Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 328 (2007) 984–994
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Polynomial approximation in Banach spaces ☆

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Received 8 February 2006
Available online 5 July 2006
Submitted by R.M. Aron

Abstract

We define new classes of topological algebras of holomorphic functions on open subsets of Banach spaces, and on open subsets of dual Banach spaces. We investigate properties and derive results concerning polynomial approximation on such algebras. We give an explicit description of their spectra, derive results on finitely generated ideals, and theorems of Banach–Stone type. We show that under certain conditions on the open subset U, this new algebra coincides with $\mathcal{H}_{wu}(U)$, deriving new results on $\mathcal{H}_{wu}(U)$.

Keywords: Holomorphic functions; Banach spaces; Schauder basis; Polynomially convex domain; Spectrum

Introduction

Let E be a Banach space, let U be an open subset of E and let V be an open subset of E'. The algebras $\mathcal{H}_{wu}(U)$ and $\mathcal{H}_{w^*u}(V)$ have been studied by several authors in the last years [1, 2,4,6,7,9]. In this work, we define similar classes of holomorphic functions. More specifically, let $\mathcal{H}_{wuk}(U)$ denote the set of all holomorphic functions $f : U \to \mathbb{C}$ that are weakly uniformly continuous on each weakly compact subset of U, and let $\mathcal{H}_{w^*uk}(V)$ denote the set of all holomorphic functions $g : V \to \mathbb{C}$ that are weak-star uniformly continuous on each weak-star compact subset of V. In the first section of this paper, we study properties of the algebras $\mathcal{H}_{wuk}(U)$ and $\mathcal{H}_{w^*uk}(V)$. One of the main results concerns polynomial approximation on such algebras. We show that if E is a Banach space with a shrinking Schauder basis and U is a weakly open subset of E which is polynomially convex, then $\mathcal{P}_f(E)$ is dense in $\mathcal{H}_{wuk}(U)$, for the topology of

☆ Research supported by FAPESP, Process No. 04/07441-3, Brazil.
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.05.076

uniform convergence on the weakly compact subsets of U. An analogous result is given for the algebra $\mathcal{H}_{w^*uk}(V)$. In Section 2, we give an explicit description of the spectrum of $\mathcal{H}_{wuk}(U)$, when E is a reflexive Banach space with a Schauder basis and U is a weakly open subset of E which is $\mathcal{P}_{wk}(E)$-convex. Indeed, we show that in this case the spectrum of $\mathcal{H}_{wk}(U)$ is identified with U. We also investigate whether $\mathcal{H}_{wuk}(U)$ coincides with $\mathcal{H}_{wu}(U)$. For example, if E is reflexive and U is weakly open and convex, then $\mathcal{H}_{wuk}(U) = \mathcal{H}_{wu}(U)$. We present another situation where $\mathcal{H}_{wuk}(U)$ coincides with $\mathcal{H}_{wu}(U)$. With these coincidence results we can improve results from [11,15]. In the last section of the article we present results on finitely generated ideals of the algebra $\mathcal{H}_{wuk}(U)$ and theorems of Banach–Stone type.

