ON AUGMENTED LAGRANGIAN METHODS WITH GENERAL LOWER-LEVEL CONSTRAINTS *

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Abstract. Augmented Lagrangian methods with general lower-level constraints are considered in the present research. These methods are useful when efficient algorithms exist for solving subproblems in which the constraints are only of the lower-level type. Inexact resolution of the lower-level constrained subproblems is considered. Global convergence is proved using the Constant Positive Linear Dependence constraint qualification. Conditions for boundedness of the penalty parameters are discussed. The resolution of location problems in which many constraints of the lower-level set are nonlinear is addressed, employing the Spectral Projected Gradient method for solving the subproblems. Problems of this type with more than 3×10^6 variables and 14×10^6 constraints are solved in this way, using moderate computer time. All the codes are available in *www.ime.usp.br/∼egbirgin/tango/*.

Key words. Nonlinear Programming, Augmented Lagrangian methods, global convergence, constraint qualifications, numerical experiments

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1. Introduction.

Many practical optimization problems have the form

$$\text{Minimize } f(x) \text{ subject to } x \in \Omega_1 \cap \Omega_2, \tag{1.1}$$

where the constraint set Ω_2 is such that subproblems of type

$$\text{Minimize } F(x) \text{ subject to } x \in \Omega_2 \tag{1.2}$$

are much easier than problems of type (1.1). By this we mean that there exist efficient algorithms for solving (1.2) that cannot be applied to (1.1). In these cases it is natural to address the resolution of (1.1) by means of procedures that allow one to take advantage of methods that solve (1.2). Several examples of this situation may be found in the expanded report [3].

These problems motivated us to revisit Augmented Lagrangian methods with arbitrary lower-level constraints. Penalty and Augmented Lagrangian algorithms can take advantage of the existence of efficient procedures for solving partially constrained subproblems in a natural way. For this reason, many practitioners in Chemistry, Physics, Economy and Engineering rely on empirical penalty approaches when they incorporate additional constraints to models that were satisfactorily solved by pre-existing algorithms.

The general structure of Augmented Lagrangian methods is well known [7, 22, 39]. An Outer Iteration consists of two main steps: (a) Minimize the Augmented Lagrangian on the appropriate "simple" set (Ω_2 in our case); (b) Update multipliers and penalty parameters. However, several decisions need to be taken in order to define a practical algorithm. In this paper we use the Powell-Hestenes-Rockafellar PHR Augmented Lagrangian function [33, 40, 42] (see [8] for a comparison with other

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Augmented Lagrangian functions) and we keep inequality constraints as they are, instead of replacing them by equality constraints plus bounds. So, we pay the price of having discontinuous second derivatives in the objective function of the subproblems when Ω_1 involves inequalities.

A good criterion is needed for deciding that a suitable approximate subproblem minimizer has been found at Step (a). In particular, one must decide whether subproblem minimizers must be feasible with respect to Ω_2 and which is the admissible level of infeasibility and lack of complementarity at these solutions. (Bertsekas [6] analyzed an Augmented Lagrangian method for solving (1.1) in the case in which the subproblems are solved exactly.) Moreover, simple and efficient rules for updating multipliers and penalty parameters must be given.

Algorithmic decisions are taken looking at theoretical convergence properties and practical performance. Only experience tells one which theoretical results have practical importance and which do not. Although we recognize that this point is controversial, we would like to make explicit here our own criteria:

1. External penalty methods have the property that, when one finds the *global* minimizers of the subproblems, every limit point is a global minimizer of the original problem [24]. We think that this property must be preserved by the Augmented Lagrangian counterparts. This is the main reason why, in our algorithm, we will force boundedness of the Lagrange multipliers estimates.
2. We aim feasibility of the limit points but, since this may be impossible (even an empty feasible region is not excluded) a “feasibility result” must say that limit points are stationary points for some infeasibility measure. Some methods require that a constraint qualification holds at all the (feasible or infeasible) iterates. In [15, 47] it was shown that, in such cases, convergence to infeasible points that are not stationary for infeasibility may occur.
3. Feasible limit points that satisfy a constraint qualification must be KKT. The constraint qualification must be as *weak* as possible. Therefore, under the assumption that all the *feasible* points satisfy the constraint qualification, all the feasible limit points should be KKT.
4. Theoretically, it is impossible to prove that the whole sequence generated by a general Augmented Lagrangian method converges, because multiple solutions of the subproblems may exist and solutions of the subproblems may oscillate. However, since one uses the solution of one subproblem as initial point for solving the following one, the convergence of the whole sequence generally occurs. In this case, under suitable local conditions, we must be able to prove that the penalty parameters remain bounded.

In other words, the method must have all the good global convergence properties of an external penalty method. In addition, when everything “goes well”, it must be free of the asymptotic instability caused by large penalty parameters. Since we deal with nonconvex problems, the possibility of obtaining full global convergence properties based on proximal-point arguments is out of question.

The algorithm presented in this paper satisfies those theoretical requirements. In particular, we will show that, if a feasible limit point satisfies the Constant Positive Linear Dependence (CPLD) condition, then it is a KKT point. A feasible point x of a nonlinear programming problem is said to satisfy CPLD if the existence of a nontrivial null linear combination of gradients of active constraints with nonnegative coefficients corresponding to the inequalities implies that the gradients involved in that combination are linearly dependent for all z in a neighborhood of x. The CPLD condition was introduced by Qi and Wei [41]. In [4] it was proved that CPLD is a constraint qualification, being strictly weaker than the Linear Independence Constraint Qualification (LICQ) and than the Mangasarian-Fromovitz condition (MFCQ) [36, 43]. Since CPLD is weaker than (say) LICQ, theoretical results saying that *if a limit point satisfies CPLD then it satisfies KKT* are stronger than theoretical results saying that *if a limit point satisfies LICQ then it satisfies KKT*.

Most practical nonlinear programming methods published after 2001 rely on (a combination of) sequential quadratic programming (SQP), Newton-like or barrier approaches [1, 5, 14, 16, 18, 19, 26, 27, 28, 29, 35, 38, 44, 45, 46, 48, 49, 50]. None of these methods can be easily adapted to the situation described by (1.1)-(1.2).

In the numerical experiments we will show that, in some very large scale location problems, to use a specific algorithm for convex-constrained programming [11, 12, 13, 23] for solving the subproblems in the Augmented Lagrangian context is much more efficient than using a general purpose method. We will also show that ALGENCAN (the particular implementation of the algorithm introduced in this paper for the case in which the lower-level set is a box [9]) seems to converge to global minimizers more often than IPOPT [47, 48].

This paper is organized as follows. A high-level description of the main algorithm is given in Section 2. The rigorous definition of the method is in Section 3. Section 4 is devoted to global convergence results. In Section 5 we prove boundedness of the penalty parameters. In Section 6 we show the numerical experiments. Conclusions are given in Section 7.

Notation. We denote: $\mathbb{R}_+ = \{t \in \mathbb{R} \mid t \geq 0\}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\|\cdot\|$ is an arbitrary vector norm and $[v]_i$ is the $i-$th component of the vector v. If there is no possibility of confusion we may also use the notation v_i. For all $y \in \mathbb{R}^n$, $y_+ = (\max\{0, y_1\}, \ldots, \max\{0, y_n\})^T$. If $F : \mathbb{R}^n \to \mathbb{R}^m$, $F = (f_1, \ldots, f_m)^T$, we denote $\nabla F(x) = (\nabla f_1(x), \ldots, \nabla f_m(x)) \in \mathbb{R}^{n \times m}$. If $K = \{k_0, k_1, k_2, \ldots\} \subset \mathbb{N}$ $(k_{j+1} > k_j\ \forall j)$, we denote $\lim_{k \in K} x_k = \lim_{j \to \infty} x_{k_j}$.

2. Overview of the method.

We will consider the following nonlinear programming problem:

$$\text{Minimize } f(x) \text{ subject to } h_1(x) = 0, g_1(x) \leq 0, h_2(x) = 0, g_2(x) \leq 0, \tag{2.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}, h_1 : \mathbb{R}^n \to \mathbb{R}^{m_1}, h_2 : \mathbb{R}^n \to \mathbb{R}^{m_2}, g_1 : \mathbb{R}^n \to \mathbb{R}^{p_1}, g_2 : \mathbb{R}^n \to \mathbb{R}^{p_2}$. We assume that all these functions admit continuous first derivatives on a sufficiently large and open domain. We define $\Omega_1 = \{x \in \mathbb{R}^n \mid h_1(x) = 0, g_1(x) \leq 0\}$ and $\Omega_2 = \{x \in \mathbb{R}^n \mid h_2(x) = 0, g_2(x) \leq 0\}$.

For all $x \in \mathbb{R}^n, \rho > 0, \lambda \in \mathbb{R}^{m_1}, \mu \in \mathbb{R}^{p_1}_+$ we define the Augmented Lagrangian with respect to Ω_1 [33, 40, 42] as:

$$L(x, \lambda, \mu, \rho) = f(x) + \frac{\rho}{2} \sum_{i=1}^{m_1} \left([h_1(x)]_i + \frac{\lambda_i}{\rho} \right)^2 + \frac{\rho}{2} \sum_{i=1}^{p_1} \left([g_1(x)]_i + \frac{\mu_i}{\rho} \right)_+^2. \tag{2.2}$$

The main algorithm defined in this paper will consist of a sequence of (approximate) minimizations of $L(x, \lambda, \mu, \rho)$ subject to $x \in \Omega_2$, followed by the updating of λ, μ and ρ. A version of the algorithm with several penalty parameters may be found in [3]. Each approximate minimization of L will be called an *Outer Iteration*.

After each Outer Iteration one wishes some progress in terms of *feasibility* and *complementarity*. The *infeasibility* of x with respect to the equality constraint $[h_1(x)]_i =$

0 is naturally represented by $|[h_1(x)]_i|$. The case of inequality constraints is more complicate because, besides feasibility, one expects to have a null multiplier estimate if $g_i(x) < 0$. A suitable combined measure of infeasibility and non-complementarity with respect to the constraint $[g_1(x)]_i \le 0$ comes from defining $[\sigma(x,\mu,\rho)]_i = \max\{[g_1(x)]_i, -\mu_i/\rho\}$. Since μ_i/ρ is always nonnegative, it turns out that $[\sigma(x,\mu,\rho)]_i$ vanishes in two situations: (a) when $[g_1(x)]_i = 0$; and (b) when $[g_1(x)]_i < 0$ and $\mu_i = 0$. So, roughly speaking, $|[\sigma(x,\mu,\rho)]_i|$ measures infeasibility and complementarity with respect to the inequality constraint $[g_1(x)]_i \le 0$. If, between two consecutive outer iterations, enough progress is observed in terms of (at least one of) feasibility and complementarity, the penalty parameter will not be updated. Otherwise, the penalty parameter is increased by a fixed factor.

