The Effect of Combination Functions on the Complexity of Relational Bayesian Networks

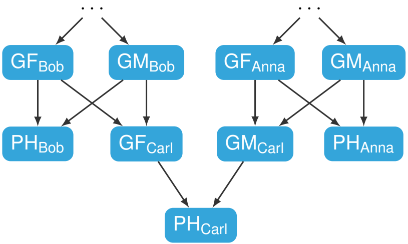
Denis D. Mauá and Fabio G. Cozman

Universidade de São Paulo, Brazil

September 7, 2016

Solution to many tasks comprises building a concrete Bayesian network from some template model for each given domain

- Object recognition
- Viral marketing
- Entity resolution
- Social network analysis
- ...



“with probability α, an individual carries disease if both parents have a certain gene”

Relational Bayesian Networks

- declarative approach for specifying abstract models
- as expressive as other probabilistic relational languages
- explicit representation of repetition, determinism and context-specific independence, which can be used to speed up inference
- clear and rigorously defined syntax/semantics

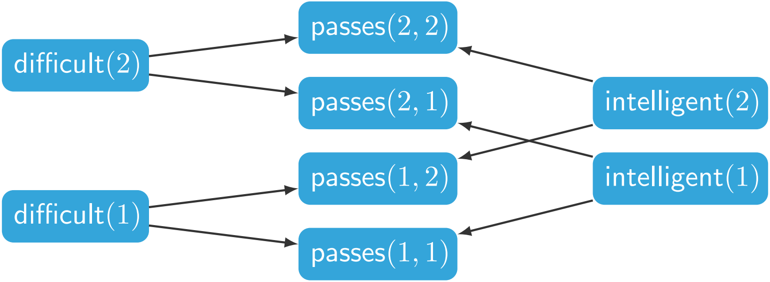
- Complexity of inference with RBNs has not been thoroughly examined yet
- In particular, the effect of combination functions, which allow summarizing information from different individuals
- This work: Complexity of marginal inference as parametrized by combination functions

Relational Bayesian Network

Acyclic directed graph where each node is a relation symbol annotated with a probability formula over atoms, numbers and combination functions

$\Pr(\text{difficult}) = 0.3$ $\qquad\qquad$ $\Pr(\text{intelligent}) = 0.6$

$$\Pr(\text{passes}|\text{difficult}, \text{intelligent}) =$$
$$(0.4 \cdot \text{difficult}(X) + 0.95(1 - \text{diffilcult}(X))) \cdot \text{intelligent}(Y) +$$
$$(0.1 \cdot \text{difficult}(X) + 0.5(1 - \text{difficult}(X)))(1 - \text{intelligent}(Y))$$



$$
\begin{aligned}
&\Pr(\textsf{difficult}(1)) = \Pr(\textsf{difficult}(2)) = 0.3\\
&\Pr(\textsf{intelligent}(1)) = \Pr(\textsf{intelligent}(2)) = 0.6\\
&\Pr(\textsf{passes}(1,2) | \textsf{difficult}(1), \textsf{intelligent}(2)) =\\
&\qquad (0.4\textsf{difficult}(1) + 0.95(1 - \textsf{diffilcult}(1)))\textsf{intelligent}(2) +\\
&\qquad (0.1\textsf{difficult}(1) + 0.5(1 - \textsf{difficult}(1)))(1 - \textsf{intelligent}(2))
\end{aligned}
$$

Probability Formula

- A rational $q \in [0,1]$ or an atom $r(X_1, \ldots, X_{|r|})$
- Convex combination $F_1 F_2 + (1 - F_1) F_3$
- Combination expression:

$$\mathrm{comb}\{F_1, \ldots, F_k | Y_1, \ldots, Y_m; \alpha\},$$

  where α is an equality constraint such as

$$X = Y \text{ or } (\neg X = Y) \vee (Y = Z \wedge X = Z)$$

  - Summarize information about individuals

Example:

$$F(X) = \mathrm{mean}\{0.6 \cdot r(X) + 0.7 \cdot \max\{1 - s(X,Y) | X; X = X\} | Y, Z; Y \neq X \wedge Z \neq X\}$$

INTERPRETATION

- Set of constants $\mathcal{D}$ (domain)
- Map μ:
  - relation symbol r into relation $r^\mu \subseteq \mathcal{D}^{|r|}$
  - equality constraint into standard semantics
  - probability formula F into function $F^\mu : \mathcal{D}^n \rightarrow [0,1]$, where n is the number of free variables in F

PROBABILITY FORMULA

$$q \xrightarrow{\mu} q$$

$$r(X_1, \ldots, X_{|r|}) \xrightarrow{\mu} F^\mu(a) = \begin{cases} 1 & \text{if } a \in r^\mu \\ 0 & \text{if } a \notin r^\mu \end{cases}$$

$$F_1F_2 + (1 - F_1)F_3 \xrightarrow{\mu} F_1^\mu(a)F_2^\mu(a) + (1 - F_1^\mu(a))F_3^\mu(a)$$