1. Polynomial approximation in Banach spaces

We refer to [8,10] for background information on infinite-dimensional complex analysis. Let E be a complex Banach space and let U be an open subset of E. For each $x \in U$, we denote by $d_U(x)$ the distance from x to the boundary of U. For each $n \in \mathbb{N}$, let $U_n = \{x \in U\colon \|x\| < n$ and $d_U(x) > 2^{-n}\}$. We denote by $\mathcal{H}_w(U)$ the set of all $f \in \mathcal{H}(U)$ that are weakly continuous on each U_n, $\mathcal{H}_{wu}(U)$ is the set of all $f \in \mathcal{H}(U)$ that are weakly uniformly continuous on each U_n, and $\mathcal{H}_b(U)$ is the set of all $f \in \mathcal{H}(U)$ that are bounded on each U_n. We have that $\mathcal{H}_{wu}(U) \subset \mathcal{H}_b(U)$ [3, Lemma 2.2] for every open subset U. If V is an open subset of E', let $\mathcal{H}_{w^*}(V)$ denote the set of all $g \in \mathcal{H}(V)$ that are weak-star continuous on each V_n, and let $\mathcal{H}_{w^*u}(V)$ denote the set of all $g \in \mathcal{H}(V)$ that are weak-star uniformly continuous on each V_n. Let $\mathcal{K}_w(U) = \{A \subset U\colon A$ is weakly compact$\}$ and let $\mathcal{K}_{w^*}(V) = \{B \subset V\colon B$ is weak-star compact$\}$. It is clear that $\mathcal{K}_w(U)$ (respectively $\mathcal{K}_{w^*}(V)$) covers U (respectively V). If U is weakly open and if V is weak-star open, then the elements of $\mathcal{K}_w(U)$ and $\mathcal{K}_{w^*}(V)$ have an useful property, that we present in the next lemma. We denote by $\mathcal{V}_w(E)$ (respectively $\mathcal{V}_{w^*}(E')$) the set of all neighborhoods of zero in E (respectively in E') with respect to the weak topology $\sigma(E, E')$ (respectively weak-star topology $\sigma(E', E)$).

Lemma 1.1. *Let E be a Banach space, let U be a weakly open subset of E and let V be an open subset of E'. Then*

(1) *For each $A \in \mathcal{K}_w(U)$ there exists $W \in \mathcal{V}_w(E)$ such that $A + W \subset U$.*
(2) *For each $B \in \mathcal{K}_{w^*}(V)$ there exists $W \in \mathcal{V}_{w^*}(E')$ such that $B + W \subset V$.*

Proof. (1) Since U is weakly open, for each $x \in A$, there exist $W_x, \widetilde{W}_x \in \mathcal{V}_w(E)$ such that $W_x + W_x \subset \widetilde{W}_x$ and $x + \widetilde{W}_x \subset U$. Since A is weakly compact, we can find $x_1, \dots, x_n \in A$ and $W_1, \dots, W_n \in \mathcal{V}_w(E)$ such that $A \subset (x_1 + W_1) \cup \dots \cup (x_n + W_n) \subset U$. If we take $W = W_1 \cap \dots \cap W_n$, then it is easy to see that $A + W \subset U$.

(2) The proof of (1) applies. □

We denote $\mathcal{P}_f(E) = \bigoplus_{m\in\mathbb{N}} \mathcal{P}_f(^m E)$, $\mathcal{P}_w(E) = \mathcal{P}(E) \cap \mathcal{H}_w(E)$ and $\mathcal{P}_{wu}(E) = \mathcal{P}(E) \cap \mathcal{H}_{wu}(E)$. Actually, the two last sets coincide, i.e., $\mathcal{P}_w(E) = \mathcal{P}_{wu}(E)$ [2, Theorem 2.9]. Let U be an open subset of E and let $\mathcal{H}_{wuk}(U) = \{f \in \mathcal{H}(U)\colon f$ is weakly uniformly continuous on each $A \in \mathcal{K}_w(U)\}$. Note that if U is weakly open, then each weakly compact subset of U is contained in some U_n, and hence $\mathcal{H}_{wu}(U) \subset \mathcal{H}_{wuk}(U)$. It is also clear that $\mathcal{P}_f(E) \subset \mathcal{P}_{wu}(E) \subset \mathcal{H}_{wuk}(U)$. Following [1], we say that a polynomial $P \in \mathcal{P}_{f^*}(E')$ if and only if P is a finite linear combination of products of weak-star continuous linear functionals on E'. Note that each weak-star continuous linear functional of E' is an evaluation at some point in E. We also denote $\mathcal{P}_{w^*}(E') = \mathcal{P}(E') \cap \mathcal{H}_{w^*}(E')$ and $\mathcal{P}_{w^*u}(E') = \mathcal{P}(E') \cap \mathcal{H}_{w^*u}(E')$, but it is clear that the last two sets coincide, i.e., $\mathcal{P}_{w^*}(E') = \mathcal{P}_{w^*u}(E')$. Let V be an open subset of E' and let $\mathcal{H}_{w^*uk}(V) = \{g \in \mathcal{H}(V)\colon g$ is weak-star uniformly continuous on each $B \in \mathcal{K}_{w^*}(V)\}$. Note that $\mathcal{H}_{w^*uk}(E') = \mathcal{H}_{w^*u}(E')$, and $\mathcal{P}_{f^*}(E') \subset \mathcal{P}_{w^*u}(E') \subset \mathcal{H}_{w^*uk}(V)$. If V is weak-star open, then $\mathcal{H}_{w^*u}(V) \subset \mathcal{H}_{w^*uk}(V)$. If E is reflexive, then $\mathcal{H}_{w^*uk}(V) = \mathcal{H}_{wuk}(V)$. We endow $\mathcal{H}_{wuk}(U)$ (respectively $\mathcal{H}_{w^*uk}(V)$) with the topology of uniform convergence on the elements of $\mathcal{K}_w(U)$ (respectively $\mathcal{K}_{w^*}(V)$), and we denote this topology by τ_k (respectively τ_{k^*}). It is clear that $(\mathcal{H}_{wuk}(U), \tau_k)$ (respectively $(\mathcal{H}_{w^*uk}(V), \tau_{k^*})$) is a locally m-convex algebra. In the next example we present a coincidence result concerning the algebras $\mathcal{H}_{wuk}(U)$ and $\mathcal{H}_{wu}(U)$.

