural choices) the region to which it belongs can be computed in constant time as

$$Reg(x) = (p(x_1), \ldots, p(x_n)), \tag{4}$$

where

$$p(a) = \max\{0, \min\{\lfloor a/\delta \rfloor, N_{reg} + 1\}\}. \tag{5}$$

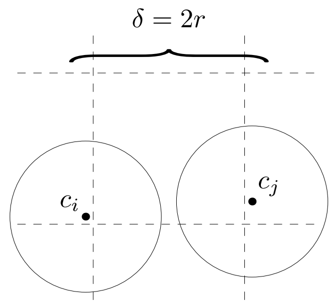


Figure 2: If two items are in non-adjacent regions then they do not overlap. In the picture, $d(c_i, c_j) \geq \delta = 2r$, so the items do not overlap.

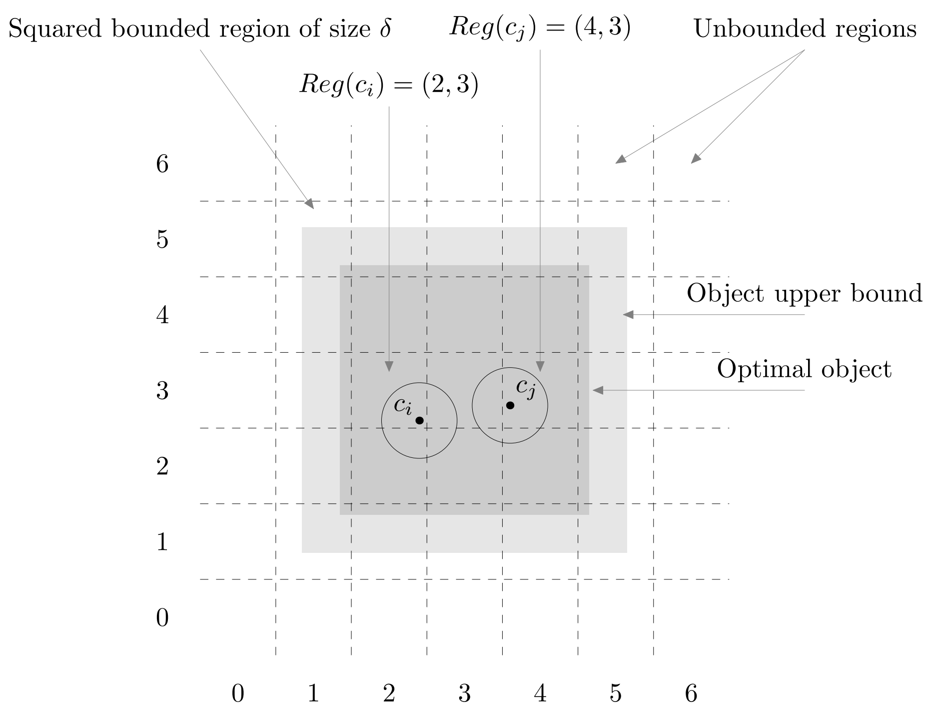


Figure 3: The space is partitioned into regions in such a way that the object is covered by squared regions. Items whose centers are in non-adjacent regions can't overlap. So, to verify overlappings, we just need to consider pairs of items in the same region or in adjacent regions.

The method starts with an empty data structure that represents the partition. Each region has a list of the items whose centers are in the region. There is a matrix with $(N_{reg}+2)^2$ elements with pointers to these lists. There is also a list with the non-empty regions (regions that have at least an item). See Figure 4. All these lists start empty. Given the centers $c_1, c_2, \dots, c_N$ of the circular items, these are the steps to compute the left hand side of (3):

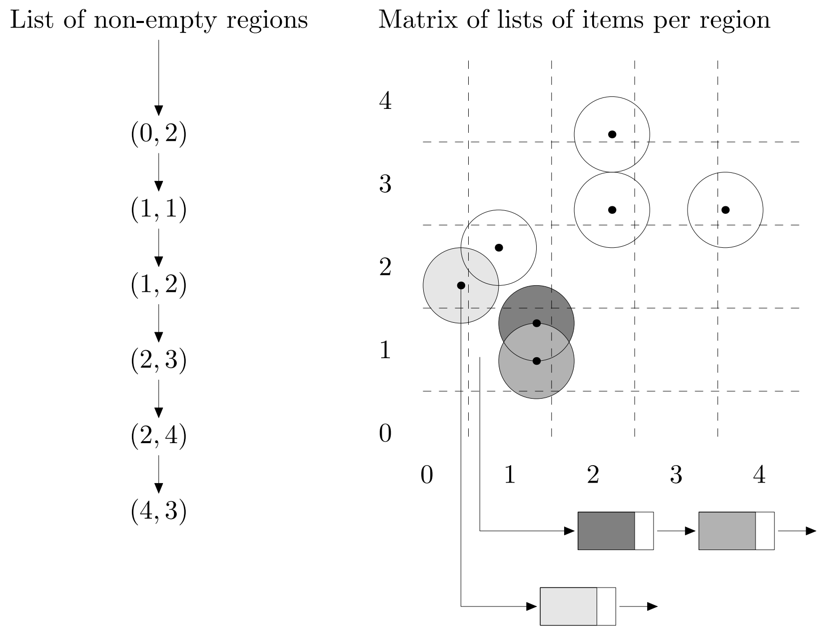


Figure 4: Data structure for the overlapping evaluation. Each region has a list of the items within it. There is also a list of the non-empty regions.

**Overlapping evaluation.**

**Step 0:** $sum \leftarrow 0$.

**Step 1:** For each item $i = 1, \dots, N$ do:

**Step 1.1:** Determine its region $Reg(c_i)$ given by (4)–(5).

**Step 1.2:** Add item $i$ to the list of items whose centers are in $Reg(c_i)$.

**Step 1.3:** If this is the first item assigned to $Reg(c_i)$, add $Reg(c_i)$ to the list of non-empty regions.

**Step 2:** For each non-empty region $R$ and for each item $i$ in $R$ do:

**Step 2.1:** For each item $j$ in $R$ or in a region adjacent to $R$ do:

**Step 2.1.1:** Compute $sum \leftarrow sum + \max\{0, (r_i + r_j)^2 - d(c_i, c_j)^2\}^2$.

**Step 3:** Clear the data structure and return $sum$.

**Remark:** At Step 2.1, in order to avoid computing all distances twice, two actions are taken: (i) Inside the same region, compute the distances from item $i$ just to the items that appears after $i$ in the list of items of region $Reg(c_i)$; (ii) Regarding adjacent regions, just a subset of the adjacent regions needs to be considered, as the other ones will be considered when the regions interchange their roles (see Figure 5).

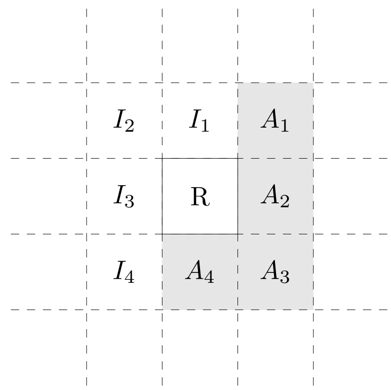


Figure 5: When the items in region $R$ are being considered, just the distances to the items in regions $A_1$, $A_2$, $A_3$ and $A_4$ need to be computed. The distances to the items in regions $I_1$, $I_2$, $I_3$ and $I_4$ are computed when the items in $I_k, k = 1, 2, 3, 4$, play the central role. In $\mathbb{R}^3$ the number of adjacent regions to consider is 13. The total number of adjacent regions in $\mathbb{R}^3$ and $\mathbb{R}^2$ are 26 and 8, respectively.

Assuming that the number of items in each region is constant (independent of $N$) then, clearly, the complexity of the whole process is $O(N)$. The assumption of the constant number of items per region is very reasonable as, when the method approaches the solution or even in a random uniformly distributed initial configuration, the items are well distributed in the space.

In order to assess the practical impact of the partitioning approach in the overlapping evaluation, we compare its CPU time against the CPU time of the naive approach for an increasing number of randomly distributed identical circular items. Figure 6 shows the results. As expected, while the CPU time of the partitioning approach can be linearly approximated by $Time(N) \approx 0.00026943\ N\ ms$, the CPU time of the naive approach can be quadratically approximated by $Time(N) \approx 0.0000028301\ N^2\ ms$.

We also performed the same experiment considering randomly distributed different-sized circular items with uniformly distributed radii within the interval $[R_{\min}, R_{\max}]$ and $R_{\max}/R_{\min} = 10, 100, 1000$. The figures were virtualy the same. The explanation for this behaviour relies on the fact that, although a larger number of comparisons is done, while $N$ goes to infinity the number of neighbour items remains constant. Of course, extreme cases with just one large item and many small items such that the object is partitioned in almost a unique region will make the partitioning approach to behave as the naive approach (or even worse considering the overhead of the data structures manipulation).

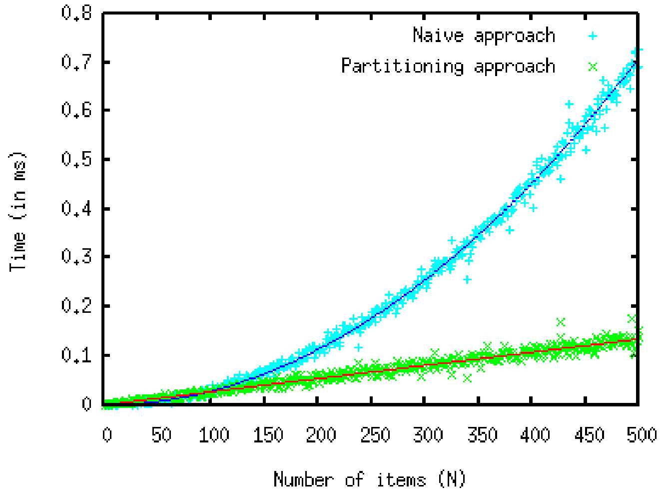


Figure 6: Practical performance of the naive and the partitioning approaches for the overlapping evaluation of an increasing number of randomly distributed identical items.