The rules for updating the multipliers need some discussion. In principle, we adopt the classical first-order correction rule [33, 40, 43] but, in addition, we impose that the multiplier estimates must be bounded. So, we will explicitly project the estimates on a compact box after each update. The reason for this decision was already given in the introduction: we want to preserve the property of external penalty methods that global minimizers of the original problem are obtained if each outer iteration computes a global minimizer of the subproblem. This property is maintained if the quotient of *the square* of each multiplier estimate over the penalty parameter tends to zero when the penalty parameter tends to infinity. We were not able to prove that this condition holds automatically for usual estimates and, in fact, we conjecture that it does not. Therefore, we decided to force the boundedness condition. The price paid by this decision seems to be moderate: in the proof of the boundedness of penalty parameters we will need to assume that the true Lagrange multipliers are within the bounds imposed by the algorithm. Since "large Lagrange multipliers" is a symptom of "near-nonfulfillment" of the Mangasarian-Fromovitz constraint qualification, this assumption seems to be compatible with the remaining ones that are necessary to prove penalty boundedness.

3. Description of the Augmented Lagrangian algorithm. In this section we provide a detailed description of the main algorithm. Approximate solutions of the subproblems are defined as points that satisfy the conditions (3.1)–(3.4) below. These formulae are relaxed KKT conditions of the problem of minimizing L subject to $x \in \Omega_2$. The first-order approximations of the multipliers are computed at Step 3. Lagrange multipliers estimates are denoted λ_k and μ_k whereas their safeguarded counterparts are $\bar\lambda_k$ and $\bar\mu_k$. At Step 4 we update the penalty parameters according to the progress in terms of feasibility and complementarity.

Algorithm 3.1.

Let $x_0 \in I\!R^n$ be an arbitrary initial point. The given parameters for the execution of the algorithm are: $\tau \in [0,1), \gamma > 1, \rho_1 > 0$, $-\infty < [\bar\lambda_{\min}]_i \le [\bar\lambda_{\max}]_i < \infty \;\forall\; i = 1,\dots,m_1$, $0 \le [\bar\mu_{\max}]_i < \infty \;\forall\; i = 1,\dots,p_1$, $[\bar\lambda_1]_i \in [[\bar\lambda_{\min}]_i, [\bar\lambda_{\max}]_i] \;\forall\; i = 1,\dots,m_1$, $[\bar\mu_1]_i \in [0,[\bar\mu_{\max}]_i] \;\forall\; i = 1,\dots,p_1$. Finally, $\{\varepsilon_k\} \subset I\!R_+$ is a sequence of tolerance parameters such that $\lim_{k\to\infty} \varepsilon_k = 0$.

Step 1. *Initialization*

Set $k \leftarrow 1$. For $i = 1,\dots,p_1$, compute $[\sigma_0]_i = \max\{0, [g_1(x_0)]_i\}$.

Step 2. *Solving the subproblem*

Compute (if possible) $x_k \in I\!R^n$ such that there exist $v_k \in I\!R^{m_2}, u_k \in I\!R^{p_2}$ satisfying

$$\|\nabla L(x_k, \bar\lambda_k, \bar\mu_k, \rho_k) + \sum_{i=1}^{m_2} [v_k]_i \nabla [h_2(x_k)]_i + \sum_{i=1}^{p_2} [u_k]_i \nabla [g_2(x_k)]_i\| \le \varepsilon_{k,1}, \tag{3.1}$$

$$[u_k]_i \ge 0 \text{ and } [g_2(x_k)]_i \le \varepsilon_{k,2} \text{ for all } i = 1, \dots, p_2, \tag{3.2}$$

$$[g_2(x_k)]_i < -\varepsilon_{k,2} \Rightarrow [u_k]_i = 0 \text{ for all } i = 1, \dots, p_2, \tag{3.3}$$

$$\|h_2(x_k)\| \le \varepsilon_{k,3}, \tag{3.4}$$

where $\varepsilon_{k,1}, \varepsilon_{k,2}, \varepsilon_{k,3} \ge 0$ are such that $\max\{\varepsilon_{k,1}, \varepsilon_{k,2}, \varepsilon_{k,3}\} \le \varepsilon_k$. If it is not possible to find x_k satisfying (3.1)–(3.4), stop the execution of the algorithm.

Step 3. *Estimate multipliers*

For all $i = 1, \dots, m_1$, compute

$$[\lambda_{k+1}]_i = [\bar\lambda_k]_i + \rho_k [h_1(x_k)]_i, \tag{3.5}$$

$$[\bar\lambda_{k+1}]_i \in [[\bar\lambda_{\min}]_i, [\bar\lambda_{\max}]_i]. \tag{3.6}$$

(Usually, $[\bar\lambda_{k+1}]_i$ will be the projection of $[\lambda_{k+1}]_i$ on the interval $[[\bar\lambda_{\min}]_i, [\bar\lambda_{\max}]_i]$.) For all $i = 1, \dots, p_1$, compute

$$[\mu_{k+1}]_i = \max\{0, [\bar\mu_k]_i + \rho_k [g_1(x_k)]_i\}, \quad [\sigma_k]_i = \max\left\{[g_1(x_k)]_i, -\frac{[\bar\mu_k]_i}{\rho_k}\right\}, \tag{3.7}$$

$$[\bar\mu_{k+1}]_i \in [0, [\bar\mu_{\max}]_i].$$

(Usually, $[\bar\mu_{k+1}]_i = \min\{[\mu_{k+1}]_i, [\bar\mu_{\max}]_i\}$.)

Step 4. *Update the penalty parameter*

If $\max\{\|h_1(x_k)\|_\infty, \|\sigma_k\|_\infty\} \le \tau \max\{\|h_1(x_{k-1})\|_\infty, \|\sigma_{k-1}\|_\infty\}$, then define $\rho_{k+1} = \rho_k$. Else, define $\rho_{k+1} = \gamma \rho_k$.

Step 5. *Begin a new outer iteration*

Set $k \leftarrow k + 1$. Go to Step 2.

4. Global convergence. In this section we assume that the algorithm does not stop at Step 2. In other words, it is always possible to find x_k satisfying (3.1)-(3.4). Problem-dependent sufficient conditions for this assumption can be given in many cases.

We will also assume that at least a limit point of the sequence generated by Algorithm 3.1 exists. A sufficient condition for this is the existence of $\varepsilon > 0$ such that the set $\{x \in I\!R^n \mid g_2(x) \le \varepsilon, \|h_2(x)\| \le \varepsilon\}$ is bounded. This condition may be enforced adding artificial simple constraints to the set Ω_2.

Global convergence results that use the CPLD constraint qualification are stronger than previous results for more specific problems: In particular, Conn, Gould and Toint [21] and Conn, Gould, Sartenaer and Toint [20] proved global convergence of Augmented Lagrangian methods for equality constraints and linear constraints assuming linear independence of all the gradients of active constraints at the limit points. Their assumption is much stronger than our CPLD assumptions. On one hand, the CPLD assumption is weaker than LICQ (for example, CPLD always holds when the constraints are linear). On the other hand, our CPLD assumption involves only feasible points instead of all possible limit points of the algorithm.

Convergence proofs for Augmented Lagrangian methods with equalities and box constraints using CPLD were given in [2].

We are going to investigate the status of the limit points of sequences generated by Algorithm 3.1. Firstly, we will prove a result on the feasibility properties of a limit point. Theorem 4.1 shows that, either a limit point is feasible or, if the CPLD constraint qualification with respect to Ω_2 holds, it is a KKT point of the sum of squares of upper-level infeasibilities.

THEOREM 4.1. *Let $\{x_k\}$ be a sequence generated by Algorithm 3.1. Let x_* be a limit point of $\{x_k\}$. Then, if the sequence of penalty parameters $\{\rho_k\}$ is bounded, the limit point x_* is feasible. Otherwise, at least one of the following possibilities hold:*

(i) x_* is a KKT point of the problem

$$\text{(4.1)}\quad \text{Minimize } \frac{1}{2}\left[\sum_{i=1}^{m_1}[h_1(x)]_i^2 + \sum_{i=1}^{p_1}\max\{0,[g_1(x)]_i\}^2\right] \text{ subject to } x \in \Omega_2.$$

(ii) x_* does not satisfy the CPLD constraint qualification associated with Ω_2.

Proof. Let K be an infinite subsequence in $I\!N$ such that $\lim_{k\in K} x_k = x_*$. Since $\varepsilon_k \to 0$, by (3.2) and (3.4), we have that $g_2(x_*) \le 0$ and $h_2(x_*) = 0$. So, $x_* \in \Omega_2$.

Now, we consider two possibilities: (a) the sequence $\{\rho_k\}$ is bounded; and (b) the sequence $\{\rho_k\}$ is unbounded. Let us analyze first Case (a). In this case, from some iteration on, the penalty parameters are not updated. Therefore, $\lim_{k\to\infty}\|h_1(x_k)\| = \lim_{k\to\infty}\|\sigma_k\| = 0$. Thus, $h_1(x_*) = 0$. Now, if $[g_1(x_*)]_j > 0$ then $[g_1(x_k)]_j > c > 0$ for $k \in K$ large enough. This would contradict the fact that $[\sigma_k]_j \to 0$. Therefore, $[g_1(x_*)]_i \le 0 \quad \forall i = 1,\ldots,p_1$.

Since $x_* \in \Omega_2$, $h_1(x_*) = 0$ and $g_1(x_*) \le 0$, x_* is feasible. Therefore, we proved the desired result in the case that $\{\rho_k\}$ is bounded.

Consider now Case (b). So, $\{\rho_k\}_{k\in K}$ is not bounded. By (2.2) and (3.1), we have:

$$\text{(4.2)}\quad \begin{aligned}&\nabla f(x_k) + \textstyle\sum_{i=1}^{m_1}([\bar\lambda_k]_i + \rho_k[h_1(x_k)]_i)\nabla[h_1(x_k)]_i + \sum_{i=1}^{p_1}\max\{0,[\bar\mu_k]_i\\ &+\rho_k[g_1(x_k)]_i\}\nabla[g_1(x_k)]_i + \textstyle\sum_{i=1}^{m_2}[v_k]_i\nabla[h_2(x_k)]_i + \sum_{j=1}^{p_2}[u_k]_j\nabla[g_2(x_k)]_j = \delta_k,\end{aligned}$$

where, since $\varepsilon_k \to 0$, $\lim_{k\in K}\|\delta_k\| = 0$.