Example 1.2. Let E be a reflexive Banach space and let U be a convex weakly open subset of E. Then $\mathcal{H}_{wuk}(U) = \mathcal{H}_{wu}(U)$.

Proof. First we observe that since U is convex, then U_n is convex for every $n \in \mathbb{N}$, and hence $\overline{U_n}^w = \overline{U_n} \subset U$. Since E is reflexive, we have that $\overline{U_n}^w$ is w-compact, and consequently $\overline{U_n}^w \in \mathcal{K}_w(U)$. Then it follows that $\mathcal{H}_{wuk}(U) \subset \mathcal{H}_{wu}(U)$. $\square$

Let E be a Banach space with a Schauder basis $(e_n)_{n\in\mathbb{N}}$ and let $(\varphi_n)_{n\in\mathbb{N}}$ be the corresponding linear functionals. For each $n \in \mathbb{N}$, T_n denotes the canonical projection $T_n : E \to E$, where $T_n(x) = T_n(\sum_{j=1}^{\infty} \varphi_j(x)e_j) = \sum_{j=1}^{n} \varphi_j(x)e_j$. We say that a Schauder basis is *shrinking* if the corresponding linear functionals $(\varphi_n)_{n\in\mathbb{N}}$ form a Schauder basis in E'. In this case, S_n denotes the canonical projection $S_n : E' \to E'$, where $S_n(\varphi) = \sum_{j=1}^{n} \varphi(e_j)\varphi_j$, for each $\varphi \in E'$. It is known that the sequence $(T_n)_{n\in\mathbb{N}}$ converges to the identity operator, uniformly on the compact subsets of E. The same result cannot hold if we replace *compact* by *bounded* subsets of E, when E is infinite-dimensional. Indeed, if it was true, then the identity operator would be a compact operator, which is a contradiction. But in the next proposition we show a weaker result of this kind.

Proposition 1.3. *Let E be a Banach space with a shrinking Schauder basis. Then*

(1) *T_n converges to the identity operator, weakly uniformly on the bounded subsets of E.*
(2) *S_n converges to the identity operator, weak-star uniformly on the bounded subsets of E'.*