## 2.2 Object dimension and placing constraints

In this subsection we focus on the problem of fitting the items within the object while minimizing the object area or volume, and, in the cases in which it is not equivalent, the object perimeter or surface area. We consider several kinds of 2D and 3D objects.

If the object is a circle, to minimize its area is equivalent to minimize its radius $R$. Moreover, the fact that a circular item with radius $r_i$ belongs to the circular object (which, without loss of generality, can be considered centered at the origin) can be modeled as $d(c_i, 0)^2 \leq (R - r_i)^2$ and $R \geq r_i$. In the case of a squared object, to minimize its area is equivalent to minimize its side $L$. Moreover, assuming that the bottom-left corner of the object is fixed at the origin and that its sides are parallel to the axis, the fitting of a circular item with radius $r_i$ within the squared object with side $L$ can be modeled as $r_i \leq c_i \leq L - r_i$. The cases in which the object is an equilateral triangle, a rectangle or a strip are analogous. To consider the 3D objects counterparts is also analogous. The rectangle, the cuboid and the cylinder are the only cases we consider for which their dimensions depend on more than one parameter and, consequently, minimizing the area or volume is not equivalent to minimizing the perimeter or surface area. Tables 1 and 2 give the model for each 2D and 3D problem, respectively, as well as the number of variables and constraints. The extension to consider convex objects represented by linear constraints is trivial. General convex 2D regions were also considered in [11, 12].

According to the typology recently introduced in [37], problems in Tables 1 and 2 (identified by object type) can be classified as follows: (i) two-dimensional circular ODP (Open Dimension Problem): circle, square, 2D strip, rectangle and equilateral triangle; (ii) three-dimensional spherical ODP: sphere, cube, 3D strip, cuboid, tetrahedron, pyramid and cylinder.

| Object type | Model | # of variables | # of constraints |
|---|---|---|---|
| Circle | Min $R$<br>s.t. $(c_i^x)^2 + (c_i^y)^2 \le (R - r_i)^2,\ \forall\, i$<br>$R \ge r_{\max} \equiv \max_{i=1,\dots,N} \{r_i\}$<br>non-overlapping constraint (3) | $2N+1$ | $N+2$<br>(1 is a box constraint) |
| Square | Min $L$<br>s.t. $r_i \le c_i^x \le L - r_i,\ \forall\, i$<br>$r_i \le c_i^y \le L - r_i,\ \forall\, i$<br>non-overlapping constraint (3) | $2N+1$ | $4N+1$<br>($2N$ are box constraints) |
| Strip | Min $W$<br>s.t. $r_i \le c_i^x \le L - r_i,\ \forall\, i$<br>$r_i \le c_i^y \le W - r_i,\ \forall\, i$<br>non-overlapping constraint (3) | $2N+1$ | $4N+1$<br>($3N$ are box constraints) |
| Rectangle | Min $LW$ or Min $L+W$<br>s.t. $r_i \le c_i^x \le L - r_i,\ \forall\, i$<br>$r_i \le c_i^y \le W - r_i,\ \forall\, i$<br>non-overlapping constraint (3) | $2N+2$ | $4N+1$<br>($2N$ are box constraints) |
| Equilateral triangle | Min $L$<br>s.t. $c_i^y \ge r_i,\ \forall\, i$<br>$6c_i^x + 2\sqrt{3}c_i^y \le 3L - 4\sqrt{3}r_i,\ \forall\, i$<br>$-6c_i^x + 2\sqrt{3}c_i^y \le 3L - 4\sqrt{3}r_i,\ \forall\, i$<br>non-overlapping constraint (3) | $2N+1$ | $3N+1$<br>($N$ are box constraints) |

Table 1: 2D models for minimizing the object dimension when the object is a circle, a square, a strip, a rectangle and an equilateral triangle. (In the nonlinear programming context, box (or bound) constraints are considered easy constraints.)

# 3 Numerical experiments

We are interested in finding a global solution of the proposed nonlinear programming models. To increase the probability of finding a global solution, we run a local solver starting from several random initial points. Given the number of items to be packed and the shape of the object, we first compute an upper bound for the parameter (or parameters) that defines the object dimension. Then, we randomly distribute the items within the overestimated object and run the local solver starting from this randomly generated configuration as initial guess. This process is repeated until a maximum allowed CPU time $T$ is exceeded. The best local solution is returned as a solution.

We chose Algencan [2, 1, 10, 7] as the local solver. Algencan is a recently introduced Augmented Lagrangian method for smooth general-constrained minimization. The method is fully described in [2] where extensive numerical experiments assess its reliability. Algencan is available as a part of the Tango Project (see the web site [6]). In the present implementation Algencan uses Gencan [9] to solve the bound-constrained subproblems. Gencan is an active-set method for bound-constrained minimization. Gencan adopts the leaving-face criterion of [8], that employs the spectral projected gradients defined in [13, 14]. For the internal-to-the-face minimization Gencan uses a general algorithm with a line search that combines backtracking and extrapolation. In the present available implementation, Gencan employs, for the direction chosen at each step inside the faces, a truncated-Newton approach with incremental quotients to approximate the matrix-vector products and memoryless BFGS preconditioners [10].

All the experiments were run on a 2GHz AMD Opteron 244 processor, 2Gb of RAM memory

| Object type | Model | # of variables | # of constraints |
|---|---|---|---|
| Sphere | Min $R$<br>s.t. $(c_i^x)^2 + (c_i^y)^2 + (c_i^z)^2 \le (R - r_i)^2,\ \forall i$<br>$R \ge r_{\max} \equiv \max_{i=1,\ldots,N} \{r_i\}$<br>non-overlapping constraint (3) | $3N+1$ | $N+2$<br>(1 is a box constraint) |
| Cube | Min $L$<br>s.t. $r_i \le c_i^x \le L - r_i,\ \forall i$<br>$r_i \le c_i^y \le L - r_i,\ \forall i$<br>$r_i \le c_i^z \le L - r_i,\ \forall i$<br>non-overlapping constraint (3) | $3N+1$ | $6N+1$<br>($3N$ are box constraints) |
| 3D strip | Min $H$<br>s.t. $r_i \le c_i^x \le L - r_i,\ \forall i$<br>$r_i \le c_i^y \le W - r_i,\ \forall i$<br>$r_i \le c_i^z \le H - r_i,\ \forall i$<br>non-overlapping constraint (3) | $3N+1$ | $6N+1$<br>($5N$ are box constraints) |
| Cuboid | Min $LWH$ or Min $LW + LH + WH$<br>s.t. $r_i \le c_i^x \le L - r_i,\ \forall i$<br>$r_i \le c_i^y \le W - r_i,\ \forall i$<br>$r_i \le c_i^z \le H - r_i,\ \forall i$<br>non-overlapping constraint (3) | $3N+3$ | $6N+1$<br>($3N$ are box constraints) |
| Tetrahedron | Min $L$<br>s.t. $2\sqrt{2}c_i^x - 2\sqrt{6}c_i^y + 2c_i^z \le \sqrt{6}L - 6r_i\ \forall i$<br>$2\sqrt{2}c_i^x + 2\sqrt{6}c_i^y + 2c_i^z \le \sqrt{6}L - 6r_i\ \forall i$<br>$-2\sqrt{2}c_i^x + c_i^z + r_i \le 0,\ \forall i$<br>$c_i^z \ge 0,\ \forall i$<br>non-overlapping constraint (3) | $3N+1$ | $4N+1$<br>($N$ are box constraints) |
| Pyramid | Min $L$<br>s.t. $2c_i^x + \sqrt{2}c_i^z \le L - \sqrt{6}r_i,\ \forall i$<br>$-2c_i^x + \sqrt{2}c_i^z \le L - \sqrt{6}r_i,\ \forall i$<br>$2c_i^y + \sqrt{2}c_i^z \le L - \sqrt{6}r_i,\ \forall i$<br>$-2c_i^y + \sqrt{2}c_i^z \le L - \sqrt{6}r_i,\ \forall i$<br>$c_i^z \ge 0,\ \forall i$<br>non-overlapping constraint (3) | $3N+1$ | $5N+1$<br>($N$ are box constraints) |
| Cylinder | Min $R^2H$ or Min $R(R+H)$<br>s.t. $(c_i^x)^2 + (c_i^y)^2 \le (R - r_i)^2,\ \forall i$<br>$R \ge r_{\max} \equiv \max_{i=1,\ldots,N} \{r_i\}$<br>$r_i \le c_i^z \le H - r_i,\ \forall i$<br>non-overlapping constraint (3) | $3N+2$ | $3N+2$<br>($N+1$ are box constraints) |

Table 2: 3D models for minimizing the object dimension when the object is a sphere, a cube, a 3D strip, a cuboid, a pyramid with equilateral triangles as faces and a square as base, a regular tetrahedron and a cylinder.

and Linux operating system. Codes are in Fortran77 and the compiler option "-O4" was adopted.

In a first set of experiments, we fixed $T = 1$ hour and $T = 1.5$ hours and solved all the 2D and 3D instances with up 50 unitary-radius items, respectively. In addition to the previous set of problems, we fixed $T = 4$ hours and solved all the instances with $55, 60, \dots, 95$ and 100 unitary-radius items. Tables 5, 6 and 7 show the solution the method found for each combination of number of items and 2D and 3D object type, respectively. When the object is a circle or a square, the obtained solutions coincide (up to a prescribed tolerance) with the ones reported in [32]. For equilateral triangles with up to 15 items, the obtained results also coincide with the ones reported in [16]. For all the other 2D and 3D objects there are no previously reported results. (The results reported in [32] for rectangular objects assume that weight/width = 0.1.) Figures 7 and 8 illustrates a few selected solutions (3D figures were generated using VMD [23] and Raster3D [4]).