PROBABILITY FORMULA

If $F = \text{comb}\{F_1, \ldots, F_k | Y_1, \ldots, Y_m; \alpha\}$, then $F^\mu(a) = \text{comb}(\mathcal{Q})$ where

- comb is the combination function
- $\mathcal{Q}$ is the multiset containing a number $F_i(a, b)$ for every $(a, b) \in \alpha^\mu$

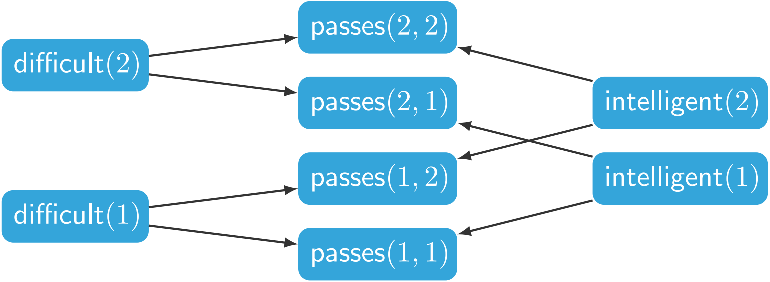
Example:

- $\mathcal{D} = \{1, 2\};\ \text{max}\{\frac{1}{3}, \frac{2}{3}, 1 | Y; Y = Y\} \xrightarrow{\mu} \max\{\frac{1}{3}, \frac{2}{3}, 1, \frac{1}{3}, \frac{2}{3}, 1\}$

Given a domain $\mathcal{D}$, an RBN with graph $G = (V, A)$ induces a probability distribution over interpretations μ by

$$\Pr(\mu|\mathcal{D}) = \prod_{r \in V} \underbrace{\prod_{a \in r^\mu} F_r(a) \prod_{a \notin r^\mu} (1 - F_r(a))}_{\Pr(r^\mu|\{s^\mu : s \in \mathrm{pa}(r)\}, \mathcal{D})},$$

where occurrences of $s \in \mathrm{pa}(r)$ in probability formula $F_r(a)$ are interpreted according to s^μ



$$F_{\text{passes}}(1,2) = \\ (0.4\text{difficult}(1) + 0.95(1 - \text{diffilcult}(1)))\text{intelligent}(2) + \\ (0.1\text{difficult}(1) + 0.5(1 - \text{difficult}(1)))(1 - \text{intelligent}(2))$$

Complexity of Marginal Inference

Given:

- Relational Bayesian Network (graph and probability formulas)
- domain $\mathcal{D}$ specified as a list of elements
- ground atom $r(a)$

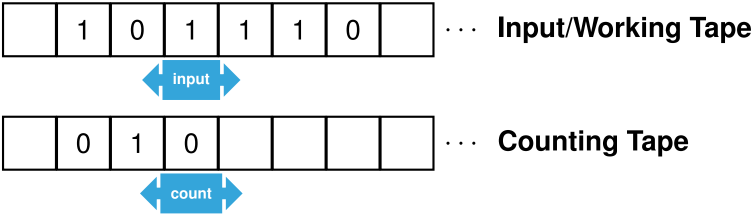
Compute:

$$\Pr(r(a) = 1) = \sum_{\mu : a \in r^{\mu}} \Pr(\mu | \mathcal{D})$$

Note: $\Pr(r_1(a_1) = \alpha_1, \ldots, r_n(a_n) = \alpha_n)$ can be computed as $\Pr(r(a) = 1)$ by defining $F_r = [1-]r_1(X_1) \cdot \cdots \cdot [1-]r_n(_n)$; conditional probability can be computed with two such calls

COUNTING TURING MACHINE

Non-deterministic Turing machine that writes on a separate tape and in binary notation the number of accepting paths



#P

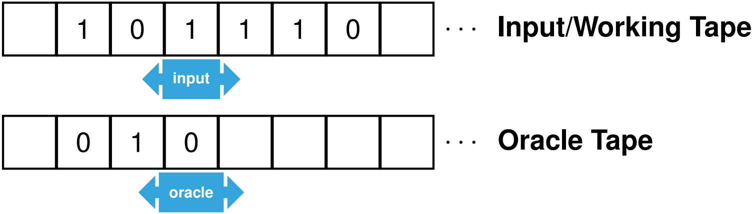
integer-valued functions computed by counting Turing machine with polynomial steps

#EXP

integer-valued functions computed by counting Turing machine with exponential steps

ORACLE TURING MACHINE

Turing machine which can query an "oracle", which then reads/writes content from/to the oracle tape in one step



ORACLE COMPLEXITY CLASSES:

For complexity classes A and B, say that a problem is in A^B if it can be computed by an A-complete problem with access to an oracle that is complete for B. Examples: P^{NP}, $\mathsf{NP}^{\mathsf{PSPACE}}$.