If $[g_2(x_*)]_i < 0$, there exists $k_1 \in I\!N$ such that $[g_2(x_k)]_i < -\varepsilon_k$ for all $k \ge k_1, k \in K$. Therefore, by (3.3), $[u_k]_i = 0$ for all $k \in K, k \ge k_1$. Thus, by $x_* \in \Omega_2$ and (4.2), for all $k \in K, k \ge k_1$ we have that

$$\begin{aligned}&\nabla f(x_k) + \textstyle\sum_{i=1}^{m_1}([\bar\lambda_k]_i + \rho_k[h_1(x_k)]_i)\nabla[h_1(x_k)]_i + \sum_{i=1}^{p_1}\max\{0,[\bar\mu_k]_i\\ &+\rho_k[g_1(x_k)]_i\}\nabla[g_1(x_k)]_i + \textstyle\sum_{i=1}^{m_2}[v_k]_i\nabla[h_2(x_k)]_i + \sum_{[g_2(x_*)]_j=0}[u_k]_j\nabla[g_2(x_k)]_j = \delta_k.\end{aligned}$$

Dividing by ρ_k we get:

$$\begin{aligned}&\frac{\nabla f(x_k)}{\rho_k} + \sum_{i=1}^{m_1}\left(\frac{[\bar\lambda_k]_i}{\rho_k} + [h_1(x_k)]_i\right)\nabla[h_1(x_k)]_i + \sum_{i=1}^{p_1}\max\left\{0,\frac{[\bar\mu_k]_i}{\rho_k} + [g_1(x_k)]_i\right\}\nabla[g_1(x_k)]_i\\ &+ \sum_{i=1}^{m_2}\frac{[v_k]_i}{\rho_k}\nabla[h_2(x_k)]_i + \sum_{[g_2(x_*)]_j=0}\frac{[u_k]_j}{\rho_k}\nabla[g_2(x_k)]_j = \frac{\delta_k}{\rho_k}.\end{aligned}$$

By Caratheodory's Theorem of Cones (see [7], page 689) there exist $\widehat{I}_k \subset \{1,\ldots,m_2\}$, $\widehat{J}_k \subset \{j \mid [g_2(x_*)]_j = 0\}$, $[\widehat{v}_k]_i$, $i \in \widehat{I}_k$ and $[\widehat{u}_k]_j \ge 0$, $j \in \widehat{J}_k$ such that the vectors

$\{\nabla[h_2(x_k)]_i\}_{i\in\widehat{I}_k} \cup \{\nabla[g_2(x_k)]_j\}_{j\in\widehat{J}_k}$ are linearly independent and

$$\frac{\nabla f(x_k)}{\rho_k} + \sum_{i=1}^{m_1}\left(\frac{[\bar\lambda_k]_i}{\rho_k} + [h_1(x_k)]_i\right)\nabla[h_1(x_k)]_i + \sum_{i=1}^{p_1}\max\left\{0, \frac{[\bar\mu_k]_i}{\rho_k} + [g_1(x_k)]_i\right\}\nabla[g_1(x_k)]_i$$
$$+ \sum_{i\in\widehat{I}_k}[\widehat{v}_k]_i\nabla[h_2(x_k)]_i + \sum_{j\in\widehat{J}_k}[\widehat{u}_k]_j\nabla[g_2(x_k)]_j = \frac{\delta_k}{\rho_k}. \tag{4.3}$$

Since there exist a finite number of possible sets $\widehat{I}_k, \widehat{J}_k$, there exists an infinite set of indices K_1 such that $K_1 \subset \{k \in K \mid k \geq k_1\}$, $\widehat{I}_k = \widehat{I}$, and

$$\widehat{J} = \widehat{J}_k \subset \{j \mid [g_2(x_*)]_j = 0\} \tag{4.4}$$

for all $k \in K_1$. Then, by (4.3), for all $k \in K_1$ we have:

$$\frac{\nabla f(x_k)}{\rho_k} + \sum_{i=1}^{m_1}\left(\frac{[\bar\lambda_k]_i}{\rho_k} + [h_1(x_k)]_i\right)\nabla[h_1(x_k)]_i + \sum_{i=1}^{p_1}\max\left\{0, \frac{[\bar\mu_k]_i}{\rho_k} + [g_1(x_k)]_i\right\}\nabla[g_1(x_k)]_i$$
$$+ \sum_{i\in\widehat{I}}[\widehat{v}_k]_i\nabla[h_2(x_k)]_i + \sum_{j\in\widehat{J}}[\widehat{u}_k]_j\nabla[g_2(x_k)]_j = \frac{\delta_k}{\rho_k}, \tag{4.5}$$

and the gradients

$$\{\nabla[h_2(x_k)]_i\}_{i\in\widehat{I}} \cup \{\nabla[g_2(x_k)]_j\}_{j\in\widehat{J}} \text{ are linearly independent.} \tag{4.6}$$

We consider, again, two cases: (1) the sequence $\{\|(\widehat{v}_k, \widehat{u}_k)\|, k \in K_1\}$ is bounded; and (2) the sequence $\{\|(\widehat{v}_k, \widehat{u}_k)\|, k \in K_1\}$ is unbounded. If the sequence $\{\|(\widehat{v}_k, \widehat{u}_k)\|\}_{k\in K_1}$ is bounded, and $\widehat{I} \cup \widehat{J} \neq \emptyset$, there exist $(\widehat{v}, \widehat{u}), \widehat{u} \geq 0$ and an infinite set of indices $K_2 \subset K_1$ such that $\lim_{k\in K_2}(\widehat{v}_k, \widehat{u}_k) = (\widehat{v}, \widehat{u})$. Since $\{\rho_k\}$ is unbounded, by the boundedness of $\bar\lambda_k$ and $\bar\mu_k$, $\lim[\bar\lambda_k]_i/\rho_k = 0 = \lim[\bar\mu_k]_j/\rho_k$ for all i, j. Therefore, by $\delta_k \to 0$, taking limits for $k \in K_2$ in (4.5), we obtain:

$$\sum_{i=1}^{m_1}[h_1(x_*)]_i\nabla[h_1(x_*)]_i + \sum_{i=1}^{p_1}\max\{0, [g_1(x_*)]_i\}\nabla[g_1(x_*)]_i$$
$$+ \sum_{i\in\widehat{I}}\widehat{v}_i\nabla[h_2(x_*)]_i + \sum_{j\in\widehat{J}}\widehat{u}_j\nabla[g_2(x_*)]_j = 0. \tag{4.7}$$

If $\widehat{I} \cup \widehat{J} = \emptyset$ we obtain $\sum_{i=1}^{m_1}[h_1(x_*)]_i\nabla[h_1(x_*)]_i + \sum_{i=1}^{p_1}\max\{0, [g_1(x_*)]_i\}\nabla[g_1(x_*)]_i = 0$.

Therefore, by $x_* \in \Omega_2$ and (4.4), x_* is a KKT point of (4.1).

Finally, assume that $\{\|(\widehat{v}_k, \widehat{u}_k)\|\}_{k\in K_1}$ is unbounded. Let $K_3 \subset K_1$ be such that $\lim_{k\in K_3}\|(\widehat{v}_k, \widehat{u}_k)\| = \infty$ and $(\widehat{v}, \widehat{u}) \neq 0, \widehat{u} \geq 0$ such that $\lim_{k\in K_3}\frac{(\widehat{v}_k, \widehat{u}_k)}{\|(\widehat{v}_k, \widehat{u}_k)\|} = (\widehat{v}, \widehat{u})$. Dividing both sides of (4.5) by $\|(\widehat{v}_k, \widehat{u}_k)\|$ and taking limits for $k \in K_3$, we deduce that $\sum_{i\in\widehat{I}}\widehat{v}_i\nabla[h_2(x_*)]_i + \sum_{j\in\widehat{J}}\widehat{u}_j\nabla[g_2(x_*)]_j = 0$. But $[g_2(x_*)]_j = 0$ for all $j \in \widehat{J}$. Then, by (4.6), x_* does not satisfy the CPLD constraint qualification associated with the set Ω_2. This completes the proof. □

Roughly speaking, Theorem 4.1 says that, if x_* is not feasible, then (very likely) it is a local minimizer of the upper-level infeasibility, subject to lower-level feasibility. From the point of view of optimality, we are interested in the status of feasible limit points. In Theorem 4.2 we will prove that, under the CPLD constraint qualification, feasible limit points are stationary (KKT) points of the original problem. Since CPLD is strictly weaker than the Mangasarian-Fromovitz (MF) constraint qualification, it turns out that the following theorem is stronger than results where KKT conditions are proved under MF or regularity assumptions.

THEOREM 4.2. *Let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence generated by Algorithm 3.1. Assume that $x_* \in \Omega_1 \cap \Omega_2$ is a limit point that satisfies the CPLD constraint qualification related to $\Omega_1 \cap \Omega_2$. Then, x_* is a KKT point of the original problem (2.1). Moreover, if x_* satisfies the Mangasarian-Fromovitz constraint qualification and $\{x_k\}_{k\in K}$ is a subsequence that converges to x_*, the set*

$$\{\|\lambda_{k+1}\|, \|\mu_{k+1}\|, \|v_k\|, \|u_k\|\}_{k\in K} \text{ is bounded.} \tag{4.8}$$

Proof. For all $k \in \mathbb{N}$, by (3.1), (3.3), (3.5) and (3.7), there exist $u_k \in \mathbb{R}^{p_2}_+$, $\delta_k \in \mathbb{R}^n$ such that $\|\delta_k\| \le \varepsilon_k$ and

$$\begin{aligned} &\nabla f(x_k) + \textstyle\sum_{i=1}^{m_1} [\lambda_{k+1}]_i \nabla [h_1(x_k)]_i + \sum_{i=1}^{p_1} [\mu_{k+1}]_i \nabla [g_1(x_k)]_i \\ &\quad + \textstyle\sum_{i=1}^{m_2} [v_k]_i \nabla [h_2(x_k)]_i + \sum_{j=1}^{p_2} [u_k]_j \nabla [g_2(x_k)]_j = \delta_k. \end{aligned} \tag{4.9}$$

By (3.7), $\mu_{k+1} \in \mathbb{R}^{p_1}_+$ for all $k \in \mathbb{N}$. Let $K \subset \mathbb{N}$ be such that $\lim_{k\in K} x_k = x_*$. Suppose that $[g_2(x_*)]_i < 0$. Then, there exists $k_1 \in \mathbb{N}$ such that $\forall k \in K, k \ge k_1, [g_2(x_k)]_i < -\varepsilon_k$. Then, by (3.3), $[u_k]_i = 0 \quad \forall k \in K, k \ge k_1$.

Let us prove now that a similar property takes place when $[g_1(x_*)]_i < 0$. In this case, there exists $k_2 \ge k_1$ such that $[g_1(x_k)]_i < c < 0 \ \ \forall k \in K, k \ge k_2$.

We consider two cases: (1) $\{\rho_k\}$ is unbounded; and (2) $\{\rho_k\}$ is bounded. In the first case we have that $\lim_{k\in K} \rho_k = \infty$. Since $\{[\bar{\mu}_k]_i\}$ is bounded, there exists $k_3 \ge k_2$ such that, for all $k \in K, k \ge k_3$, $[\bar{\mu}_k]_i + \rho_k [g_1(x_k)]_i < 0$. By the definition of μ_{k+1} this implies that $[\mu_{k+1}]_i = 0 \quad \forall k \in K, k \ge k_3$.