Proof. (1) We must show that for each bounded subset B of E, $\varphi \in E'$ and $\varepsilon > 0$, there is an integer $n_0 \in \mathbb{N}$ such that $(*)$ $\sup_{x\in B} |\varphi(T_n(x) - x)| < \varepsilon$, for all $n \geqslant n_0$. It is clear that $\varphi(x - T_n(x)) = \sum_{j=n+1}^{\infty} \varphi_j(x)\varphi(e_j)$, for each $x \in E$, $\varphi \in E'$ and $n \in \mathbb{N}$. Since $(\varphi_j)_{j\in\mathbb{N}}$ is a Schauder basis for E', given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|\sum_{j=n+1}^{\infty} \varphi(e_j)\varphi_j\| < \varepsilon$, for $n \geqslant n_0$, or in other words, $\sup_{x\in B_E} |\sum_{j=n+1}^{\infty} \varphi(e_j)\varphi_j(x)| < \varepsilon$, for $n \geqslant n_0$, which is precisely $(*)$ for $B = B_E$. Now let B be a bounded subset of E and $r > 0$ be such that $B \subset rB_E$. Then $\sup_{x\in B} |\sum_{j=n+1}^{\infty} \varphi(e_j)\varphi_j(x)| \leqslant \sup_{x\in B_E} |\sum_{j=n+1}^{\infty} \varphi(e_j)\varphi_j(rx)| < r\varepsilon$, for all $n \geqslant n_0$.

(2) Let $B \subset E'$ be a bounded subset, $x \in E$ and $\varepsilon > 0$. Let $r > 0$ be such that $B \subset B_{E'}(0, r)$. Since $(e_n)_{n\in\mathbb{N}}$ is a Schauder basis for E, there exists $n_0 \in \mathbb{N}$ such that $\|\sum_{j=n+1}^{\infty} \varphi_j(x)e_j\| < \frac{\varepsilon}{r}$, for all $n \geqslant n_0$. If we write $\varphi = \sum_{j=1}^{\infty} \varphi(e_j)\varphi_j$, then

$$\sup_{\varphi\in B}\bigl|S_n(\varphi)(x) - \varphi(x)\bigr| = \sup_{\varphi\in B}\left|\sum_{j=n+1}^{\infty} \varphi(e_j)\varphi_j(x)\right| = \sup_{\varphi\in B}\left|\varphi\left(\sum_{j=n+1}^{\infty} \varphi_j(x)e_j\right)\right|$$
$$\leqslant \sup_{\varphi\in B}\|\varphi\|\left\|\sum_{j=n+1}^{\infty} \varphi_j(x)e_j\right\| < r\cdot\frac{\varepsilon}{r} = \varepsilon, \quad \text{for } n \geqslant n_0. \qquad \square$$

From now on, if the proof for the weak-star case in E' is not given, then it is because it follows the same arguments of the proof given for the weak case in E. As a consequence of Proposition 1.3, we have the following corollaries.

Corollary 1.4. *Let E be a Banach space with a shrinking Schauder basis. Then $\mathcal{P}_f(E)$ is norm-dense in $\mathcal{P}_w(E)$ and $\mathcal{P}_{f^*}(E')$ is norm-dense in $\mathcal{P}_{w^*}(E')$.*

Proof. Let $c > 1$ be such that $\|T_n\| \leqslant c$, for all $n \in \mathbb{N}$. Let $B = B(0, r)$, $C = B(0, cr)$ and $P \in \mathcal{P}_w(E) = \mathcal{P}_{wu}(E)$. Given $\varepsilon > 0$, there exists $W \in \mathcal{V}_w(E)$ such that if $x, y \in C$ and $x - y \in W$ then $|P(x) - P(y)| < \varepsilon$. By Proposition 1.3, there exists $n_0 \in \mathbb{N}$ such that $T_n(x) - x \in W$, for all $x \in B$ and $n \geqslant n_0$. Consequently $|P \circ T_n(x) - P(x)| < \varepsilon$, for all $x \in B$ and $n \geqslant n_0$. Observe now that $P \circ T_n \in \mathcal{P}_f(E)$, for all $n \in \mathbb{N}$. $\square$

Let A be a subset of a Banach space E, and $\mathcal{F} \subset \mathcal{P}(E)$. Then the *$\mathcal{F}$-hull of A* is the set $\widehat{A}_{\mathcal{F}} = \{x \in E\colon |f(x)| \leqslant \sup_A |f|, \text{ for all } f \in \mathcal{F}\}$.