To give an idea of the computational cost of the present approach, Table 8 presents a few figures related to the 2D problems with circular and squared objects. The table shows the total number of nonlinear programming problems that were solved and the elapsed CPU time until the best solution was found. (The remaining time, to complete the maximum allowed CPU time $T$, was spent just to confirm that a better solution could not be found.) The computational effort of the method deserves some explanation. Larger the number of items, larger the amount of (undesired) local minimizers of the models. So, when the number of items increases, the simple multi-start global optimization strategy needs more local minimizations to find the "global" minimizer. The combination of the present approach with more sophisticated global optimization techniques might improve the computational performance of the method.

To evaluate the quality of the obtained solutions for the problems that minimize the object area (volume), we compared them with a simple lower bound given by the sum of the items area (volume). To compute a lower bound on the perimeter (surface area) of an object, we proceeded as follows. First, a lower bound $a_{lb}$ ($v_{lb}$) on its area (volume) was computed. Then, we analiticaly solved the problem of minimizing its perimeter (surface area) subject to the object area (volume) being greater than or equal to $a_{lb}$ ($v_{lb}$) and fitting at least the largest item. Table 3 shows, for each type of object, the average relative distance to the lower bound, computed as

$$\text{relative distance} = \frac{\text{solution found} - \text{lower bound}}{\text{lower bound}},$$

and the average density of the packings obtained. Note that, as it is known that optimal solutions were found for circular and squared objects (and also for equilateral triangular objects up to 15 items), the comparison against the figures (average relative distance to the bound and average packing density) of those cases can be used to evaluate the quality of the other cases. However, the comparison must be done with care, as it is expected, for example, that the density of a packing within a square to be worse than the density of a packing within a rectangle.

In another set of experiments, we fixed $T = 24$ hours and tested the behaviour of the proposed method in the strip packing problems with different-sized circular and spherical items considered in [35] and [36], respectively. The number of items varies from 25 to 60 and the value of the fixed dimensions of the strips as well as the radii of the items can be found in [36, 35]. Table 4 shows the results. The table shows a lower bound based on the areas or volumes ratio, the solutions obtained in [36, 35] and by the present approach, the total number of nonlinear programming

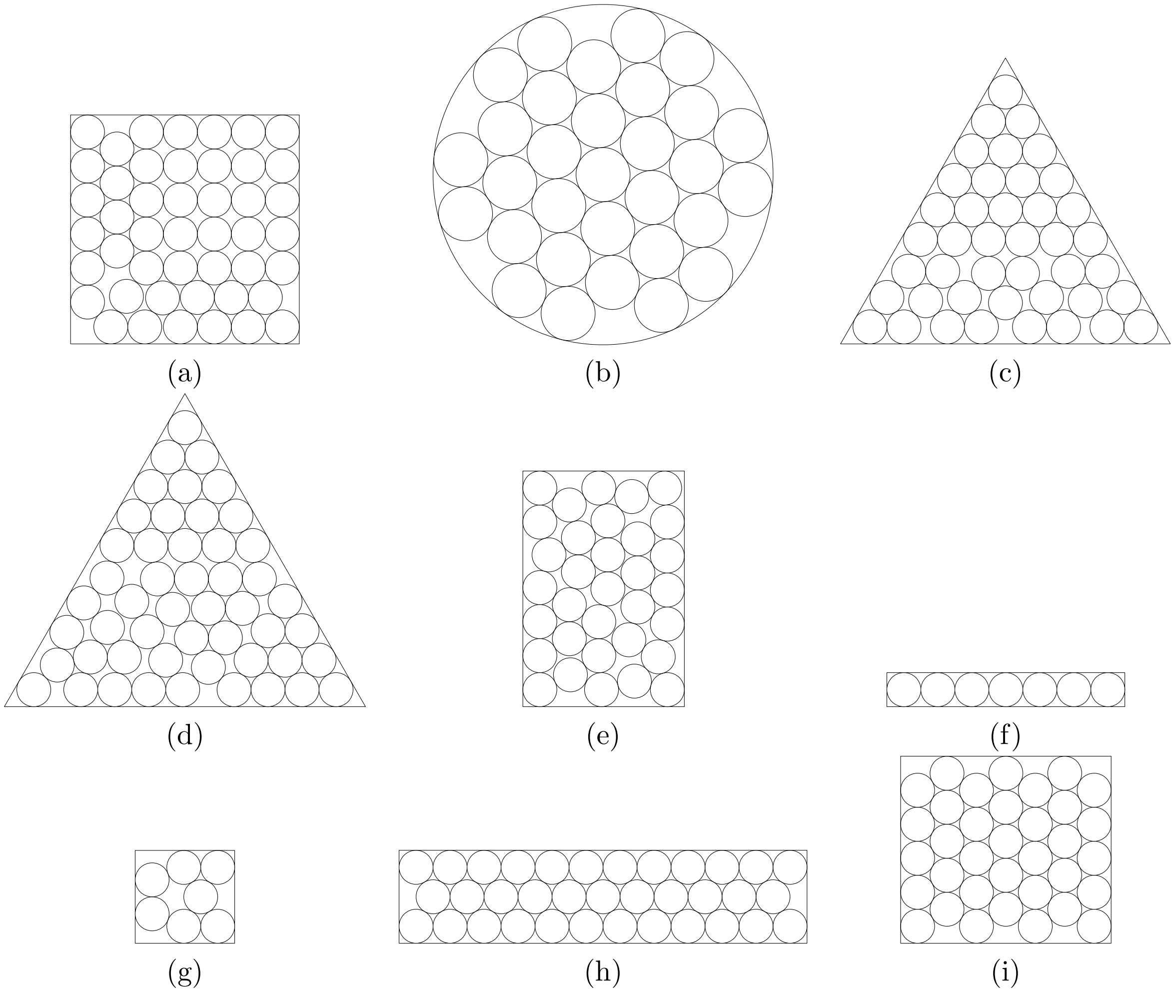


Figure 7: Selected pictures of 2D unitary-radius circle packing problems: (a) 46 circles in a square of size 13.4626063029, (b) 31 circles in a circle of radius 6.2907435849, (c)-(d) 42 and 50 circles in equilateral triangles of size 19.4045801371 and 21.2440452846, respectively, (e) 33 circles in a strip of fixed length 9.5 and width 13.8305751217, (f)-(g) 7 circles within a rectangle, minimizing area ($13.998199825 \times 2.0$) and perimeter ($5.8609504075 \times 5.4637640106$), respectively, (h)-(i) 35 circles within a rectangle, minimizing area ($23.9970048173 \times 5.463736504$) and perimeter ($12.3910375234 \times 10.9989629363$), respectively.

problems that were solved and the elapsed CPU time until the best solution was found. Figure 9 displays the graphical representation of the solutions. While the present approach failed to obtain good quality solutions in the two 2D problems, it was able to find better solutions in four over the six 3D problems.

As final examples of the wide range of applicability the present unified approach can have, Figure 10a shows the solution found for the problem of packing 100 different-sized spheres in a