Consider now the case in which $\{\rho_k\}$ is bounded. In this case, $\lim_{k\to\infty} [\sigma_k]_i = 0$. Therefore, since $[g_1(x_k)]_i < c < 0$ for $k \in K$ large enough, $\lim_{k\in K} [\bar{\mu}_k]_i = 0$. So, for $k \in K$ large enough, $[\bar{\mu}_k]_i + \rho_k [g_1(x_k)]_i < 0$. By the definition of μ_{k+1}, there exists $k_4 \ge k_2$ such that $[\mu_{k+1}]_i = 0$ for $k \in K, k \ge k_4$.

Therefore, there exists $k_5 \ge \max\{k_1, k_3, k_4\}$ such that for all $k \in K$, $k \ge k_5$,

$$[[g_1(x_*)]_i < 0 \Rightarrow [\mu_{k+1}]_i = 0] \text{ and } [[g_2(x_*)]_i < 0 \Rightarrow [u_k]_i = 0]. \tag{4.10}$$

(Observe that, up to now, we did not use the CPLD condition.) By (4.9) and (4.10), for all $k \in K, k \ge k_5$, we have:

$$\begin{aligned} &\nabla f(x_k) + \textstyle\sum_{i=1}^{m_1} [\lambda_{k+1}]_i \nabla [h_1(x_k)]_i + \sum_{[g_1(x_*)]_i = 0} [\mu_{k+1}]_i \nabla [g_1(x_k)]_i \\ &\quad + \textstyle\sum_{i=1}^{m_2} [v_k]_i \nabla [h_2(x_k)]_i + \sum_{[g_2(x_*)]_j = 0} [u_k]_j \nabla [g_2(x_k)]_j = \delta_k, \end{aligned} \tag{4.11}$$

with $\mu_{k+1} \in \mathbb{R}^{p_1}_+, u_k \in \mathbb{R}^{p_2}_+$.

By Caratheodory's Theorem of Cones, for all $k \in K, k \ge k_5$, there exist

$$\widehat{I}_k \subset \{1, \ldots, m_1\}, \widehat{J}_k \subset \{j \mid [g_1(x_*)]_j = 0\}, \hat{I}_k \subset \{1, \ldots, m_2\}, \hat{J}_k \subset \{j \mid [g_2(x_*)]_j = 0\},$$

$$[\widehat{\lambda}_k]_i \in \mathbb{R} \ \forall i \in \widehat{I}_k, [\widehat{\mu}_k]_j \ge 0 \ \forall j \in \widehat{J}_k, [\widehat{v}_k]_i \in \mathbb{R} \ \forall i \in \hat{I}_k, [\widehat{u}_k]_j \ge 0 \ \forall j \in \hat{J}_k$$

such that the vectors

$$\{\nabla [h_1(x_k)]_i\}_{i\in\widehat{I}_k} \cup \{\nabla [g_1(x_k)]_i\}_{i\in\widehat{J}_k} \cup \{\nabla [h_2(x_k)]_i\}_{i\in\hat{I}_k} \cup \{\nabla [g_2(x_k)]_i\}_{i\in\hat{J}_k}$$

are linearly independent and

$$\begin{aligned} &\nabla f(x_k) + \textstyle\sum_{i\in\widehat{I}_k} [\widehat{\lambda}_k]_i \nabla [h_1(x_k)]_i + \sum_{i\in\widehat{J}_k} [\widehat{\mu}_k]_i \nabla [g_1(x_k)]_i \\ &\quad + \textstyle\sum_{i\in\hat{I}_k} [\widehat{v}_k]_i \nabla [h_2(x_k)]_i + \sum_{j\in\hat{J}_k} [\widehat{u}_k]_j \nabla [g_2(x_k)]_j = \delta_k. \end{aligned} \tag{4.12}$$

Since the number of possible sets of indices $\widehat{I}_k, \widehat{J}_k, \hat{I}_k, \hat{J}_k$ is finite, there exists an infinite set $K_1 \subset \{k \in K \mid k \geq k_5\}$ such that $\widehat{I}_k = \widehat{I}, \widehat{J}_k = \widehat{J}, \hat{I}_k = \hat{I}, \hat{J}_k = \hat{J}$, for all $k \in K_1$.

Then, by (4.12),

$$\begin{aligned} &\nabla f(x_k) + \textstyle\sum_{i \in \widehat{I}} [\widehat{\lambda}_k]_i \nabla [h_1(x_k)]_i + \sum_{i \in \widehat{J}} [\widehat{\mu}_k]_i \nabla [g_1(x_k)]_i \\ &\quad + \textstyle\sum_{i \in \hat{I}} [\widehat{v}_k]_i \nabla [h_2(x_k)]_i + \sum_{j \in \hat{J}} [\widehat{u}_k]_j \nabla [g_2(x_k)]_j = \delta_k \end{aligned} \tag{4.13}$$

and the vectors

$$\{\nabla [h_1(x_k)]_i\}_{i \in \widehat{I}} \cup \{\nabla [g_1(x_k)]_i\}_{i \in \widehat{J}} \cup \{\nabla [h_2(x_k)]_i\}_{i \in \hat{I}} \cup \{\nabla [g_2(x_k)]_i\}_{i \in \hat{J}} \tag{4.14}$$

are linearly independent for all $k \in K_1$.

If $\widehat{I} \cup \widehat{J} \cup \hat{I} \cup \hat{J} = \emptyset$, by (4.13) and $\delta_k \to 0$ we obtain $\nabla f(x_*) = 0$. Otherwise, let us define

$$S_k = \max\{\max\{|[\widehat{\lambda}_k]_i|, i \in \widehat{I}\}, \max\{[\widehat{\mu}_k]_i, i \in \widehat{J}\}, \max\{|[\widehat{v}_k]_i|, i \in \hat{I}\}, \max\{[\widehat{u}_k]_i, i \in \hat{J}\}\}.$$

We consider two possibilities: (a) $\{S_k\}_{k \in K_1}$ has a bounded subsequence; and (b) $\lim_{k \in K_1} S_k = \infty$. If $\{S_k\}_{k \in K_1}$ has a bounded subsequence, there exists $K_2 \subset K_1$ such that $\lim_{k \in K_2} [\widehat{\lambda}_k]_i = \widehat{\lambda}_i$, $\lim_{k \in K_2} [\widehat{\mu}_k]_i = \widehat{\mu}_i \geq 0$, $\lim_{k \in K_2} [\widehat{v}_k]_i = \widehat{v}_i$, and $\lim_{k \in K_2} [\widehat{u}_k]_i = \widehat{u}_i \geq 0$. By $\varepsilon_k \to 0$ and $\|\delta_k\| \leq \varepsilon_k$, taking limits in (4.13) for $k \in K_2$, we obtain:

$$\nabla f(x_*) + \sum_{i \in \widehat{I}} \widehat{\lambda}_i \nabla [h_1(x_*)]_i + \sum_{i \in \widehat{J}} \widehat{\mu}_i \nabla [g_1(x_*)]_i + \sum_{i \in \hat{I}} \widehat{v}_i \nabla [h_2(x_*)]_i + \sum_{j \in \hat{J}} \widehat{u}_j \nabla [g_2(x_*)]_j = 0,$$

with $\widehat{\mu}_i \geq 0, \widehat{u}_i \geq 0$. Since $x_* \in \Omega_1 \cap \Omega_2$, we have that x_* is a KKT point of (2.1).

Suppose now that $\lim_{k \in K_2} S_k = \infty$. Dividing both sides of (4.13) by S_k we obtain:

$$\begin{aligned} &\frac{\nabla f(x_k)}{S_k} + \textstyle\sum_{i \in \widehat{I}} \frac{[\widehat{\lambda}_k]_i}{S_k} \nabla [h_1(x_k)]_i + \sum_{i \in \widehat{J}} \frac{[\widehat{\mu}_k]_i}{S_k} \nabla [g_1(x_k)]_i \\ &+ \textstyle\sum_{i \in \hat{I}} \frac{[\widehat{v}_k]_i}{S_k} \nabla [h_2(x_k)]_i + \sum_{j \in \hat{J}} \frac{[\widehat{u}_k]_j}{S_k} \nabla [g_2(x_k)]_j = \frac{\delta_k}{S_k}, \end{aligned} \tag{4.15}$$

where $\left|\frac{[\widehat{\lambda}_k]_i}{S_k}\right| \leq 1, \left|\frac{[\widehat{\mu}_k]_i}{S_k}\right| \leq 1, \left|\frac{[\widehat{v}_k]_i}{S_k}\right| \leq 1, \left|\frac{[\widehat{u}_k]_j}{S_k}\right| \leq 1$. Therefore, there exists $K_3 \subset K_1$ such that $\lim_{k \in K_3} \frac{[\widehat{\lambda}_k]_i}{S_k} = \widehat{\lambda}_i, \lim_{k \in K_3} \frac{[\widehat{\mu}_k]_i}{S_k} = \widehat{\mu}_i \geq 0, \lim_{k \in K_3} \frac{[\widehat{v}_k]_i}{S_k} = \widehat{v}_i, \lim_{k \in K_3} \frac{[\widehat{u}_k]_j}{S_k} = \widehat{u}_j \geq 0$. Taking limits on both sides of (4.15) for $k \in K_3$, we obtain:

$$\sum_{i \in \widehat{I}} \widehat{\lambda}_i \nabla [h_1(x_*)]_i + \sum_{i \in \widehat{J}} \widehat{\mu}_i \nabla [g_1(x_*)]_i + \sum_{i \in \hat{I}} \widehat{v}_i \nabla [h_2(x_*)]_i + \sum_{j \in \hat{J}} \widehat{u}_j \nabla [g_2(x_*)]_j = 0.$$

But the modulus of at least one of the coefficients $\widehat{\lambda}_i, \widehat{\mu}_i, \widehat{v}_i, \widehat{u}_i$ is equal to 1. Then, by the CPLD condition, the gradients

$$\{\nabla [h_1(x)]_i\}_{i \in \widehat{I}} \cup \{\nabla [g_1(x)]_i\}_{i \in \widehat{J}} \cup \{\nabla [h_2(x)]_i\}_{i \in \hat{I}} \cup \{\nabla [g_2(x)]_i\}_{i \in \hat{J}}$$

must be linearly dependent in a neighborhood of x_*. This contradicts (4.14). Therefore, the main part of the theorem is proved.

Finally, let us prove that the property (4.8) holds if x_* satisfies the Mangasarian-Fromovitz constraint qualification. Let us define

$$B_k = \max\{\|\lambda_{k+1}\|_\infty, \|\mu_{k+1}\|_\infty, \|v_k\|_\infty, \|u_k\|_\infty\}_{k \in K}.$$

If (4.8) is not true, we have that $\lim_{k \in K} B_k = \infty$. In this case, dividing both sides of (4.11) by B_k and taking limits for an appropriate subsequence, we obtain that x_* does not satisfy the Mangasarian-Fromovitz constraint qualification. ∎

5. Boundedness of the penalty parameters. When the penalty parameters associated with Penalty or Augmented Lagrangian methods are too large, the subproblems tend to be ill-conditioned and its resolution becomes harder. One of the main motivations for the development of the basic Augmented Lagrangian algorithm is the necessity of overcoming this difficulty. Therefore, the study of conditions under which penalty parameters are bounded plays an important role in Augmented Lagrangian approaches.