Corollary 1.5. *Let E be a Banach space with a shrinking Schauder basis, let A be a bounded subset of E and let B a bounded subset of E'. Then $\widehat{A}_{\mathcal{P}_f(E)} = \widehat{A}_{\mathcal{P}_w(E)}$, and $\widehat{B}_{\mathcal{P}_{f^*}(E')} = \widehat{B}_{\mathcal{P}_{w^*}(E')}$.*

Proof. The proof of [15, Lemma 2] applies, using Corollary 1.4. $\square$

Corollary 1.6. *Let E be a Banach space with a shrinking Schauder basis, let U be a weakly open subset of E and let V be a weak-star open subset of E'.*

(1) *For each $A \in \mathcal{K}_w(U)$ there exists $W \in \mathcal{V}_w(E)$ and $n_0 \in \mathbb{N}$ such that $A + W \subset U$ and $T_n(A) + W \subset U$, for all $n \geqslant n_0$. In particular, $T_n(A) \in \mathcal{K}_w(U)$, for each $n \geqslant n_0$.*
(2) *For each $B \in \mathcal{K}_{w^*}(V)$ there exists $W \in \mathcal{V}_{w^*}(E')$ and $n_0 \in \mathbb{N}$ such that $B + W \subset V$ and $S_n(B) + W \subset V$, for all $n \geqslant n_0$. In particular, $S_n(B) \in \mathcal{K}_{w^*}(V)$, for each $n \geqslant n_0$.*
(3) *The set $C = A \cup \{T_n(A)\colon n \geqslant n_0\}$ belongs to $\mathcal{K}_w(U)$.*
(4) *The set $D = B \cup \{S_n(B)\colon n \geqslant n_0\}$ belongs to $\mathcal{K}_{w^*}(V)$.*

Proof. (1) Let $A \in \mathcal{K}_w(U)$. By Lemma 1.1, we can find $W, \widetilde{W} \in \mathcal{V}_w(E)$ such that $W + W \subset \widetilde{W}$ and $A + \widetilde{W} \subset U$. By Proposition 1.3, there exists $n_0 \in \mathbb{N}$ such that $T_n(x) - x \in W$, for all $x \in A$ and $n \geqslant n_0$. This implies that $T_n(A) \subset A + W \subset U$, for all $n \geqslant n_0$, and hence $T_n(A) + W \subset U$, for all $n \geqslant n_0$.

(3) By (1), we have in particular that $C \subset U$. To show that C is weakly compact, let $(W_\alpha)_{\alpha \in \Lambda}$ be a weakly open cover for C, i.e., $C \subset \bigcup_{\alpha \in \Lambda} W_\alpha$. Since $A \subset C$ is weakly compact, there exist $\alpha_1, \dots, \alpha_k \in \Lambda$ such that $A \subset \bigcup_{i=1}^k W_{\alpha_i}$. By Lemma 1.1, let $W \in \mathcal{V}_w(E)$ be such that $A + W \subset \bigcup_{i=1}^k W_{\alpha_i}$. By Proposition 1.3, there exists $n_1 \geqslant n_0$ such that $T_n(x) - x \in W$, for all $x \in A$ and $n \geqslant n_1$, which implies that $T_n(x) \in \bigcup_{i=1}^k W_{\alpha_i}$, for all $x \in A$ and $n \geqslant n_1$. Now it is clear that $T_n(A)$, $n = n_0, \dots, n_1$ is contained in a finite subfamily of $(W_\alpha)_{\alpha \in \Lambda}$. $\square$

Proposition 1.7. *Let E be a Banach space with a shrinking Schauder basis, let U be a weakly open subset of E, and let V be a weak-star open subset of E'. Let $f \in \mathcal{H}_{wuk}(U)$ and let $g \in \mathcal{H}_{w^*uk}(V)$. Then*

(1) *For each* $A \in \mathcal{K}_w(U)$ *and* $\varepsilon > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that*

$$\sup_{x\in A}\bigl|f\bigl(T_n(x)\bigr) - f(x)\bigr| < \varepsilon, \quad \textit{for all } n \geqslant n_0.$$