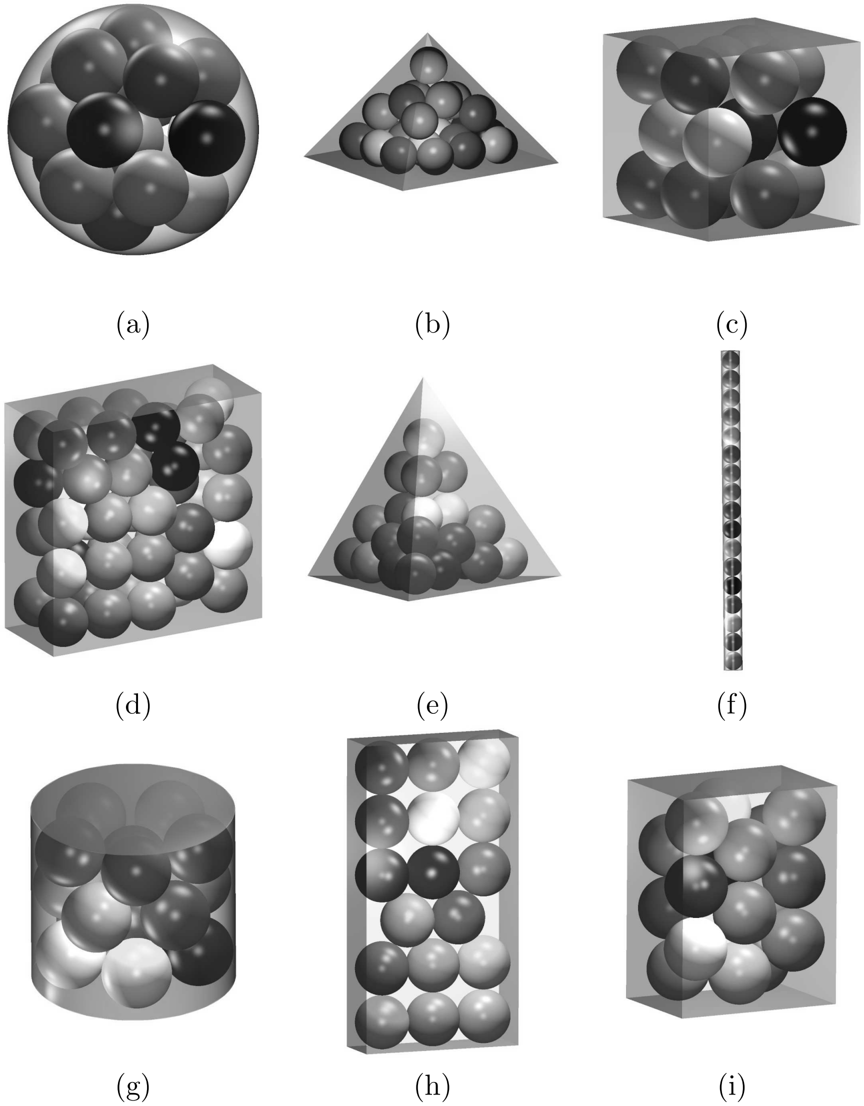


Figure 8: Selected pictures of 3D unitary-radius sphere packing problems: (a) 17 spheres in a sphere of radius 3.2711271196, (b) 24 spheres in a squared pyramid of size 9.8116682857, (c) 18 spheres in a cube of size 5.3279843248, (d) 44 spheres in a three-dimensional strip of fixed length 9.5, fixed width 9.5 and height 3.9867870680, (e) 23 spheres in a regular tetrahedron of size 12.2090353736, (f)-(g) 17 spheres within a cylinder minimizing the volume (height 33.9972524429 and radius 1.0) and the surface area (height 5.2362405719 and radius 2.9018413949), respectively, and (h)-(i) 17 spheres within a cuboid minimizing the volume ($11.4636013874 \times 5.9997955157 \times 2.0$) and the surface area ($5.4638212027 \times 6.9742074355 \times 3.7319493416$), respectively.

| Object Type | | Average distance to lower bound | Average density |
|---|---|---|---|
| 2D | Circle | 0.3143 | 0.7661 |
| | Square | 0.3091 | 0.7681 |
| | Strip | 0.4216 | 0.7494 |
| | Rectangle (area) | 0.2197 | 0.8206 |
| | Rectangle (perimeter) | 0.1259 | 0.7919 |
| | Equilateral triangle | 0.2878 | 0.7846 |
| 3D | Sphere | 1.0598 | 0.4997 |
| | Cube | 1.0497 | 0.4960 |
| | Strip | 2.9543 | 0.4082 |
| | Cuboid (volume) | 0.8211 | 0.5501 |
| | Cuboid (surface area) | 0.5877 | 0.5364 |
| | Tetrahedron | 1.1965 | 0.4706 |
| | Pyramid | 1.2277 | 0.4622 |
| | Cylinder (volume) | 0.4999 | 0.6667 |
| | Cylinder (surface area) | 0.5742 | 0.5228 |

Table 3: Statistics related to the solutions found for the 2D and 3D instances with up 50 unitary-radius items.

tetrahedron. Finally, we also consider the case in which the object is given by a set of linear equations. In this case, we will define the objective of the problem as minimizing the object dimension (area, perimeter, volume or surface area) by preserving its "shape and proportion among sides or faces". In this case, the problem can be posed as

$$\begin{array}{ll} \text{Minimize} & L \\ \text{s.t.} & Ac_i + r_i u \leq Lb, \ \forall\, i \\ & \text{non-overlapping constraint (3)} \end{array} \tag{6}$$

where $A \in I\!R^{k \times n}$, $b, u = (\|a_1\|_2, \ldots, \|a_k\|_2)^T \in I\!R^k$, $a_j$ is the $j$-th row of $A$ and $k$ is the number of constraints that represent the object. As an example (see Figure 10b), consider the three-dimensional object given by:

$$A = \begin{pmatrix} -16 & & 4 \\ & 48 & 24 \\ 16 & & 24 \\ & -48 & 24 \\ & & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 160 \\ 96 \\ 32 \\ 96 \\ 0 \end{pmatrix}. \tag{7}$$

# 4 Final remarks

We introduced a large variety of twice-differentiable nonlinear programming models for the problem of minimizing the object dimension in 2D and 3D packing problems. Identical and

| Problem | | Lower bound | Solutions found | | Effort measurements | |
|---|---|---|---|---|---|---|
| | | | In [36, 35] | Using GENPACK | Number of trials | CPU time is secs. |
| 2D | (a) | 12.2332 | 14.3785 | 14.9509618123 | 529 | 184.70 |
| | (b) | 14.5496 | 17.1968 | 18.0340842834 | 54221 | 27735.22 |
| 3D | (c) | 5.0712 | 9.8667 | 9.7941763980 | 58563 | 29320.06 |
| | (d) | 5.7561 | 9.6220 | 11.0128604540 | 33918 | 52060.52 |
| | (e) | 4.8487 | 9.4728 | 9.3089999467 | 26851 | 41308.01 |
| | (f) | 5.8572 | 11.0862 | 11.0962093979 | 26806 | 85617.91 |
| | (g) | 6.1663 | 11.6453 | 11.6210793508 | 6418 | 28081.08 |
| | (h) | 6.8115 | 12.8415 | 12.7215414636 | 3829 | 19170.46 |

Table 4: Performance of the method in the strip packing problems with different-sized circular and spherical items from [36, 35].

different-sized items were considered. We implemented an efficient methodology to reduce the computational cost of computing the overlapping. A practical method was used to solve all the proposed models, attesting its applicability. Moreover, the presented methodology is fully parallelizable. The combination of the present approach with clever problem-dependent global optimization techniques like the one developed in [34, 36, 35] would be a line for future research.

The complete Fortran 77 sources codes of the algorithms and models presented in this paper are available in `http://www.ime.usp.br/∼egbirgin/`.

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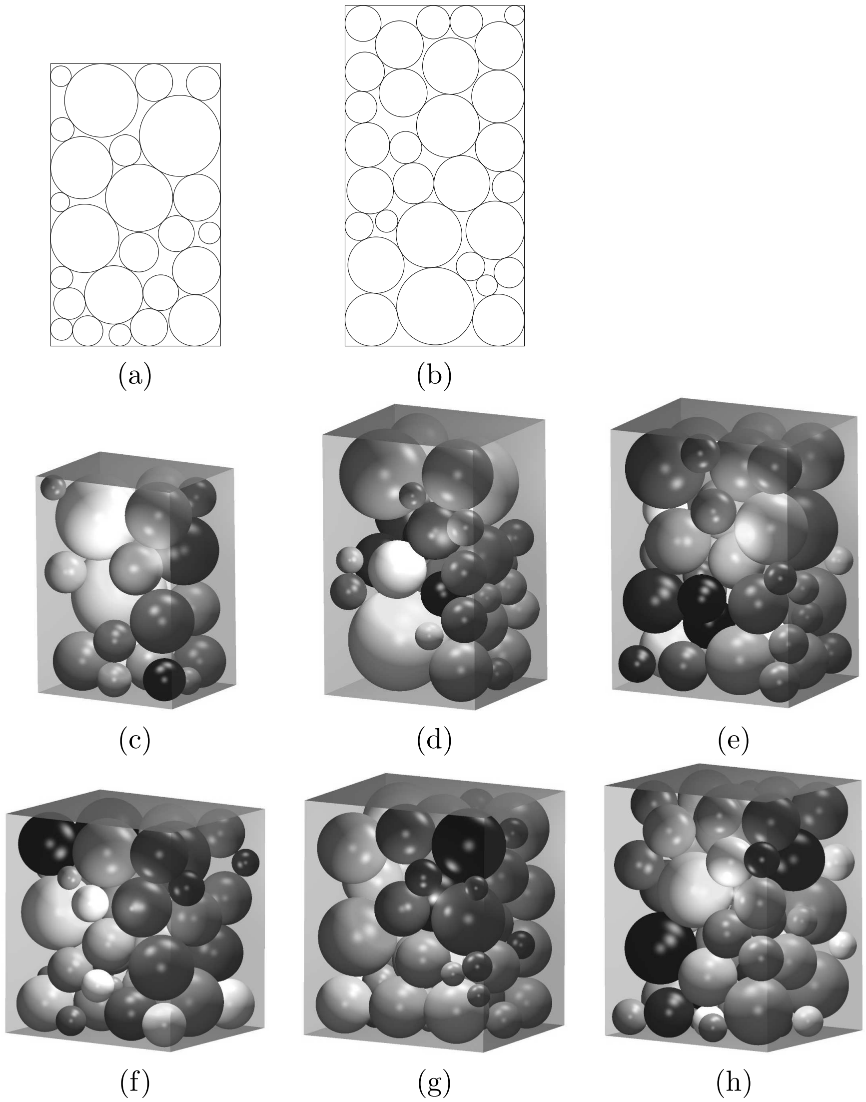
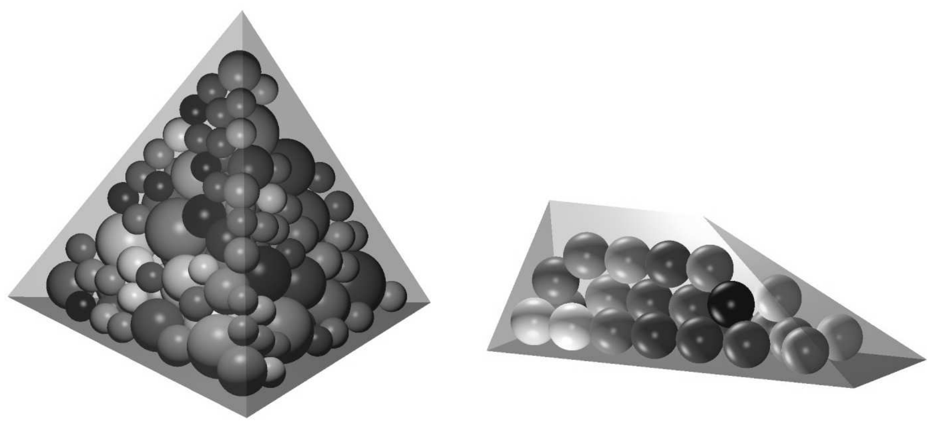



Figure 9: Problems of packing different-sized circular and spherical items within strips from [36, 35].