5.1. Equality constraints. We will consider first the case $p_1 = p_2 = 0$.

Let $f : I\!R^n \to I\!R, h_1 : I\!R^n \to I\!R^{m_1}, h_2 : I\!R^n \to I\!R^{m_2}$. We address the problem

$$\text{Minimize } f(x) \text{ subject to } h_1(x) = 0,\ h_2(x) = 0. \tag{5.1}$$

The Lagrangian function associated with problem (5.1) is given by $L_0(x, \lambda, v) = f(x) + \langle h_1(x), \lambda\rangle + \langle h_2(x), v\rangle$, for all $x \in I\!R^n$, $\lambda \in I\!R^{m_1}$, $v \in I\!R^{m_2}$.

Algorithm 3.1 will be considered with the following standard definition for the safeguarded Lagrange multipliers.

Definition. *For all $k \in I\!N$, $i = 1, \ldots, m_1$, $[\bar\lambda_{k+1}]_i$ will be the projection of $[\lambda_{k+1}]_i$ on the interval $[[\bar\lambda_{\min}]_i, [\bar\lambda_{\max}]_i]$.*

We will use the following assumptions:

Assumption 1. *The sequence $\{x_k\}$ is generated by the application of Algorithm 3.1 to problem (5.1) and $\lim_{k\to\infty} x_k = x_*$.*

Assumption 2. *The point x_* is feasible ($h_1(x_*) = 0$ and $h_2(x_*) = 0$).*

Assumption 3. *The gradients $\nabla[h_1(x_*)]_1, \ldots, \nabla[h_1(x_*)]_{m_1}, \nabla[h_2(x_*)]_1, \ldots, \nabla[h_2(x_*)]_{m_2}$ are linearly independent.*

Assumption 4. *The functions f, h_1 and h_2 admit continuous second derivatives in a neighborhood of x_*.*

Assumption 5. *The second order sufficient condition for local minimizers ([25], page 211) holds with Lagrange multipliers $\lambda_* \in I\!R^{m_1}$ and $v_* \in I\!R^{m_2}$.*

Assumption 6. *For all $i = 1, \ldots, m_1$, $[\lambda_*]_i \in ([\bar\lambda_{\min}]_i, [\bar\lambda_{\max}]_i)$.*

PROPOSITION 5.1. *Suppose that Assumptions 1, 2, 3 and 6 hold. Then, $\lim_{k\to\infty} \lambda_k = \lambda_*$, $\lim_{k\to\infty} v_k = v_*$ and $\bar\lambda_k = \lambda_k$ for k large enough.*

Proof. The proof of the first part follows from the definition of λ_{k+1}, the stopping criterion of the subproblems and the linear independence of the gradients of the constraints at x_*. The second part of the thesis is a consequence of $\lambda_k \to \lambda_*$, using Assumption 6 and the definition of $\bar\lambda_{k+1}$. □

LEMMA 5.2. *Suppose that Assumptions 3 and 5 hold. Then, there exists $\bar\rho > 0$ such that, for all $\pi \in [0, 1/\bar\rho]$, the matrix*

$$\begin{pmatrix} \nabla^2_{xx} L_0(x_*, \lambda_*, v_*) & \nabla h_1(x_*) & \nabla h_2(x_*) \\ \nabla h_1(x_*)^T & -\pi I & 0 \\ \nabla h_2(x_*)^T & 0 & 0 \end{pmatrix}$$

is nonsingular.

Proof. The matrix is trivially nonsingular for $\pi = 0$. So, the thesis follows by the continuity of the matricial inverse. □

LEMMA 5.3. *Suppose that Assumptions 1–5 hold. Let $\bar\rho$ be as in Lemma 5.2. Suppose that there exists $k_0 \in I\!N$ such that $\rho_k \geq \bar\rho$ for all $k \geq k_0$. Define*

$$\alpha_k = \nabla L(x_k, \bar\lambda_k, \rho_k) + \nabla h_2(x_k) v_k, \tag{5.2}$$

$$\beta_k = h_2(x_k). \tag{5.3}$$

Then, there exists $M > 0$ such that, for all $k \in I\!N$,

$$\|x_k - x_*\| \le M \max\left\{\frac{\|\bar\lambda_k - \lambda_*\|_\infty}{\rho_k}, \|\alpha_k\|, \|\beta_k\|\right\}, \tag{5.4}$$

$$\|\lambda_{k+1} - \lambda_*\| \le M \max\left\{\frac{\|\bar\lambda_k - \lambda_*\|_\infty}{\rho_k}, \|\alpha_k\|, \|\beta_k\|\right\}. \tag{5.5}$$

Proof. Define, for all $k \in I\!N$,

$$t_k = (\bar\lambda_k - \lambda_*)/\rho_k, \tag{5.6}$$

$$\pi_k = 1/\rho_k. \tag{5.7}$$

By (3.5), (5.2) and (5.3), $\nabla L(x_k, \bar\lambda_k, \rho_k) + \nabla h_2(x_k) v_k - \alpha_k = 0$, $\lambda_{k+1} = \bar\lambda_k + \rho_k h_1(x_k)$ and $h_2(x_k) - \beta_k = 0$ for all $k \in I\!N$.

Therefore, by (5.6) and (5.7), we have that $\nabla f(x_k) + \nabla h_1(x_k)\lambda_{k+1} + \nabla h_2(x_k) v_k - \alpha_k = 0$, $h_1(x_k) - \pi_k \lambda_{k+1} + t_k + \pi_k \lambda_* = 0$ and $h_2(x_k) - \beta_k = 0$ for all $k \in I\!N$. Define, for all $\pi \in [0, 1/\bar\rho]$, $F_\pi : I\!R^n \times I\!R^{m_1} \times I\!R^{m_2} \times I\!R^{m_1} \times I\!R^n \times I\!R^{m_2} \to I\!R^n \times I\!R^{m_1} \times I\!R^{m_2}$ by

$$F_\pi(x, \lambda, v, t, \alpha, \beta) = \begin{pmatrix} \nabla f(x) + \nabla h_1(x)\lambda + \nabla h_2(x) v - \alpha \\ [h_1(x)]_1 - \pi[\lambda]_1 + [t]_1 + \pi[\lambda_*]_1 \\ \cdot \\ \cdot \\ \cdot \\ [h_1(x)]_{m_1} - \pi[\lambda]_{m_1} + [t]_{m_1} + \pi[\lambda_*]_{m_1} \\ h_2(x) - \beta \end{pmatrix}.$$

Clearly,

$$F_{\pi_k}(x_k, \lambda_{k+1}, v_k, t_k, \alpha_k, \beta_k) = 0 \tag{5.8}$$

and, by Assumptions 1 and 2,

$$F_\pi(x_*, \lambda_*, v_*, 0, 0, 0) = 0 \ \ \forall \pi \in [0, 1/\bar\rho]. \tag{5.9}$$

Moreover, the Jacobian matrix of F_π with respect to (x, λ, v) computed at $(x_*, \lambda_*, v_*, 0, 0, 0)$ is:

$$\begin{pmatrix} \nabla^2_{xx} L_0(x_*, \lambda_*, v_*) & \nabla h_1(x_*) & \nabla h_2(x_*) \\ \nabla h_1(x_*)^T & -\pi I & 0 \\ \nabla h_2(x_*)^T & 0 & 0 \end{pmatrix}.$$

By Lemma 5.2, this matrix is nonsingular for all $\pi \in [0, 1/\bar\rho]$. By continuity, the norm of its inverse is bounded in a neighborhood of $(x_*, \lambda_*, v_*, 0, 0, 0)$ uniformly with respect to $\pi \in [0, 1/\bar\rho]$. Moreover, the first and second derivatives of F_π are also bounded in a neighborhood of $(x_*, \lambda_*, v_*, 0, 0, 0)$ uniformly with respect to $\pi \in [0, 1/\bar\rho]$. Therefore, the bounds (5.4) and (5.5) follow from (5.8) and (5.9) by the Implicit Function Theorem and the Mean Value Theorem of Integral Calculus. □

THEOREM 5.4. *Suppose that Assumptions 1–6 are satisfied by the sequence generated by Algorithm 3.1 applied to the problem (5.1). In addition, assume that there* *exists a sequence $\eta_k \to 0$ such that $\varepsilon_k \le \eta_k \|h_1(x_k)\|_\infty$ for all $k \in I\!N$. Then, the sequence of penalty parameters $\{\rho_k\}$ is bounded.*

Proof. Assume, by contradiction, that $\lim_{k\to\infty} \rho_k = \infty$. Since $h_1(x_*) = 0$, by the continuity of the first derivatives of h_1 there exists $L > 0$ such that, for all $k \in I\!N$, $\|h_1(x_k)\|_\infty \le L\|x_k - x_*\|$. Therefore, by the hypothesis, (5.4) and Proposition 5.1, we have that $\|h_1(x_k)\|_\infty \le LM \max\left\{\frac{\|\lambda_k-\lambda_*\|_\infty}{\rho_k}, \eta_k\|h_1(x_k)\|_\infty\right\}$ for k large enough. Since η_k tends to zero, this implies that

$$\|h_1(x_k)\|_\infty \le LM\frac{\|\lambda_k - \lambda_*\|_\infty}{\rho_k} \tag{5.10}$$

for k large enough.

By (3.6) and Proposition 5.1, we have that $\lambda_k = \lambda_{k-1} + \rho_{k-1}h_1(x_{k-1})$ for k large enough. Therefore,

$$\|h_1(x_{k-1})\|_\infty = \frac{\|\lambda_k - \lambda_{k-1}\|_\infty}{\rho_{k-1}} \ge \frac{\|\lambda_{k-1} - \lambda_*\|_\infty}{\rho_{k-1}} - \frac{\|\lambda_k - \lambda_*\|_\infty}{\rho_{k-1}}. \tag{5.11}$$

Now, by (5.5), the hypothesis of this theorem and Proposition 5.1, for k large enough we have: $\|\lambda_k - \lambda_*\|_\infty \le M\left(\frac{\|\lambda_{k-1}-\lambda_*\|_\infty}{\rho_{k-1}} + \eta_{k-1}\|h_1(x_{k-1})\|_\infty\right)$. This implies that $\frac{\|\lambda_{k-1}-\lambda_*\|_\infty}{\rho_{k-1}} \ge \frac{\|\lambda_k-\lambda_*\|_\infty}{M} - \eta_{k-1}\|h_1(x_{k-1})\|_\infty$. Therefore, by (5.11), $(1+\eta_{k-1})\|h_1(x_{k-1})\|_\infty \ge \|\lambda_k - \lambda_*\|_\infty\left(\frac{1}{M} - \frac{1}{\rho_{k-1}}\right) \ge \frac{1}{2M}\|\lambda_k - \lambda_*\|_\infty$. Thus, $\|\lambda_k - \lambda_*\|_\infty \le 3M\|h_1(x_{k-1})\|_\infty$ for k large enough. By (5.10), we have that $\|h_1(x_k)\|_\infty \le \frac{3LM^2}{\rho_k}\|h_1(x_{k-1})\|_\infty$. Therefore, since $\rho_k \to \infty$, there exists $k_1 \in I\!N$ such that $\|h_1(x_k)\|_\infty \le \tau\|h_1(x_{k-1})\|_\infty$ for all $k \ge k_1$. So, $\rho_{k+1} = \rho_k$ for all $k \ge k_1$. Thus, $\{\rho_k\}$ is bounded. □

5.2. General constraints. In this subsection we address the general problem (2.1). As in the case of equality constraints, we adopt the following definition for the safeguarded Lagrange multipliers in Algorithm 3.1.