(2) *For each* $B \in \mathcal{K}_{w^*}(V)$ *and* $\varepsilon > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that*

$$\sup_{y'\in B}\bigl|g\bigl(S_n(y')\bigr) - g(y')\bigr| < \varepsilon, \quad \textit{for all } n \geqslant n_0.$$

Proof. Let $A \in \mathcal{K}_w(U)$. By Corollary 1.6, there is an integer $n_1 \in \mathbb{N}$ such that $A \cup \{T_n(A): n \geqslant n_1\} = C \in \mathcal{K}_w(U)$. Since $f \in \mathcal{H}_{wuk}(U)$, there exists $W \in \mathcal{V}_w(E)$ such that if $x, y \in C$ and $x - y \in W$ then $|f(x) - f(y)| < \varepsilon$. For this W there is $n_2 \in \mathbb{N}$ such that $T_n(x) - x \in W$, for all $x \in C$ and $n \geqslant n_2$. Let $n_0 = \max\{n_1, n_2\}$, $x \in A$ and $n \geqslant n_0$. Then $x, T_n(x) \in C$, $T_n(x) - x \in W$ and consequently $|f(T_n(x)) - f(x)| < \varepsilon$. $\square$

Proposition 1.7 roughly means that $f \circ T_n$ *converges to* f *uniformly on the elements of* $\mathcal{K}_w(U)$. But this would be an abuse of language, since not all compositions $f \circ T_n$ are well defined, for every $n \in \mathbb{N}$. Next theorem is our first important result concerning the algebras $\mathcal{H}_{wuk}(U)$ and $\mathcal{H}_{w^*uk}(V)$.

Theorem 1.8. *Let E be a Banach space with a shrinking Schauder basis, let U be a polynomially convex weakly open subset of E and let V be a polynomially convex weak-star open subset of E'. Then $\mathcal{P}_f(E)$ is τ_k-dense in $\mathcal{H}_{wuk}(U)$, and $\mathcal{P}_{f^*}(E')$ is τ_{k^*}-dense in $\mathcal{H}_{w^*uk}(V)$.*

Proof. Let $A \in \mathcal{K}_w(U)$, $f \in \mathcal{H}_{wuk}(U)$ and $\varepsilon > 0$ be given. By Corollary 1.6 and Proposition 1.7 we can find an integer $n_0 \in \mathbb{N}$ such that

$$T_{n_0}(A) \in \mathcal{K}_w(U) \quad \text{and} \quad \bigl|f \circ T_{n_0}(x) - f(x)\bigr| < \frac{\varepsilon}{2}, \quad \text{for all } x \in A. \tag{1}$$

Since U is polynomially convex, we have that $U \cap T_{n_0}(E)$ is polynomially convex in $T_{n_0}(E)$ [10, Examples 25.2(d)]. On the other hand, it is clear that $T_{n_0}(A)$ is a compact subset of $U \cap T_{n_0}(E)$. Then it follows by [10, Theorem 25.4] that there exists $P \in \mathcal{P}(T_{n_0}(E))$ such that $|P(y) - f(y)| < \varepsilon/2$, uniformly on $y \in T_{n_0}(A)$, or in other words

$$\sup_{x\in A}\bigl|P \circ T_{n_0}(x) - f \circ T_{n_0}(x)\bigr| < \frac{\varepsilon}{2}. \tag{2}$$

Now the conclusion follows by (1) and (2). $\square$

The first assertion in Corollary 1.4 is known when E' has the approximation property (see [3, Proposition 2.7]). The second assertion in Theorem 1.8 is known when $V = E'$ and E has the approximation property (see [1, Theorem 5.2]). But the proof given here when E has a shrinking Schauder basis, is much simpler.

2. Characterization of the spectrum

In this section we present several applications of the results of the previous section. The results concern new classes of open subsets of Banach spaces. Definitions 2.1 are inspired by [15, Definitions 1].