(a) (b)

Figure 10: (a) Packing of 100 different-sized spheres within a tetrahedron. There are 50 unitary-radius spheres and 50 spheres with radius $1.025, 1.050, \ldots, 2.250$. The dimension of the tetrahedron is given by $L = 27.3333464485$. (b) Packing of 25 unitary-radius spheres into a 3D object given by the arbitrary linear constraints (6–7). Solution: $L = 1.4999323248$.

| Number of items | Circle $R$ | Square $L$ | Strip $L$ | Triangle $L$ | Rectangle (area) $L$ | Rectangle (area) $W$ | Rectangle (perimeter) $L$ | Rectangle (perimeter) $W$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1.0000000000 | 1.9999986363 | 2.0000000000 | 3.4641016146 | 2.0000000000 | 2.0000000000 | 2.0000000000 | 2.0000000000 |
| 2 | 1.9996222546 | 3.4137477127 | 2.0000000000 | 5.4633621099 | 3.9994547642 | 2.0000000000 | 3.9992604948 | 2.0000000000 |
| 3 | 2.1544558927 | 3.9313808636 | 1.9999911315 | 5.4636533311 | 2.0000000000 | 5.9991345164 | 3.9995011744 | 3.7317273490 |
| 4 | 2.4139375051 | 3.9996299811 | 2.0000000000 | 6.9274637829 | 3.9996565400 | 3.9996565400 | 3.9996302816 | 3.9996302816 |
| 5 | 2.7010098451 | 4.8279613489 | 2.6952686399 | 7.4634468126 | 9.9986261610 | 2.0000000000 | 3.9998611307 | 5.4633052124 |
| 6 | 2.9996508504 | 5.3277596117 | 3.3224368208 | 7.4636709802 | 3.9997150028 | 5.9994299921 | 3.9997760936 | 5.9993282244 |
| 7 | 2.9997761921 | 5.7315971190 | 3.5608566245 | 8.9273837492 | 13.9981998250 | 2.0000000000 | 5.8609504075 | 5.4637640106 |
| 8 | 3.3044206851 | 5.8632353309 | 3.6884793600 | 9.2930737754 | 7.9993979180 | 3.9997993110 | 5.9995344103 | 5.4637633218 |
| 9 | 3.6127740710 | 5.9995518554 | 3.9993016842 | 9.4634763151 | 5.9996222843 | 5.9996222843 | 5.9995521684 | 5.9995521684 |
| 10 | 3.8126788316 | 6.7466961038 | 4.6951121788 | 9.4636358192 | 9.9992644508 | 3.9998161178 | 5.9996594681 | 7.1956196062 |
| 11 | 3.9228091423 | 7.0210502213 | 5.1211545417 | 10.7276943421 | 5.4634407024 | 7.9987047809 | 5.9991924047 | 7.4624830643 |
| 12 | 4.0286834176 | 7.1437936464 | 5.3765047895 | 10.9264246875 | 7.9986295356 | 5.9990863749 | 7.9982454190 | 5.9991228063 |
| 13 | 4.2349654525 | 7.4630289424 | 5.8523347064 | 11.4044366767 | 2.0000000000 | 25.9938432497 | 7.4627500826 | 7.4627500827 |
| 14 | 4.3275521837 | 7.7309165102 | 5.9980828464 | 11.4625024049 | 9.9984346514 | 5.4635325793 | 7.9984352858 | 7.1953665961 |
| 15 | 4.5206216997 | 7.8626415945 | 6.6937425787 | 11.4627797055 | 7.9988353651 | 7.4630923328 | 7.9986635705 | 7.4630647263 |
| 16 | 4.6143481044 | 7.9989556782 | 7.0649643624 | 12.7110751779 | 3.7318388122 | 16.9966762489 | 7.9986841453 | 7.9986841451 |
| 17 | 4.7912398107 | 8.5309003541 | 7.4495386792 | 12.9261773819 | 5.4635902166 | 11.9981939658 | 8.9131925338 | 7.9543346939 |
| 18 | 4.8627806310 | 8.6552029299 | 7.8517294049 | 13.2910260295 | 3.7318475438 | 18.9964440023 | 7.9990444062 | 8.9268910590 |
| 19 | 4.8629278433 | 8.9061300551 | 7.9976715063 | 13.4456687039 | 7.4632082242 | 9.9985864965 | 9.4625659570 | 7.9990156750 |
| 20 | 5.1212784390 | 8.9768656326 | 8.6935864074 | 13.4625490829 | 5.4636308244 | 13.9979711566 | 8.9272322541 | 8.9986390784 |
| 21 | 5.2512555209 | 9.3563084678 | 9.1690829563 | 13.4627429100 | 5.4636886104 | 14.9975772973 | 9.2167842122 | 9.4339701956 |
| 22 | 5.4390007858 | 9.4622675308 | 9.4490464873 | 14.6101271902 | 3.7318614400 | 22.9960005512 | 9.9407136592 | 8.9271510823 |
| 23 | 5.5445265607 | 9.7309291904 | 9.8510937710 | 14.8805620808 | 5.4636615016 | 15.9977610644 | 8.9273391176 | 9.9986129160 |
| 24 | 5.6508749142 | 9.8625920424 | 9.9971058011 | 14.9264967836 | 5.4637089641 | 16.9973842151 | 9.4629665387 | 9.9986287544 |
| 25 | 5.7520474056 | 9.9989092425 | 10.6930072130 | 15.2915870167 | 3.7318697115 | 25.9956839018 | 10.9982515685 | 8.9275611421 |
| 26 | 5.8274395455 | 10.3761296161 | 11.1687967336 | 15.4568486004 | 5.4636857741 | 17.9975609058 | 10.6588451987 | 9.9409595292 |
| 27 | 5.9054946032 | 10.4788058721 | 11.4484657320 | 15.4625744338 | 5.4637257927 | 18.9971982173 | 10.6591024240 | 9.9988547158 |
| 28 | 6.0141566559 | 10.6754220133 | 11.8505484609 | 15.4627172406 | 11.9985828103 | 8.9273321305 | 9.9989366878 | 10.9268949096 |
| 29 | 6.1377763523 | 10.8137503471 | 11.9966464202 | 16.6030248601 | 5.4637056254 | 19.9973689183 | 11.4625890050 | 9.9989080012 |
| 30 | 6.1970915627 | 10.9072687500 | 12.6932092975 | 16.7281252215 | 20.9970183919 | 5.4637400405 | 10.6592978781 | 10.9986516064 |
| 31 | 6.2907435849 | 11.1915352105 | 13.1685323354 | 16.9263595271 | 15.9979597338 | 7.4633846242 | 10.9270095378 | 11.4247556202 |
| 32 | 6.4287631231 | 11.3803881756 | 13.4480140426 | 17.2453120324 | 21.9971838652 | 5.4637222711 | 11.9404852062 | 10.6593828953 |
| 33 | 6.4856320243 | 11.4625012404 | 13.8472008358 | 17.4039294262 | 13.9984103697 | 8.9273990398 | 11.9985032730 | 10.6594384363 |
| 34 | 6.6101463641 | 11.7309249171 | 13.9962078766 | 17.4587344689 | 7.1956075610 | 17.9977228032 | 11.9986028314 | 10.9272272456 |
| 35 | 6.6964365742 | 11.8625361315 | 14.6929957747 | 17.4625898173 | 23.9970048173 | 5.4637365040 | 12.3910375234 | 10.9989629363 |
| 36 | 6.7459641444 | 11.9988792034 | 15.1690615764 | 17.4626981458 | 5.4637630915 | 24.9966743828 | 10.6595680742 | 12.9983311035 |
| 37 | 6.7579310827 | 12.1804216686 | 15.4475690694 | 18.5287831373 | 5.4637358338 | 25.8578787169 | 12.3031569983 | 11.9965783735 |
| 38 | 6.9610838204 | 12.2374434375 | 15.8270155555 | 18.7266870206 | 5.4637488666 | 25.9968310515 | 11.9817725699 | 12.3908769558 |
| 39 | 7.0571911486 | 12.2887512850 | 15.9957923056 | 18.9140027395 | 5.4637726295 | 26.9965090852 | 10.6596663838 | 13.9981945067 |
| 40 | 7.1231096220 | 12.6267382011 | 16.6927575874 | 18.9265358552 | 20.9974831288 | 7.1956410497 | 12.9268799669 | 11.9988610018 |
| 41 | 7.2592804090 | 12.7456257952 | 17.1686399573 | 19.2916298956 | 5.4637597441 | 27.9966619868 | 12.9406947738 | 12.3912989627 |
| 42 | 7.3462119818 | 12.8518646718 | 17.4477019156 | 19.4045801371 | 7.1956025496 | 21.9971608203 | 12.3913577634 | 12.9986607790 |
| 43 | 7.4192209173 | 13.0972405183 | 17.8098929473 | 19.4596912793 | 17.9980933123 | 8.9274901429 | 13.5989755952 | 12.3917670632 |
| 44 | 7.4972867738 | 13.1939383256 | 17.9953961237 | 19.4625983940 | 29.9964971470 | 5.4637694187 | 12.3914982006 | 13.9404577794 |
| 45 | 7.5722836395 | 13.3803865970 | 18.6925377387 | 19.4626504950 | 5.4637888824 | 30.9961898134 | 13.9814312021 | 12.3913995511 |
| 46 | 7.6494228038 | 13.4626063029 | 19.1682027497 | 20.5249300923 | 7.1956672420 | 23.9972574437 | 13.9985220814 | 12.6593249459 |
| 47 | 7.7233640079 | 13.6751033111 | 19.4475892668 | 20.7011666871 | 5.4637781024 | 31.9963361352 | 12.9271460134 | 13.9985889130 |
| 48 | 7.7905553343 | 13.8047806547 | 19.8069204856 | 20.8799581334 | 19.9979450130 | 8.9275232377 | 12.9989194323 | 14.1231541809 |
| 49 | 7.8860817787 | 13.9472637804 | 19.9948589623 | 20.9264402987 | 5.4637306400 | 33.9945655425 | 12.6173802384 | 15.0112526170 |
| 50 | 7.9468087710 | 14.0083589613 | 20.6923384169 | 21.2440452148 | 5.4637859581 | 33.9961786161 | 13.6373330997 | 14.3820364308 |
| 55 | 8.2101683532 | 14.6923955360 | 22.6835225197 | 21.4622516388 | 8.9275832375 | 22.9976175240 | 14.1524486932 | 14.9442580942 |
| 60 | 8.6454213199 | 15.3755552334 | 24.6785675081 | 22.9265606115 | 17.9983545775 | 12.3913421248 | 14.1235217033 | 15.9985449327 |
| 65 | 9.0166711986 | 15.8203131072 | 26.6507629243 | 23.4622014647 | 5.4638164895 | 43.9954346000 | 15.7232054364 | 15.9601095466 |
| 70 | 9.3451963876 | 16.4998033416 | 28.6306926662 | 24.8807696211 | 8.9276367873 | 28.9972283659 | 15.8552623767 | 16.9113958554 |
| 75 | 9.6711933041 | 17.0956713219 | 30.6445646992 | 25.4558282921 | 8.9276510682 | 30.9971052432 | 17.9965380801 | 15.8417107227 |
| 80 | 9.9673713500 | 17.4293700602 | 32.6135139169 | 26.5919337671 | 8.9276638633 | 32.9969839021 | 18.9231113668 | 15.9766418153 |
| 85 | 10.1624343053 | 17.9585744521 | 34.5997299255 | 27.2455664377 | 34.9968654011 | 8.9276756174 | 17.9969843318 | 17.8547723425 |
| 90 | 10.5452581347 | 18.6092727313 | 36.6010872992 | 27.4621886433 | 10.6597334412 | 30.9978987556 | 19.1320378453 | 17.9971314230 |
| 95 | 10.8393458264 | 19.0830329823 | 38.5727471051 | 28.7283473468 | 12.3916727778 | 27.9982222555 | 19.9950093368 | 18.0780528627 |
| 100 | 11.0816840023 | 19.4522055083 | 40.5514750264 | 29.3893128146 | 12.3916727778 | 27.9982222555 | 19.5888129645 | 19.3655902612 |