Definition. *For all $k \in I\!N$, $i = 1, \dots, m_1$, $j = 1, \dots, p_1$, $[\bar\lambda_{k+1}]_i$ will be the projection of $[\lambda_{k+1}]_i$ on the interval $[[\bar\lambda_{\min}]_i, [\bar\lambda_{\max}]_i]$ and $[\bar\mu_{k+1}]_j$ will be the projection of $[\mu_{k+1}]_j$ on $[0, [\bar\mu_{\max}]_j]$.*

The technique for proving boundedness of the penalty parameter consists of reducing (2.1) to a problem with (only) equality constraints. The equality constraints of the new problem will be the active constraints at the limit point x_*. After this reduction, the boundedness result is deduced from Theorem 5.4. The sufficient conditions are listed below.

Assumption 7. *The sequence $\{x_k\}$ is generated by the application of Algorithm 3.1 to problem (2.1) and $\lim_{k\to\infty} x_k = x_*$.*

Assumption 8. *The point x_* is feasible ($h_1(x_*) = 0$, $h_2(x_*) = 0$, $g_1(x_*) \le 0$ and $g_2(x_*) \le 0$.)*

Assumption 9. *The gradients $\{\nabla[h_1(x_*)]_i\}_{i=1}^{m_1}$, $\{\nabla[g_1(x_*)]_i\}_{[g_1(x_*)]_i=0}$, $\{\nabla[h_2(x_*)]_i\}_{i=1}^{m_2}$, $\{\nabla[g_2(x_*)]_i\}_{[g_2(x_*)]_i=0}$ are linearly independent. (LICQ holds at x_*.)*

Assumption 10. *The functions f, h_1, g_1, h_2 and g_2 admit continuous second derivatives in a neighborhood of x_*.*

Assumption 11. *Define the tangent subspace T as the set of all $z \in I\!R^n$ such that $\nabla h_1(x_*)^T z = \nabla h_2(x_*)^T z = 0$, $\langle\nabla[g_1(x_*)]_i, z\rangle = 0$ for all i such that $[g_1(x_*)]_i = 0$ and*

$\langle \nabla [g_2(x_*)]_i, z\rangle = 0$ *for all* i *such that* $[g_2(x_*)]_i = 0$. *Then, for all* $z \in T, z \neq 0$,

$$\langle z, [\nabla^2 f(x_*) + \textstyle\sum_{i=1}^{m_1} [\lambda_*]_i \nabla^2 [h_1(x_*)]_i + \sum_{i=1}^{p_1} [\mu_*]_i \nabla^2 [g_1(x_*)]_i$$
$$+ \textstyle\sum_{i=1}^{m_2} [v_*]_i \nabla^2 [h_2(x_*)]_i + \sum_{i=1}^{p_2} [u_*]_i \nabla^2 [g_2(x_*)]_i] z\rangle > 0.$$

Assumption 12. *For all* $i = 1, \dots, m_1, j = 1, \dots, p_1$, $[\lambda_*]_i \in ([\bar\lambda_{\min}]_i, [\bar\lambda_{\max}]_i)$, $[\mu_*]_j \in [0, [\bar\mu_{\max}]_j)$.

Assumption 13. *For all* i *such that* $[g_1(x_*)]_i = 0$, *we have* $[\mu_*]_i > 0$.

Observe that Assumption 13 imposes strict complementarity related only to the constraints in the upper-level set. In the lower-level set it is admissible that $[g_2(x_*)]_i = [u_*]_i = 0$. Observe, too, that Assumption 11 is weaker than the usual second-order sufficiency assumption, since the subspace T is orthogonal to the gradients of *all* active constraints, and no exception is made with respect to active constraints with null multiplier $[u_*]_i$. In fact, Assumption 11 is not a second-order sufficiency assumption for local minimizers. It holds for the problem of minimizing $x_1 x_2$ subject to $x_2 - x_1 \le 0$ at $(0, 0)$ although $(0, 0)$ is not a local minimizer of this problem.

THEOREM 5.5. *Suppose that Assumptions 7–13 are satisfied. In addition, assume that there exists a sequence* $\eta_k \to 0$ *such that* $\varepsilon_k \le \eta_k \max\{\|h_1(x_k)\|_\infty, \|\sigma_k\|_\infty\}$ *for all* $k \in I\!N$. *Then, the sequence of penalty parameters* $\{\rho_k\}$ *is bounded.*

Proof. Without loss of generality, assume that: $[g_1(x_*)]_i = 0$ if $i \le q_1$, $[g_1(x_*)]_i < 0$ if $i > q_1$, $[g_2(x_*)]_i = 0$ if $i \le q_2$, $[g_2(x_*)]_i < 0$ if $i > q_2$. Consider the auxiliary problem:

$$\text{Minimize } f(x) \text{ subject to } H_1(x) = 0, H_2(x) = 0, \tag{5.12}$$

where $H_1(x) = \begin{pmatrix} h_1(x) \\ [g_1(x)]_1 \\ \vdots \\ [g_1(x)]_{q_1} \end{pmatrix}$, $H_2(x) = \begin{pmatrix} h_2(x) \\ [g_2(x)]_1 \\ \vdots \\ [g_2(x)]_{q_2} \end{pmatrix}$.

By Assumptions 7–11, x_* satisfies the Assumptions 2–5 (with H_1, H_2 replacing h_1, h_2). Moreover, by Assumption 8, the multipliers associated to (2.1) are the Lagrange multipliers associated to (5.12).

As in the proof of (4.10) (first part of the proof of Theorem 4.2), we have that, for k large enough: $[[g_1(x_*)]_i < 0 \Rightarrow [\mu_{k+1}]_i = 0]$ and $[[g_2(x_*)]_i < 0 \Rightarrow [u_k]_i = 0]$.

Then, by (3.1), (3.5) and (3.7),

$$\|\nabla f(x_k) + \textstyle\sum_{i=1}^{m_1} [\lambda_{k+1}]_i \nabla [h_1(x_k)]_i + \sum_{i=1}^{q_1} [\mu_{k+1}]_i \nabla [g_1(x_k)]_i$$
$$+ \textstyle\sum_{i=1}^{m_2} [v_k]_i \nabla [h_2(x_k)]_i + \sum_{i=1}^{q_2} [u_k]_i \nabla [g_2(x_k)]_i\| \le \varepsilon_k$$

for k large enough.

By Assumption 9, taking appropriate limits in the inequality above, we obtain that $\lim_{k\to\infty} \lambda_k = \lambda_*$ and $\lim_{k\to\infty} \mu_k = \mu_*$.

In particular, since $[\mu_*]_i > 0$ for all $i \le q_1$,

$$[\mu_k]_i > 0 \tag{5.13}$$

for k large enough.

Since $\lambda_* \in (\bar\lambda_{\min}, \bar\lambda_{\max})^{m_1}$ and $[\mu_*]_i < [\bar\mu_{\max}]_i$, we have that $[\bar\mu_k]_i = [\mu_k]_i$, $i = 1, \dots, q_1$ and $[\bar\lambda_k]_i = [\lambda_k]_i$, $i = 1, \dots, m_1$ for k large enough.

Let us show now that the updating formula (3.7) for $[\mu_{k+1}]_i$, provided by Algorithm 3.1, coincides with the updating formula (3.5) for the corresponding multiplier in the application of the algorithm to the auxiliary problem (5.12).

In fact, by (3.7) and $[\bar{\mu}_k]_i = [\mu_k]_i$, we have that, for k large enough, $[\mu_{k+1}]_i = \max\{0, [\mu_k]_i + \rho_k[g_1(x_k)]_i\}$. But, by (5.13), $[\mu_{k+1}]_i = [\mu_k]_i + \rho_k[g_1(x_k)]_i, i = 1, \ldots, q_1$, for k large enough.

In terms of the auxiliary problem (5.12) this means that $[\mu_{k+1}]_i = [\mu_k]_i + \rho_k[H_1(x_k)]_i, i = 1, \ldots, q_1$, as we wanted to prove.

Now, let us analyze the meaning of $[\sigma_k]_i$. By (3.7), we have: $[\sigma_k]_i = \max\{[g_1(x_k)]_i, -[\bar{\mu}_k]_i/\rho_k\}$ for all $i = 1, \ldots, p_1$. If $i > q_1$, since $[g_1(x_*)]_i < 0$, $[g_1]_i$ is continuous and $[\bar{\mu}_k]_i = 0$, we have that $[\sigma_k]_i = 0$ for k large enough. Now, suppose that $i \le q_1$. If $[g_1(x_k)]_i < -\frac{[\bar{\mu}_k]_i}{\rho_k}$, then, by (3.7), we would have $[\mu_{k+1}]_i = 0$. This would contradict (5.13). Therefore, $[g_1(x_k)]_i \ge -\frac{[\bar{\mu}_k]_i}{\rho_k}$ for k large enough and we have that $[\sigma_k]_i = [g_1(x_k)]_i$. Thus, for k large enough,

$$H_1(x_k) = \begin{pmatrix} h_1(x_k) \\ \sigma_k \end{pmatrix}. \tag{5.14}$$

Therefore, the test for updating the penalty parameter in the application of Algorithm 3.1 to (5.12) coincides with the updating test in the application of the algorithm to (2.1). Moreover, formula (5.14) also implies that the condition $\varepsilon_k \le \eta_k \max\{\|\sigma_k\|_\infty, \|h_1(x_k)\|_\infty\}$ is equivalent to the hypothesis $\varepsilon_k \le \eta_k\|H_1(x_k)\|_\infty$ assumed in Theorem 5.4.

This completes the proof that the sequence $\{x_k\}$ may be thought as being generated by the application of Algorithm 3.1 to (5.12). We proved that the associated approximate multipliers and the penalty parameters updating rule also coincide. Therefore, by Theorem 5.4, the sequence of penalty parameters is bounded, as we wanted to prove. □

Remark. The results of this section provide a theoretical answer to the following practical question: What happens if the box chosen for the safeguarded multipliers estimates is too small? The answer is: the box should be large enough to contain the "true" Lagrange multipliers. If it is not, the global convergence properties remain but, very likely, the sequence of penalty parameters will be unbounded, leading to hard subproblems and possible numerical instability. In other words, if the box is excessively small, the algorithm tends to behave as an external penalty method. This is exactly what is observed in practice.

