A PRACTICE-BASED THEORY OF MATHEMATICAL KNOWLEDGE FOR TEACHING
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A central challenge of professional education is to prepare novices for skilful beginning practice. Doing this depends on robust theory about the relationship between teaching practice and teacher education. Theories of instruction can help provide a foundation for a professional curriculum centred on learning the practice of teaching. Teachers need to know and be able to use mathematics in the work of teaching pupils and they need to be able to carry off specific instructional practices that enable pupils to learn. This paper reports from a set of coordinated studies that have developed a practice-based theory of mathematical knowledge for teaching and theories about ways to teach it, designed assessments keyed to this practice-based theory of teachers’ usable knowledge of mathematics, and developed a coding scheme for appraising the mathematical quality of instruction.

INTRODUCTION
A central challenge of professional education is to prepare novices for skilful beginning practice. Doing this depends on robust theory about the relationship between teaching practice and teacher education. Theories of instruction provide a foundation for a professional curriculum centred on learning the practice of teaching.

One particularly vexing problem has been to determine the mathematical knowledge needed for teaching. Teachers need to do more than simply know mathematics. They need to be able to use mathematics in the work of teaching pupils. Research on teachers’ mathematical knowledge has investigated what practicing teachers do know, and our literature is replete with studies that show the lacks in teachers’ content knowledge. Others have created lists of what teachers should know, based usually on the curriculum they are responsible for teaching or on expert opinion about what would be good for teachers to know. These lists are not empirically connected to the work of teaching - that is, we do not know whether teachers who know these things actually teach better than those who do not. Although important, neither of these approaches to identifying the mathematical knowledge needed for teaching is tied to the work of teaching, and hence, neither is warranted by the demands of the work.

The introduction of pedagogical content knowledge (Shulman, 1986) as a special domain of teacher knowledge was important for distinguishing the personal knowledge of content (knowing content for oneself) from the special amalgam of content and pedagogy needed to teach the subject. Important here was the naming of
a type of content knowledge uniquely needed by teachers—a subject-matter-based form of professional knowledge. The notion of pedagogical content knowledge quickly caught the imagination of researchers, not only in mathematics but also in science education. Still, the term was underdefined, and scholars and teacher educators used the notion in different ways.

**MATHEMATICAL KNOWLEDGE FOR TEACHING**

Our research group decided to investigate the question more directly by asking, “What do teachers do in teaching mathematics, and how does what they do demand mathematical reasoning, insight, understanding, and skill?” We oriented our investigation of the mathematical knowledge needed for teaching in studies of the practice of teaching. We sought to uncover the ways in which mathematics is involved in contending with the regular day-to-day, moment-to-moment demands of teaching. Our analyses lay the foundation for a *practice-based theory of mathematical knowledge for teaching* (Ball & Bass, 2003). This approach can be seen as a kind of job analysis, similar to analyses done of other mathematically intensive occupations that range from nursing and engineering physics (Hoyle, Noss, & Pozzi, 2001; Noss, Healy, & Hoyles, 1997) to carpentry and waiting tables.

By “mathematical knowledge for teaching” (MKT), we mean the mathematical knowledge needed to carry out the work of teaching mathematics. We focus on the tasks involved in teaching and analyse the mathematical demands of these tasks. Obviously, because teaching involves showing pupils how to solve problems, answering learners’ questions, and checking their work, mathematical knowledge for teaching requires understanding the school curriculum. However, it also requires mathematical understanding beyond what can be seen on the tables of contents of school textbooks or in curriculum frameworks.

The fundamental questions that orient this theoretical approach are:

1. What are the recurrent tasks and problems of teaching mathematics? What do teachers do as they teach mathematics?
2. What mathematical knowledge, skills, and sensibilities are required to manage these tasks?

By “teaching,” we mean everything that teachers do to support the learning of their pupils. Clearly we mean the interactive work of teaching lessons in classrooms, and all the tasks that arise in the course of that work. But we also mean planning for those lessons, evaluating pupils’ work, writing and grading assessments, explaining learners’ progress to parents, making and managing homework, attending to issues of equity, and justifying one’s decisions to the school head.

Central to the progress of this work has been a large longitudinal database, documenting an entire year of the mathematics teaching in a grade 3 public school classroom. The records collected across that year include videotapes and audiotapes
A MULTI-DIMENSIONAL MODEL

We define the mathematical knowledge we are studying as mathematical knowledge “entailed by teaching”—in other words, mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to pupils. To avoid a narrow perspective, we use a generous conception of “need” that allows for the perspective, habits of mind, and appreciation that matter for effective teaching of the discipline.

This program of research, based on studies of practice, and tested empirically (Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2004, Hill, Rowan, & Ball, 2005) has yielded a multi-dimensional model of the mathematical knowledge needed for the work of teaching, depicted in Figure 1. We use the generic label (“content knowledge”) to highlight the conceptual structure of the theory, and to encourage scholars in other disciplines to test its broader applicability.

Note the labels, “subject matter knowledge” and “pedagogical content knowledge” that head the model. On the left hand side of the diagram is the mathematical knowing in teaching that is purely mathematics; the right hand side comprises mixtures of knowing about pupils or knowing ways of teaching in the context of particular mathematical topics. By “common content knowledge” (CCK), we mean the knowledge of mathematics needed in common with others who also know and use mathematics. This is distinct from “specialized content knowledge” (SCK), which is a form of knowing mathematics not needed by those who do not teach. This specialized kind of knowing includes being able to figure out whether an unusual learner solution works in general or is just a fluke, for example, or considering the differences and equivalences among several different representations for a division problem.
Knowledge at the mathematical horizon” is the understanding of the broader set of mathematical ideas to which a particular idea connects. It is the sort of understanding that gives teachers peripheral vision for where they are and where their pupils are heading, to be conscious of the consequences of how ideas are represented, or the later development that is enabled — or possibly impeded — by decisions within the current work. The domains represented on the right of the diagram (“knowledge of content and students” [KCS], “knowledge of teaching and students” [KCT]; and “knowledge of curriculum) comprise the special amalgams that are deeply embedded in the work of teaching—knowing, for example, what makes a topic difficult for pupils, or the ways in which learners tend to develop understanding of a particular idea, or ways to sequence and structure the development of a mathematical topic, including representations likely to help pupils learn.

CONCLUDING SUMMARY

In summary, our approach to developing theory about the mathematical knowledge needed for teaching is framed in relation to practice. It is centered on the work and its demands rather than on what teachers do know or might need to know. Working from practice, we ask about the situations that arise in teaching that require teachers to use mathematics. Still, despite our expressed intention to focus on knowledge use, our categories remain rather static. Needed is to understand how teachers reason about and deploy mathematical ideas in their work. This includes skills, habits, sensibilities, and ways of reasoning as well as “knowledge.” The questions we pose are intended to measure teacher knowledge in the context of its use, but how knowledge is actually used and what features of teacher thinking shape its use remain tacit and unexamined. How to capture the common and specialized aspects of the mathematical reasoning of teaching practice, as well as how different categories of knowledge come into play in the course of teaching, is a next important step in this program of work.