Table 5: Numerical results for the 2D problems.

| Number of items | Sphere $R$ | Cube $L$ | Cuboid (volume) $L$ | $W$ | $H$ | Cuboid (surface) $L$ | $W$ | $H$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1.0000005143 | 1.9999987021 | 1.9999994082 | 1.9999994082 | 1.9999994082 | 1.9999998015 | 1.9999998015 | 1.9999998015 |
| 2 | 1.9999202433 | 3.1546039826 | 2.0000000000 | 3.9998520179 | 2.0000000000 | 3.9998520181 | 2.0000000000 | 2.0000000000 |
| 3 | 2.1546210616 | 3.4141527373 | 2.0000000000 | 5.9997650951 | 2.0000000000 | 2.0000000000 | 2.0000000000 | 5.9997650951 |
| 4 | 2.2246870768 | 3.4141480640 | 3.9999067784 | 3.9999067784 | 2.0000000000 | 3.9999153028 | 2.0000000000 | 3.9999153028 |
| 5 | 2.4141267898 | 3.7887647862 | 9.9996271139 | 2.0000000000 | 2.0000000000 | 5.4639348176 | 2.0000000000 | 3.9999512713 |
| 6 | 2.4141694573 | 3.8855478230 | 3.9999226428 | 5.9998452847 | 2.0000000000 | 3.9999355322 | 2.0000000000 | 5.9998549455 |
| 7 | 2.5911991946 | 3.9977338382 | 3.9999400751 | 2.0000000000 | 7.4638804452 | 3.9999563663 | 2.0000000000 | 7.4638868064 |
| 8 | 2.6452822222 | 3.9999345267 | 3.9999353642 | 3.9999353642 | 3.9999353642 | 5.9998908944 | 2.0000000000 | 5.4640108186 |
| 9 | 2.7320034623 | 4.3093402273 | 5.9998707285 | 5.9998707285 | 2.0000000000 | 5.9998870712 | 2.0000000000 | 5.9998870712 |
| 10 | 2.8324167450 | 4.6666081946 | 3.9999370664 | 9.9997482632 | 2.0000000000 | 3.7320243621 | 3.7320243621 | 5.9998991367 |
| 11 | 2.9019556947 | 4.8159601052 | 2.0000000000 | 5.4639222153 | 7.9996484758 | 5.4638610588 | 2.0000000000 | 7.9994404199 |
| 12 | 2.9020105031 | 4.8280686639 | 3.9998824683 | 3.9998824683 | 5.9997649343 | 7.9994337904 | 2.0000000000 | 5.9996461265 |
| 13 | 2.9997977416 | 4.8282131208 | 2.0000000000 | 5.9997946298 | 8.9278206589 | 3.9999337537 | 3.9999337537 | 6.8277518754 |
| 14 | 3.0910230631 | 4.8282912625 | 2.0000000000 | 9.9995751531 | 5.4639471485 | 5.9997880092 | 3.9999467126 | 4.8281141461 |
| 15 | 3.1415101582 | 5.1997445225 | 7.4638276614 | 2.0000000000 | 7.9996838961 | 5.9997645604 | 3.7319723788 | 5.4638832882 |
| 16 | 3.2155339532 | 5.2964155013 | 3.7319816590 | 8.9996807054 | 3.7319816590 | 6.2423258359 | 3.4142320441 | 6.2423258359 |
| 17 | 3.2711271196 | 5.2996749577 | 5.9997955157 | 2.0000000000 | 11.4636013874 | 6.9742074355 | 3.7319493416 | 5.4638212027 |
| 18 | 3.3188360708 | 5.3279843248 | 9.9996565716 | 3.7319855151 | 3.7319855151 | 5.4639276546 | 3.7319915897 | 6.9996882715 |
| 19 | 3.3858513801 | 5.4586383844 | 9.4637456920 | 7.9997083203 | 2.0000000000 | 4.8282103240 | 3.9999557695 | 7.9996695729 |
| 20 | 3.4733633429 | 5.6048721029 | 3.7319889320 | 10.9996304432 | 3.7319889320 | 7.1958225538 | 3.7319840261 | 5.9998330126 |
| 21 | 3.4862092764 | 5.6431452803 | 5.4639638751 | 7.9997917071 | 3.7319840662 | 7.9996463417 | 3.7320008721 | 5.4639501784 |
| 22 | 3.5796881818 | 5.7710452688 | 5.9998723682 | 5.9998723682 | 4.8282875226 | 5.4639607819 | 3.9999031097 | 7.9996349464 |
| 23 | 3.6273889908 | 5.8199237336 | 7.6565874480 | 4.8283548413 | 4.8283548413 | 7.1956140331 | 3.7319880108 | 6.9995678216 |
| 24 | 3.6852536176 | 5.8633943713 | 5.4639744767 | 8.9997659859 | 3.7319892222 | 7.1958834279 | 3.7320005199 | 6.9997694833 |
| 25 | 3.6872896999 | 5.9589662351 | 5.9998979005 | 8.9279025044 | 3.7319795011 | 8.9277722105 | 3.7319917901 | 5.9998728297 |
| 26 | 3.7471886302 | 5.9952148260 | 3.7319762835 | 9.9742707123 | 5.4639484659 | 7.1959593106 | 3.7320172791 | 7.9658996899 |
| 27 | 3.8132974881 | 5.9998355203 | 9.9997461864 | 5.4639818590 | 3.7319928014 | 9.9995542616 | 3.7320141670 | 5.4639826430 |
| 28 | 3.8415149358 | 6.2421317695 | 7.9998295947 | 7.1959606322 | 3.7319897620 | 7.1959176362 | 3.7320092538 | 7.9997307495 |
| 29 | 3.8769556364 | 6.2423824504 | 3.7319826950 | 10.9742467685 | 5.4639615210 | 8.9278148456 | 3.7320060834 | 6.9997214200 |
| 30 | 3.9163640724 | 6.2424721518 | 10.9997251845 | 5.4639882208 | 3.7319958897 | 8.9278471493 | 3.7320065066 | 6.9998178874 |
| 31 | 3.9506163857 | 6.2425174557 | 7.9996954592 | 5.4639083563 | 5.4638915004 | 8.9996802550 | 3.7320300978 | 7.1959753668 |
| 32 | 3.9873139176 | 6.2425285827 | 5.4639777146 | 7.9998611848 | 5.4639777146 | 7.9997796243 | 5.4639770041 | 5.4639770041 |
| 33 | 4.0197789249 | 6.4689813902 | 11.9997062916 | 5.4639933032 | 3.7319983540 | 9.8130876582 | 3.7320307315 | 7.1960389489 |
| 34 | 4.0475629785 | 6.5735658875 | 5.4639912617 | 12.8589488963 | 3.7319936763 | 9.9664645648 | 3.7320116113 | 7.4612564245 |
| 35 | 4.0842793950 | 6.5931612342 | 5.4639946940 | 12.9996092372 | 3.7319987413 | 9.0706267144 | 4.8284135759 | 6.2425430006 |
| 36 | 4.1128778440 | 6.6970889746 | 5.4639977043 | 12.9996874801 | 3.7320004885 | 9.9996505797 | 3.7320211822 | 7.1959652997 |
| 37 | 4.1546179855 | 6.7083709308 | 7.6566693014 | 7.6566712985 | 4.8283733467 | 7.6565995398 | 4.8284183539 | 7.6565980201 |
| 38 | 4.1575074608 | 6.7093947495 | 4.8283780629 | 7.6566840081 | 7.6566840081 | 7.6566177480 | 4.8284198756 | 7.6566177480 |
| 39 | 4.2238096708 | 6.7739983433 | 5.4640014512 | 13.9996696629 | 3.7320023049 | 7.9997871781 | 5.4640126543 | 6.9978553162 |
| 40 | 4.2551873286 | 6.7998570644 | 7.1959946200 | 10.9997719786 | 3.7320006207 | 7.9997867024 | 5.2659037464 | 7.3884098761 |
| 41 | 4.2961778465 | 6.9039667243 | 5.4639942369 | 9.9998247954 | 5.4639942369 | 7.9996918020 | 5.4639887407 | 7.1958647739 |
| 42 | 4.3080119020 | 6.9906644142 | 7.1959834279 | 7.9998806718 | 5.4639906152 | 7.1959619143 | 5.4639991620 | 7.9998289682 |
| 43 | 4.3528504653 | 7.0610542561 | 11.9992433167 | 7.1959553361 | 3.7319926669 | 7.9998102754 | 5.4640058977 | 7.4638869282 |
| 44 | 4.3827055183 | 7.0991542522 | 11.9997554903 | 7.1960017830 | 3.7320029043 | 7.4638315875 | 5.7112633939 | 7.9998952839 |
| 45 | 4.4068820245 | 7.1269867567 | 10.9997985680 | 5.4640016367 | 5.4640016367 | 6.9998991121 | 5.4639942298 | 8.9279089439 |
| 46 | 4.4409860772 | 7.1396025069 | 7.1959915445 | 12.9657366193 | 3.7320002792 | 8.9944778512 | 5.4640390355 | 7.1960161968 |
| 47 | 4.4739825106 | 7.1447631846 | 12.9997061426 | 7.1960058286 | 3.7320057053 | 8.9996324205 | 5.4640009566 | 7.1959237566 |
| 48 | 4.4961576629 | 7.2254788705 | 7.1960079302 | 12.9997392813 | 3.7320048642 | 8.9997922699 | 5.4640138768 | 7.1959859052 |
| 49 | 4.5191083524 | 7.3396050716 | 11.9995595224 | 5.4639889656 | 5.4639845156 | 8.9279159160 | 5.2659430685 | 7.9998131188 |
| 50 | 4.5504157703 | 7.3606467872 | 11.9997927906 | 5.4640048144 | 5.4640048144 | 7.6566831702 | 6.2425762929 | 7.6566831702 |
| 55 | 4.6849920452 | 7.6497254384 | 11.9997854621 | 8.9280146257 | 3.7320062218 | 9.9997441915 | 5.4640300646 | 7.4639258867 |
| 60 | 4.7748065618 | 7.6567160689 | 10.9998345830 | 7.1960136241 | 5.4640104777 | 8.9998349213 | 5.4640250495 | 8.9279785504 |
| 65 | 4.9241573261 | 7.9389780072 | 8.9280289549 | 13.9997569867 | 3.7320096198 | 9.9994110181 | 5.4640152915 | 8.9278768296 |
| 70 | 5.0328821287 | 8.1572766782 | 7.6567464615 | 10.4850757364 | 6.2425743011 | 10.4849680177 | 6.2426124333 | 7.6567505843 |
| 75 | 5.1642307312 | 8.2424863321 | 10.9998561341 | 8.9280296531 | 5.4640168915 | 11.1805242617 | 5.5583422100 | 9.1634519725 |
| 80 | 5.2752317706 | 8.5400043137 | 7.6567555710 | 11.8992578456 | 6.2425798506 | 11.9991661607 | 5.4639641616 | 8.9277971955 |
| 85 | 5.3792538717 | 8.6924540639 | 11.9996488835 | 7.1959794230 | 7.1959936564 | 10.9993312884 | 5.4640567077 | 10.6601156192 |
| 90 | 5.4771826035 | 8.8678323052 | 13.3134414952 | 7.6567626911 | 6.2425842090 | 8.9269377677 | 7.1941376138 | 10.3636092004 |
| 95 | 5.5716172496 | 9.0184678703 | 8.9280290941 | 13.9997110994 | 5.4640176971 | 9.0875358264 | 6.3481408059 | 12.2608997182 |
| 100 | 5.6661018170 | 9.1663256540 | 8.9280659172 | 14.9594160981 | 5.4640342817 | 10.4850433554 | 7.6567796673 | 9.0709141038 |