6. Numerical experiments. For solving unconstrained and bound-constrained subproblems we use Gencan [9] with second derivatives and a CG-preconditioner [10]. Algorithm 3.1 with Gencan will be called Algencan. For solving the convex-constrained subproblems that appear in the large location problems, we use the Spectral Projected Gradient method SPG [11, 12, 13]. The resulting Augmented Lagrangian algorithm is called Alspg. In general, it would be interesting to apply Alspg to any problem such that the selected lower-level constraints define a convex set for which it is easy (cheap) to compute the projection of an arbitrary point. The codes are free for download in *www.ime.usp.br/~egbirgin/tango/*. They are written in Fortran 77 (double precision). Interfaces of Algencan with AMPL, CUTEr, C/C++, Python and R (language and environment for statistical computing) are available and interfaces with Matlab and Octave are being developed.

For the practical implementation of Algorithm 3.1, we set $\tau = 0.5$, $\gamma = 10$, $\bar{\lambda}_{\min} = -10^{20}$, $\bar{\mu}_{\max} = \bar{\lambda}_{\max} = 10^{20}$, $\varepsilon_k = 10^{-4}$ for all k, $\bar{\lambda}_1 = 0, \bar{\mu}_1 = 0$ and $\rho_1 =$

$\max\left\{10^{-6}, \min\left\{10, \frac{2|f(x_0)|}{\|h_1(x_0)\|^2+\|g_1(x_0)_+\|^2}\right\}\right\}$. As stopping criterion we used $\max(\|h_1(x_k)\|_\infty, \|\sigma_k\|_\infty) \leq 10^{-4}$. The condition $\|\sigma_k\|_\infty \leq 10^{-4}$ guarantees that, for all $i = 1, \ldots, p_1$, $g_i(x_k) \leq 10^{-4}$ and that $[\mu_{k+1}]_i = 0$ whenever $g_i(x_k) < -10^{-4}$. This means that, approximately, feasibility and complementarity hold at the final point. Dual feasibility with tolerance 10^{-4} is guaranteed by (3.1) and the choice of ε_k. All the experiments were run on a 3.2 GHz Intel(R) Pentium(R) with 4 processors, 1Gb of RAM and Linux Operating System. Compiler option "-O" was adopted.

6.1. Testing the theory. In Discrete Mathematics, experiments should reproduce exactly what theory predicts. In the continuous world, however, the situation changes because the mathematical model that we use for proving theorems is not exactly isomorphic to the one where computations take place. Therefore, it is always interesting to interpret, in finite precision calculations, the continuous theoretical results and to verify to what extent they are fulfilled.

Some practical results presented below may be explained in terms of a simple theoretical result that was tangentially mentioned in the introduction: If, at Step 2 of Algorithm 3.1, one computes a global minimizer of the subproblem and the problem (2.1) is feasible, then every limit point is a global minimizer of (2.1). This property may be easily proved using boundedness of the safeguarded Lagrange multipliers by means of external penalty arguments. Now, algorithms designed to solve reasonably simple subproblems usually include practical procedures that actively seek function decrease, beyond the necessity of finding stationary points. For example, efficient line-search procedures in unconstrained minimization and box-constrained minimization usually employ aggressive extrapolation steps [9], although simple backtracking is enough to prove convergence to stationary points. In other words, from good subproblem solvers one expects much more than convergence to stationary points. For this reason, we conjecture that Augmented Lagrangian algorithms like ALGENCAN tend to converge to global minimizers more often than SQP-like methods. In any case, these arguments support the necessity of developing global-oriented subproblem solvers.

Experiments in this subsection were made using the AMPL interfaces of ALGENCAN (considering all the constraints as upper-level constraints) and IPOPT. Presolve AMPL option was disabled to solve the problems exactly as they are. The ALGENCAN parameters and stopping criteria were the ones stated at the beginning of this section. For IPOPT we used all its default parameters (including the ones related to stopping criteria). The random generation of initial points was made using the function `Uniform01()` provided by AMPL. When generating several random initial points, the seed used to generate the i-th random initial point was set to i.

Example 1: *Convergence to KKT points that do not satisfy MFCQ.*

$$\begin{array}{ll} \text{Minimize} & x_1 \\ \text{subject to} & x_1^2 + x_2^2 \leq 1, \\ & x_1^2 + x_2^2 \geq 1. \end{array}$$

The global solution is $(-1, 0)$ and no feasible point satisfies the Mangasarian-Fromovitz Constraint Qualification, although all feasible points satisfy CPLD. Starting with 100 random points in $[-10, 10]^2$, ALGENCAN converged to the global solution in all the cases. Starting from $(5, 5)$ convergence occurred using 14 outer iterations. The final penalty parameter was 4.1649E-01 (the initial one was 4.1649E-03) and the final multipliers were 4.9998E-01 and 0.0000E+00. IPOPT also found the global solution in all the cases and used 25 iterations when starting from $(5, 5)$.

Example 2: *Convergence to a non-KKT point.*

$$\begin{array}{ll} \text{Minimize} & x \\ \text{subject to} & x^2 = 0, \\ & x^3 = 0, \\ & x^4 = 0. \end{array}$$

Here the gradients of the constraints are linearly dependent for all $x \in I\!R$. In spite of this, the only point that satisfies Theorem 4.1 is $x = 0$. Starting with 100 random points in $[-10, 10]$, ALGENCAN converged to the global solution in all the cases. Starting with $x = 5$ convergence occurred using 20 outer iterations. The final penalty parameter was 2.4578E+05 (the initial one was 2.4578E-05) and the final multipliers were 5.2855E+01 -2.0317E+00 and 4.6041E-01. IPOPT was not able to solve the problem in its original formulation because “Number of degrees of freedom is NIND = -2”. We modified the problem in the following way

$$\begin{array}{ll} \text{Minimize} & x_1 + x_2 + x_3 \\ \text{subject to} & x_1^2 = 0, \\ & x_1^3 = 0, \\ & x_1^4 = 0, \\ & x_i \geq 0, i = 1, 2, 3, \end{array}$$

and, after 16 iterations, IPOPT stopped near $x = (0, +\infty, +\infty)$ saying “Iterates become very large (diverging?)”.

Example 3: *Infeasible stationary points* [18, 34].

$$\begin{array}{ll} \text{Minimize} & 100(x_2 - x_1^2)^2 + (x_1 - 1)^2 \\ \text{subject to} & x_1 - x_2^2 \leq 0, \\ & x_2 - x_1^2 \leq 0, \\ & -0.5 \leq x_1 \leq 0.5, \\ & x_2 \leq 1. \end{array}$$

This problem has a global KKT solution at $x = (0, 0)$ and a stationary infeasible point at $x = (0.5, \sqrt{0.5})$. Starting with 100 random points in $[-10, 10]^2$, ALGENCAN converged to the global solution in all the cases. Starting with $x = (5, 5)$ convergence occurred using 6 outer iterations. The final penalty parameter was 1.0000E+01 (the initial one was 1.0000E+00) and the final multipliers were 1.9998E+00 and 3.3390E-03. IPOPT found the global solution starting from 84 out of the 100 random initial points. In the other 16 cases IPOPT stopped at $x = (0.5, \sqrt{0.5})$ saying “Convergence to stationary point for infeasibility” (this was also the case when starting from $x = (5, 5)$).

Example 4: *Difficult-for-barrier* [15, 18, 47].

$$\begin{array}{ll} \text{Minimize} & x_1 \\ \text{subject to} & x_1^2 - x_2 + a = 0, \\ & x_1 - x_3 - b = 0, \\ & x_2 \geq 0, \ x_3 \geq 0. \end{array}$$

In [18] we read: "This test example is from [47] and [15]. Although it is well-posed, many barrier-SQP methods ('Type-I Algorithms' in [47]) fail to obtain feasibility for a range of infeasible starting points."

We ran two instances of this problem varying the values of parameters a and b and the initial point x_0 as suggested in [18]. When $(a, b) = (1, 1)$ and $x_0 = (-3, 1, 1)$ ALGENCAN converged to the solution $\bar{x} = (1, 2, 0)$ using 2 outer iterations. The final penalty parameter was 5.6604E-01 (the initial one also was 5.6604E-01) and the final multipliers were 6.6523E-10 and -1.0000E+00. IPOPT also found the same solution using 20 iterations. When $(a, b) = (-1, 0.5)$ and $x_0 = (-2, 1, 1)$ ALGENCAN converged to the solution $\tilde{x} = (1, 0, 0.5)$ using 5 outer iterations. The final penalty parameter was 2.4615E+00 (the initial one also was 2.4615E+00) and the final multipliers were -5.0001E-01 and -1.3664E-16. On the other hand, IPOPT stopped declaring convergence to a stationary point for the infeasibility.

Example 5: *Preference for global minimizers*

$$\begin{array}{ll} \text{Minimize} & \sum_{i=1}^n x_i \\ \text{subject to} & x_i^2 = 1, i = 1, \ldots, n. \end{array}$$

Solution: $x_* = (-1, \ldots, -1)$, $f(x_*) = -n$. We set $n = 100$ and ran ALGENCAN and IPOPT starting from 100 random initial points in $[-100, 100]^n$. ALGENCAN converged to the global solution in all the cases while IPOPT never found the global solution. When starting from the first random point, ALGENCAN converged using 4 outer iterations. The final penalty parameter was 5.0882E+00 (the initial one was 5.0882E-01) and the final multipliers were all equal to 4.9999E-01.

6.2. Location problems. Here we will consider a variant of the family of *location* problems introduced in [12]. In the original problem, given a set of n_p disjoint polygons $P_1, P_2, \ldots, P_{n_p}$ in $I\!R^2$ one wishes to find the point $z^1 \in P_1$ that minimizes the sum of the distances to the other polygons. Therefore, the original problem formulation is:

$$\min_{z^i,\, i=1,\ldots,n_p} \quad \frac{1}{n_p - 1} \sum_{i=2}^{n_p} \|z^i - z^1\|_2 \quad \text{subject to} \quad z^i \in P_i, \; i = 1, \ldots, n_p.$$

In the variant considered in the present work, we have, in addition to the n_p polygons, n_c circles. Moreover, there is an ellipse which has a non empty intersection with P_1 and such that z_1 must be inside the ellipse and $z_i, i = 2, \ldots, n_p + n_c$ must be outside. Therefore, the problem considered in this work is

$$\min_{z^i,\, i=1,\ldots,n_p+n_c} \quad \frac{1}{n_c + n_p - 1} \left[\sum_{i=2}^{n_p} \|z^i - z^1\|_2 + \sum_{i=1}^{n_c} \|z^{n_p+i} - z^1\|_2 \right]$$

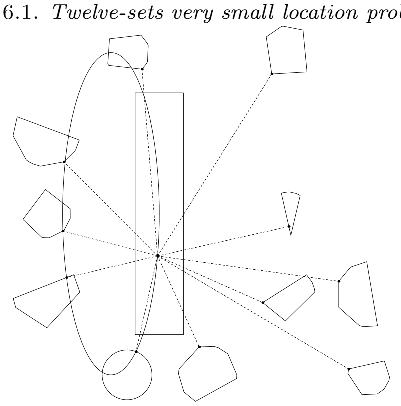
$$\begin{array}{llcll} \text{subject to} & g(z^1) & \leq & 0, & \\ & g(z^i) & \geq & 0, & i = 2, \ldots, n_p + n_c, \\ & z^i & \in & P_i, & i = 1, \ldots, n_p, \\ & z^{n_p+i} & \in & C_i, & i = 1, \ldots, n_c, \end{array}$$

where $g(x) = (x_1/a)^2 + (x_2/b)^2 - c$, and $a, b, c \in I\!R$ are positive constants. Observe that the objective function is differentiable in a large open neighborhood of the feasible region. To solve this family of problems, we will consider $g(z^1) \le 0$ and $g(z^i) \ge 0, i = 2, \ldots, n_p + n_c$ as upper-level constraints, and $z^i \in P_i, i = 1, \ldots, n_p$ and $z^{n_p+i} \in C_i, i = 1, \ldots, n_c$ as lower-level constraints. In this way the subproblems can be efficiently solved by the Spectral Projected Gradient method (SPG) [11, 12] as suggested by the experiments in [12].