$\#\Sigma_k^p = \#\mathsf{P}^{\overbrace{\mathsf{NP}^{\mathsf{NP}^{\cdots}}}^{k \text{ times}}}$

integer-valued functions computed by counting Turing machine with oracle Σ_k^p (k "stacks" of NP machines) with polynomial steps

$\#\mathsf{PH} = \#\mathsf{P}^{\mathsf{PH}}$

integer-valued functions computed by counting Turing machine with oracle PH (arbitrary stacking of NP machines) with polynomial steps

$$\mathsf{FP} \subseteq \mathsf{NP} \subseteq \#\mathsf{P} \subseteq \#\Sigma_1^p \subseteq \cdots \subseteq \#\mathsf{PH} \subseteq \underset{=\#\mathsf{PSPACE}}{\mathsf{FPSPACE}} \subseteq \#\mathsf{EXP} \subseteq \mathsf{FEXP}$$

COMPLETENESS

A problem is said complete for a class if it is the hardest problem in that class

We cannot establish e.g. #P-completeness of inference as it returns rationals (and #P does not seem to be closed under division)

EQUIVALENCE

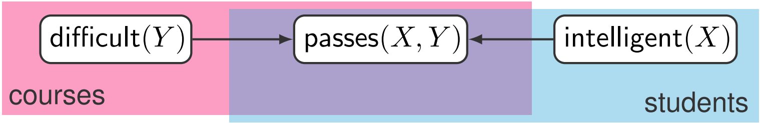
A problem A is X-equivalent for counting class X if it is X-hard (via parsimonious reduction) up to a polynomial scaling and can be solved with FP^X

Marginal inference in Bayesian networks specified by CPTs is #P-equivalent

THEOREM

Inference in RBNs without combination functions is #P-equivalent, even if the domain is specified solely by its size in binary notation

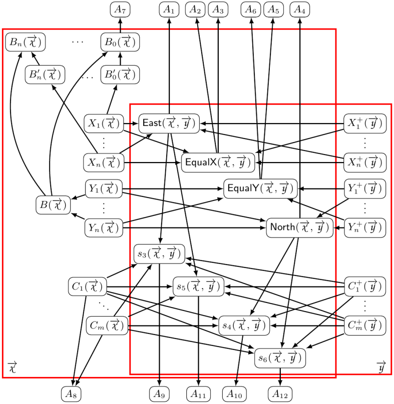
Proof: RBNs without combination functions encode plate models, which are #P-equivalent (Cozman & Mauá, 2015)



THEOREM

Inference is #EXP-equivalent when the only combination function is max.

Proof: RBNs with max combination functions encode enhanced plate models, which are #EXP-equivalent by reduction from domino tiling (Cozman & Mauá, 2015)



Exponential behavior due to unbounded arity (which allows us to specify an exponential number of relevant individuals)

THEOREM

Inference is FPSPACE*-complete when the arity is bounded and the only combination function is max.*

Proof: Hardness is by reduction from counting QBF solutions:

- Input: A formula $\varphi(X_1, \ldots, X_n) = Q_1 Y_1 Q_2 Y_2 \ldots Q_m Y_m \psi(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$, where each Q_i is either $\exists$ or $\forall$, and ψ is a 3-CNF formula over variables $X_1, \ldots, X_n, Y_1, \ldots, Y_m$.
- Output: The number of assignments to the variables $X_1, \ldots, X_n$ that satisfy φ.

NESTING LEVEL

- Nesting level of $F = q$ or $F = r(X_1, \ldots, X_n)$ is zero
- Nesting level of $F = F_1 F_2 + (1 - F_2) F_3$ is the highest nesting level of F_1, F_2, F_3
- Nesting level of $F = \text{comb}\{F_1, \ldots, F_k | Y_1, \ldots, Y_M; \alpha\}$ is the highest nesting level over all F_i, plus one.

THEOREM

Inference is $\#\Sigma_k^{\mathsf{P}}$-equivalent when arity is bounded, nesting level is at most k, and the only combination function is max.

Proof: Reduction from $\#\Pi_k$SAT (counting satisfying assignments of CNF formulas with at most k alternating quantifiers)

Other combination functions

THEOREM

Inference is FEXP*-complete when arity is bounded and combination functions are polynomial in their arguments*

Proof: Succinct specification of an exponential multiset allows exponential computation (as in succinct circuits).

Other combination functions

THEOREM

Inference is #P-equivalent when arity is bounded and combination formulas are polynomial

Proof: Ground the model into a Bayesian network with polynomial effort.

CONCLUSION

- No combination functions, $\rightarrow$ inference is #P-equivalent
- Only maximization as combination function
  - $\#$EXP-equivalent
  - FPSPACE-complete with bound on arity of relations
  - $\#\Sigma_k$-complete with bound on arity of relations and bound on number of nestings
- FEXP-complete if combination function is polynomial
- $\#P$-equivalent if probability formula is polynomial