Table 6: Numerical results for the 3D problems.

| Number of items | 3D strip | Tetrahedron | Pyramid | Cylinder (surface) | | Cylinder (volume) | |
|---|---|---|---|---|---|---|---|
| | $L$ | $L$ | $L$ | $H$ | $R$ | $H$ | $R$ |
| 1 | 2.0000000000 | 4.8989794855 | 3.8637033034 | 2.0000000000 | 1.0000000000 | 2.0000000000 | 1.0000000000 |
| 2 | 1.9999999511 | 6.8987786397 | 5.2777419314 | 2.0000000000 | 1.9998851315 | 3.9999799164 | 1.0000000000 |
| 3 | 1.9999996826 | 6.8988829315 | 5.7954078635 | 2.0000000000 | 2.1546325115 | 5.9999681193 | 1.0000000000 |
| 4 | 1.9999997558 | 6.8989186607 | 5.8636236011 | 7.9995822321 | 1.0000000000 | 7.9999582245 | 1.0000000000 |
| 5 | 1.9999998012 | 8.1648125393 | 5.8636160349 | 3.6329467608 | 2.1546454674 | 9.9999598635 | 1.0000000000 |
| 6 | 1.9999998345 | 8.7817120034 | 6.6919872393 | 3.6329550792 | 2.1546531385 | 11.9999534257 | 1.0000000000 |
| 7 | 1.9999998574 | 8.8988016274 | 7.1456607739 | 3.9999562296 | 2.3228259410 | 13.9999474063 | 1.0000000000 |
| 8 | 1.9999998750 | 8.8988805540 | 7.2778333226 | 3.6817732264 | 2.4141690175 | 15.9999417139 | 1.0000000000 |
| 9 | 1.9999998889 | 8.8989049422 | 7.5660753093 | 5.2659119030 | 2.1546760712 | 17.9999362872 | 1.0000000000 |
| 10 | 1.9999998872 | 8.8989186535 | 7.7273237304 | 3.7012882426 | 2.7012496035 | 19.9999310827 | 1.0000000000 |
| 11 | 1.9999998908 | 10.1473178899 | 7.8588071332 | 3.6532929729 | 2.8209499930 | 21.9979915089 | 1.0000000000 |
| 12 | 1.9999999018 | 10.3327450473 | 7.8632345878 | 5.3634470841 | 2.4141288826 | 23.9978597508 | 1.0000000000 |
| 13 | 1.9999998503 | 10.5039904936 | 7.8634093587 | 3.7111744027 | 2.9998379486 | 25.9977315770 | 1.0000000000 |
| 14 | 1.9999999132 | 10.7497129642 | 7.8634553793 | 3.9994148462 | 2.9961624606 | 27.9976076192 | 1.0000000000 |
| 15 | 1.9999998995 | 10.8777393372 | 8.6303103460 | 5.4025004405 | 2.7011994854 | 29.9974864572 | 1.0000000000 |
| 16 | 1.9999998967 | 10.8983529191 | 8.8467717390 | 7.0451550000 | 2.4141488724 | 31.9973681489 | 1.0000000000 |
| 17 | 1.9999999316 | 10.8985312575 | 8.9751441419 | 5.2362405719 | 2.9018413949 | 33.9972524429 | 1.0000000000 |
| 18 | 1.9999999092 | 10.8986448409 | 9.1412510454 | 5.4201459445 | 2.9567222851 | 35.9971391244 | 1.0000000000 |
| 19 | 1.9999999091 | 10.8986690234 | 9.2769864393 | 5.4223037529 | 2.9998633645 | 37.9970280076 | 1.0000000000 |
| 20 | 1.9999999175 | 10.8987161340 | 9.2776202362 | 5.4223197427 | 2.9998806834 | 39.9969189309 | 1.0000000000 |
| 21 | 1.9999999449 | 11.9107290304 | 9.5209353533 | 5.9988407441 | 2.9896028065 | 41.9968117522 | 1.0000000000 |
| 22 | 1.9999999378 | 12.1644939511 | 9.5954475686 | 6.9596198256 | 2.8210553449 | 43.9967063459 | 1.0000000000 |
| 23 | 2.5159516472 | 12.2090353736 | 9.7466361361 | 6.7723208885 | 2.9612676153 | 45.9966026003 | 1.0000000000 |
| 24 | 2.6231977042 | 12.4099719051 | 9.8116682857 | 7.1300489686 | 2.9567312663 | 47.9965004154 | 1.0000000000 |
| 25 | 2.6956448382 | 12.5741811638 | 9.8544161043 | 7.4064098865 | 2.9232129228 | 49.9963997013 | 1.0000000000 |
| 26 | 2.9617066600 | 12.6747046630 | 9.8627741922 | 7.1334431350 | 2.9998954899 | 51.9963003768 | 1.0000000000 |
| 27 | 3.0684401131 | 12.7736673573 | 9.8633027680 | 7.1334608972 | 2.9999050186 | 53.9962023683 | 1.0000000000 |
| 28 | 3.1196255500 | 12.8600469119 | 9.8634416955 | 7.9968894897 | 2.9844231581 | 55.9961056087 | 1.0000000000 |
| 29 | 3.1913632493 | 12.8842225810 | 9.8634679168 | 7.9997982380 | 3.0407002526 | 57.9960100366 | 1.0000000000 |
| 30 | 3.2086893898 | 12.8980309149 | 9.8635090748 | 7.1505964007 | 3.3044734720 | 59.9959155958 | 1.0000000000 |
| 31 | 3.2859642224 | 12.8984986467 | 10.5305062334 | 7.1507270283 | 3.3046552989 | 61.9958222347 | 1.0000000000 |
| 32 | 3.3086949340 | 12.8986547111 | 10.6235488127 | 8.8348924555 | 2.9999147311 | 63.9957298179 | 1.0000000000 |
| 33 | 3.3178381842 | 12.8986989768 | 10.7625054766 | 8.8445826634 | 2.9999150187 | 65.9956385637 | 1.0000000000 |
| 34 | 3.3204768911 | 12.8987169455 | 10.8335071538 | 8.8446035999 | 2.9999206841 | 67.9955481688 | 1.0000000000 |
| 35 | 3.3224991941 | 12.8987348085 | 11.0171361880 | 9.9954742176 | 2.9760543713 | 69.9954586825 | 1.0000000000 |
| 36 | 3.3227701051 | 13.8016667112 | 11.0838846064 | 7.1616804229 | 3.6130019962 | 71.9953700694 | 1.0000000000 |
| 37 | 3.5030791690 | 14.0412319624 | 11.1960904966 | 7.1623926235 | 3.6381953396 | 73.9952822965 | 1.0000000000 |
| 38 | 3.5474588054 | 14.1657928132 | 11.2697974969 | 8.8673378976 | 3.3046009656 | 75.9951953326 | 1.0000000000 |
| 39 | 3.6033087937 | 14.2196073802 | 11.2890658063 | 8.8676233722 | 3.3046726654 | 77.9951528898 | 1.0000000000 |
| 40 | 3.6427538524 | 14.3444779177 | 11.4043727618 | 10.5557274333 | 2.9999277002 | 79.9950671884 | 1.0000000000 |