We generated 36 problems of this class, varying n_c and n_p and choosing randomly the location of the circles and polygons and the number of vertices of each polygon. Details of the generation, including the way in which we guarantee empty intersections (in order to have differentiability everywhere), may be found in [12] and its related code (also available in *www.ime.usp.br/∼egbirgin/tango/*), where the original problem was introduced. Moreover, details of the present variant of the problem can be found within its fully commented Fortran 77 code also available in *www.ime.usp.br/∼egbirgin/tango/*. In Table 6.1 we display the main characteristics of each problem (number of circles, number of polygons, total number of vertices of the polygons, dimension of the problem and number of lower-level and upper-level constraints). Figure 6.1 shows the solution of a very small twelve-sets problem that has 24 variables, 81 lower-level constraints and 12 upper-level constraints.

The 36 problems are divided in two sets of 18 problems: small and large problems. We first solved the small problems with ALGENCAN (considering all the constraints as upper-level constraints) and ALSPG. Both methods use the Fortran 77 formulation of the problem (ALSPG needs an additional subroutine to compute the projection of an arbitrary point onto the convex set given by the lower-level constraints). In Table 6.2 we compare the performance of both methods for solving this problem. Both methods obtain feasible points and arrive to the same solutions. Due to the performance of ALSPG, we also solved the set of large problems using it. Table 6.3 shows its performance. A comparison against IPOPT was made and, while IPOPT was able to find equivalent solutions for the smaller problems, it was unable to handle the larger problems due to memory requirements.

FIG. 6.1. *Twelve-sets very small location problem.*



7. Final Remarks. In the last few years many sophisticated algorithms for nonlinear programming have been published. They usually involve combinations of interior-point techniques, sequential quadratic programming, trust regions, restoration, nonmonotone strategies and advanced sparse linear algebra procedures. See, for example [17, 28, 30, 31, 32, 37] and the extensive reference lists of these papers. Moreover, methods for solving efficiently specific problems or for dealing with special

TABLE 6.1

Location problems and their main features. The problem generation is based on a grid. The number of city-circles (n_c) and city-polygons (n_p) depend on the number of points in the grid, the probability of having a city in a grid point (procit) and the probability of a city to be a polygon (propol) or a circle ($1 - propol$). The number of vertices of a city-polygon is a random number and the total number of vertices of all the city-polygons together is totnvs. Finally, the number of variables of the problem is $n = 2(n_c + n_p)$, the number of upper-level inequality constraints is $p_1 = n_c + n_p$ and the number of lower-level inequality constraints is $p_2 = n_c + totnvs$. The total number of constraints is $p_1 + p_2$. The central rectangle is considered here a "special" city-polygon. The lower-level constraints correspond to the fact that each point must be inside a city and the upper-level constraints come from the fact that the central point must be inside the ellipse and all the others must be outside.

Problem	n_c	n_p	*totnvs*	n	p_1	p_2
1	28	98	295	252	126	323
2	33	108	432	282	141	465
3	33	108	539	282	141	572
4	33	109	652	284	142	685
5	35	118	823	306	153	858
6	35	118	940	306	153	975
7	35	118	1,057	306	153	1,092
8	35	118	1,174	306	153	1,209
9	35	118	1,291	306	153	1,326
10	35	118	1,408	306	153	1,443
11	35	118	1,525	306	153	1,560
12	35	118	1,642	306	153	1,677
13	35	118	1,759	306	153	1,794
14	35	118	1,876	306	153	1,911
15	35	118	1,993	306	153	2,028
16	35	118	2,110	306	153	2,145
17	35	118	2,227	306	153	2,262
18	35	118	2,344	306	153	2,379
19	3,029	4,995	62,301	16,048	8,024	65,330
20	4,342	7,271	91,041	23,226	11,613	95,383
21	6,346	10,715	133,986	34,122	17,061	140,332
22	13,327	22,230	278,195	71,114	35,557	291,522
23	19,808	33,433	417,846	106,482	53,241	437,654
24	29,812	50,236	627,548	160,096	80,048	657,360
25	26,318	43,970	549,900	140,576	70,288	576,218
26	39,296	66,054	825,907	210,700	105,350	865,203
27	58,738	99,383	1,241,823	316,242	158,121	1,300,561
28	65,659	109,099	1,363,857	349,516	174,758	1,429,516
29	98,004	164,209	2,052,283	524,426	262,213	2,150,287
30	147,492	245,948	3,072,630	786,880	393,440	3,220,122
31	131,067	218,459	2,730,798	699,052	349,526	2,861,865
32	195,801	327,499	4,094,827	1,046,600	523,300	4,290,628
33	294,327	490,515	6,129,119	1,569,684	784,842	6,423,446
34	261,319	435,414	5,442,424	1,393,466	696,733	5,703,743
35	390,670	654,163	8,177,200	2,089,666	1,044,833	8,567,870
36	588,251	979,553	12,244,855	3,135,608	1,567,804	12,833,106

constraints are often introduced. Many times, a particular algorithm is extremely efficient for dealing with problems of a given type, but fails (or cannot be applied) when constraints of a different class are incorporated. This situation is quite common in engineering applications. In the Augmented Lagrangian framework additional constraints are naturally incorporated to the objective function of the subproblems, which therefore preserve their constraint structure. For this reason, we conjecture that the Augmented Lagrangian approach (with general lower-level constraints) will continue to be used for many years.

This fact motivated us to improve and analyze Augmented Lagrangian methods with arbitrary lower-level constraints. From the theoretical point of view our goal was to eliminate, as much as possible, restrictive constraint qualifications. With this in mind we used, both in the feasibility proof and in the optimality proof, the Constant Positive Linear Dependence (CPLD) condition. This condition [41] has been

Table 6.2
Performance of Algencan *and* Alspg *in the set of small location problems.*

Problem	Algencan OuIt	InIt	Fcnt	Gcnt	Time	Alspg OuIt	InIt	Fcnt	Gcnt	Time	f
1	7	127	1309	142	1.21	3	394	633	397	0.10	1.7564E+01
2	6	168	1921	181	1.15	3	614	913	617	0.16	1.7488E+01
3	6	150	1818	163	1.02	3	736	1127	739	0.21	1.7466E+01
4	9	135	972	154	0.63	3	610	943	613	0.18	1.7451E+01
5	5	213	2594	224	1.16	3	489	743	492	0.15	1.7984E+01
6	5	198	2410	209	1.21	3	376	547	379	0.12	1.7979E+01
7	5	167	1840	178	1.71	3	332	510	335	0.11	1.7975E+01
8	3	212	2548	219	1.34	3	310	444	313	0.11	1.7971E+01
9	3	237	3116	244	1.49	3	676	1064	679	0.25	1.7972E+01
10	3	212	2774	219	1.27	3	522	794	525	0.20	1.7969E+01
11	3	217	2932	224	1.46	3	471	720	474	0.19	1.7969E+01
12	3	208	2765	215	1.40	3	569	872	572	0.23	1.7968E+01
13	3	223	2942	230	1.44	3	597	926	600	0.25	1.7968E+01
14	3	272	3981	279	2.12	3	660	1082	663	0.29	1.7965E+01
15	3	278	3928	285	2.20	3	549	834	552	0.24	1.7965E+01
16	3	274	3731	281	2.52	3	565	880	568	0.26	1.7965E+01
17	3	257	3186	264	2.31	3	525	806	528	0.24	1.7963E+01
18	3	280	3866	287	2.39	3	678	1045	681	0.32	1.7963E+01

Table 6.3
Performance of Alspg *on set of large location problems. The memory limitation (to generate and save the problems statement) is the only inconvenience for* Alspg *solving problems with higher dimension than problem 36 (approximately* 3×10^6 *variables,* 1.5×10^6 *upper-level inequality constraints, and* 1.2×10^7 *lower-level inequality constraints), since computer time is quite reasonable.*

Problem	Alspg OuIt	InIt	Fcnt	Gcnt	Time	f
19	8	212	308	220	3.46	4.5752E+02
20	8	107	186	115	2.75	5.6012E+02
21	9	75	149	84	3.05	6.8724E+02
22	7	80	132	87	5.17	4.6160E+02
23	7	71	125	78	7.16	5.6340E+02
24	8	53	106	61	8.72	6.9250E+02
25	8	55	124	63	8.00	4.6211E+02
26	7	63	127	70	12.56	5.6438E+02
27	9	80	155	89	19.84	6.9347E+02
28	8	67	138	75	22.24	4.6261E+02
29	7	54	107	61	27.36	5.6455E+02
30	9	95	179	104	51.31	6.9382E+02
31	7	59	111	66	39.12	4.6280E+02
32	7	66	120	73	63.35	5.6449E+02
33	9	51	113	60	85.65	6.9413E+02
34	7	58	110	65	79.38	4.6270E+02
35	7	50	104	57	107.27	5.6432E+02
36	10	56	133	66	190.59	6.9404E+02

proved to be a constraint qualification in [4] where its relations with other constraint qualifications have been given.

We provided a family of examples (Location Problems) where the potentiality of the arbitrary lower-level approach is clearly evidenced. This example represents a typical situation in applications. A specific algorithm (SPG) is known to be very efficient for a class of problems but turns out to be impossible to apply when additional constraints are incorporated. However, the Augmented Lagrangian approach is able to deal with the additional constraints taking advantage of the efficiency of SPG for solving the subproblems. In this way, we were able to solve nonlinear programming problems with more than 3,000,000 variables and 14,000,000 constraints in less than five minutes of CPU time.

Open problems related to theory and implementation of practical Augmented

Lagrangian methods may be found in the expanded report [3].

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