| 41 | 3.7221021534 | 14.4386612980 | 11.5140686475 | 10.5557491304 | 2.9999315502 | 81.9959862095 | 1.0000000000 |
| 42 | 3.8536666422 | 14.5640872361 | 11.6082371018 | 8.7907404838 | 3.5579717977 | 83.9959193185 | 1.0000000000 |
| 43 | 3.9249852736 | 14.6375723666 | 11.6669595312 | 8.8804026948 | 3.5693360047 | 85.9958535090 | 1.0000000000 |
| 44 | 3.9867870680 | 14.7494779827 | 11.7267093766 | 7.1694333250 | 3.9417801321 | 87.9957876533 | 1.0000000000 |
| 45 | 4.1307639555 | 14.8022341879 | 11.7669103726 | 8.8822182115 | 3.6130144699 | 89.9957228992 | 1.0000000000 |
| 46 | 4.2650163216 | 14.8666473324 | 11.8183969887 | 7.1833112011 | 3.9434227567 | 91.9956583385 | 1.0000000000 |
| 47 | 4.3919831741 | 14.8843583422 | 11.8389199288 | 12.2668751642 | 2.9999367635 | 93.9955940066 | 1.0000000000 |
| 48 | 4.4702282327 | 14.8888916898 | 11.8543855969 | 12.2668975375 | 2.9999394941 | 95.9955306328 | 1.0000000000 |
| 49 | 4.5479606952 | 14.8977952472 | 11.8602390935 | 8.8864887258 | 3.7543441279 | 97.9954674610 | 1.0000000000 |
| 50 | 4.6361497395 | 14.8980720507 | 11.8631629616 | 10.1548109139 | 3.6130894764 | 99.9954044402 | 1.0000000000 |
| 55 | 4.8415441305 | 14.8987433181 | 11.8635296721 | 13.9780304659 | 2.9999509413 | 109.9950972151 | 1.0000000000 |
| 60 | 5.3118320575 | 16.1509800447 | 12.7654145569 | 10.6125761426 | 3.8506973721 | 119.9947990625 | 1.0000000000 |
| 65 | 5.7279874805 | 16.5478815306 | 13.1924098309 | 10.6151070070 | 3.9236949232 | 129.9945087687 | 1.0000000000 |
| 70 | 5.9680473558 | 16.8200872202 | 13.4726646298 | 15.7352137761 | 3.3047042663 | 139.9942266038 | 1.0000000000 |
| 75 | 6.4032451608 | 16.8976376228 | 13.7256661130 | 12.3341346475 | 3.9235372581 | 149.9939509187 | 1.0000000000 |
| 80 | 6.9617540267 | 16.8986738106 | 13.8486607110 | 13.7717708962 | 3.9236584875 | 159.9936813790 | 1.0000000000 |
| 85 | 7.2908059511 | 17.6079959898 | 13.8632029690 | 13.7807630028 | 3.9237709088 | 169.9934168155 | 1.0000000000 |
| 90 | 7.4688349216 | 18.2065359049 | 13.8635411877 | 15.3806065880 | 3.9237168589 | 179.9931591781 | 1.0000000000 |
| 95 | 7.9994329832 | 18.5235744448 | 14.5689221176 | 15.1764940219 | 3.9237791996 | 189.9929003178 | 1.0000000000 |
| 100 | 8.5044325927 | 18.7913616667 | 14.9342481081 | 15.7490442120 | 3.9236938448 | 199.9926489863 | 1.0000000000 |

Table 7: Numerical results for the 3D problems (cont.).

| | Circle | | Square | |
|---|---|---|---|---|
| Number of items | Number of trials | CPU time in seconds | Number of trials | CPU time in seconds |
| 1 | 1 | 0.00 | 1 | 0.00 |
| 2 | 9 | 0.14 | 1 | 0.00 |
| 3 | 978 | 10.81 | 12 | 0.01 |
| 4 | 49382 | 270.93 | 1 | 0.00 |
| 5 | 2194 | 19.15 | 1 | 0.00 |
| 6 | 7497 | 64.19 | 38 | 0.56 |
| 7 | 200231 | 1154.91 | 16 | 1.19 |
| 8 | 194651 | 1291.75 | 50 | 5.69 |
| 9 | 99678 | 809.78 | 4 | 0.74 |
| 10 | 1457 | 15.98 | 148 | 23.07 |
| 11 | 76924 | 650.97 | 14 | 1.95 |
| 12 | 10444 | 112.07 | 22 | 5.32 |
| 13 | 9003 | 95.65 | 191 | 32.21 |
| 14 | 14242 | 180.95 | 15 | 4.60 |
| 15 | 30803 | 417.10 | 812 | 148.67 |
| 16 | 96878 | 1390.54 | 1 | 0.00 |
| 17 | 112173 | 1555.73 | 615 | 131.41 |
| 18 | 6031 | 96.12 | 52 | 13.27 |
| 19 | 399 | 6.47 | 9 | 3.20 |
| 20 | 33934 | 694.50 | 13 | 4.31 |
| 21 | 5447 | 137.53 | 208 | 60.78 |
| 22 | 24829 | 679.90 | 107 | 22.41 |
| 23 | 89 | 3.41 | 38 | 9.34 |
| 24 | 84943 | 3049.42 | 974 | 223.51 |
| 25 | 201 | 9.16 | 2 | 0.03 |
| 26 | 39288 | 1731.99 | 10 | 2.35 |
| 27 | 35280 | 1865.71 | 268 | 70.48 |
| 28 | 4386 | 252.17 | 384 | 92.88 |
| 29 | 3575 | 223.01 | 50 | 16.81 |
| 30 | 2 | 0.09 | 79 | 23.47 |
| 31 | 1 | 0.02 | 375 | 102.02 |
| 32 | 2 | 0.13 | 257 | 53.67 |
| 33 | 1167 | 89.87 | 35 | 8.58 |
| 34 | 1029 | 83.38 | 228 | 50.41 |
| 35 | 46 | 3.48 | 4564 | 1029.51 |
| 36 | 3 | 0.22 | 31 | 16.93 |
| 37 | 2306 | 189.48 | 1011 | 297.64 |
| 38 | 25562 | 2430.00 | 4803 | 1318.76 |
| 39 | 7 | 0.64 | 215 | 83.36 |
| 40 | 20665 | 2191.25 | 56 | 20.97 |
| 41 | 96 | 10.51 | 166 | 64.72 |
| 42 | 246 | 29.63 | 179 | 59.08 |
| 43 | 527 | 70.99 | 6654 | 1698.42 |
| 44 | 139 | 37.10 | 173 | 54.16 |
| 45 | 3007 | 456.26 | 192 | 39.47 |
| 46 | 32 | 4.79 | 213 | 52.13 |
| 47 | 64 | 10.05 | 475 | 119.72 |
| 48 | 2422 | 425.56 | 2748 | 753.35 |
| 49 | 10859 | 2087.76 | 3335 | 1011.08 |
| 50 | 50 | 10.02 | 1315 | 437.59 |
| 55 | 102 | 24.64 | 2 | 0.19 |
| 60 | 179 | 56.58 | 512 | 204.70 |
| 65 | 5361 | 2049.69 | 6848 | 3559.31 |
| 70 | 2647 | 1245.92 | 11360 | 6237.46 |
| 75 | 13143 | 7411.86 | 18817 | 10339.88 |
| 80 | 628 | 457.77 | 8661 | 6661.87 |
| 85 | 16759 | 13277.81 | 12261 | 11589.59 |
| 90 | 10123 | 8783.84 | 9824 | 10765.19 |
| 95 | 12937 | 13321.23 | 5790 | 7871.10 |
| 100 | 11553 | 13760.70 | 987 | 1678.37 |

Table 8: Effort measurements for the packing problems with circular and squared objects.

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