Definitions 2.1. Let E be a Banach space, let U be an open subset of E and let V be an open subset of E'. We say that:

(1) U is *$\mathcal{P}_{wk}(E)$-convex* if $\widehat{A}_{\mathcal{P}_w(E)} \cap U \in \mathcal{K}_w(U)$, for all $A \in \mathcal{K}_w(U)$.
(2) V is *$\mathcal{P}_{w^*k}(E')$-convex* if $\widehat{B}_{\mathcal{P}_{w^*}(E')} \cap V \in \mathcal{K}_{w^*}(V)$, for all $B \in \mathcal{K}_{w^*}(V)$.
(3) U is *strongly $\mathcal{P}_{wk}(E)$-convex* if $\widehat{A}_{\mathcal{P}_w(E)} \subset U$ and $\widehat{A}_{\mathcal{P}_w(E)} \in \mathcal{K}_w(U)$, for all $A \in \mathcal{K}_w(U)$.
(4) V is *strongly $\mathcal{P}_{w^*k}(E')$-convex* if $\widehat{B}_{\mathcal{P}_{w^*}(E')} \subset V$ and $\widehat{B}_{\mathcal{P}_{w^*}(E')} \in \mathcal{K}_{w^*}(V)$, for all $B \in \mathcal{K}_{w^*}(V)$.

In the next lemma we show that the last conditions of Definitions 2.1(3) and (4) are actually unnecessary.

Lemma 2.1. *Let E be a Banach space, let U be an open subset of E and let V be an open subset of E'. Let $A \in \mathcal{K}_w(U)$ and $B \in \mathcal{K}_{w^*}(V)$. If $\widehat{A}_{\mathcal{P}_w(E)} \subset U$, then $\widehat{A}_{\mathcal{P}_w(E)} \in \mathcal{K}_w(U)$; and if $\widehat{B}_{\mathcal{P}_{w^*}(E')} \subset V$, then $\widehat{B}_{\mathcal{P}_{w^*}(E')} \in \mathcal{K}_{w^*}(V)$.*

Proof. Since $\mathbb{C} \oplus E' \subset \mathcal{P}_w(E)$, we have that $\widehat{A}_{\mathcal{P}_w(E)} \subset \widehat{A}_{\mathbb{C}\oplus E'} = \overline{co}^w(A)$, where the last equality follows by [10, Proposition 11.1]. Since $\overline{co}^w(A)$ is weakly compact and $\widehat{A}_{\mathcal{P}_w(E)}$ is weakly closed, it follows that $\widehat{A}_{\mathcal{P}_w(E)} \subset U$ is weakly compact and then $\widehat{A}_{\mathcal{P}_w(E)} \in \mathcal{K}_w(U)$. The second assertion is trivial, since $\widehat{B}_{\mathcal{P}_{w^*}(E')}$ is weak-star closed and bounded, and hence weak-star compact. □

Lemma 2.2. *Let E be a Banach space, and let A be a bounded subset of E'. Then $\widehat{A}_{\mathbb{C}\oplus E} = \overline{co}^{w^*}(A)$, where $\mathbb{C} \oplus E$ denotes the set $\{a + \delta_x\colon a \in \mathbb{C},\ x \in E\} \subset E''$.*

Proof. We follow the proof of [10, Proposition 11.1(b)], applying the Hahn–Banach theorem to the locally convex space $(E', \sigma(E', E))$. □

Example 2.3. Let E be a Banach space, $P \in \mathcal{P}_f(E)$ and $Q \in \mathcal{P}_{f^*}(E')$. Then:

(1) Every convex weakly open subset of E is strongly $\mathcal{P}_{wk}(E)$-convex.
(2) Every convex weak-star open subset of E' is strongly $\mathcal{P}_{w^*k}(E')$-convex.
(3) $U = \{x \in E\colon |P(x)| < 1\}$ is a strongly $\mathcal{P}_{wk}(E)$-convex weakly open set.
(4) $V = \{x \in E'\colon |Q(x)| < 1\}$ is a strongly $\mathcal{P}_{w^*k}(E')$-convex weak-star open set.

Proof. (1) Let $A \in \mathcal{K}_w(U)$. We will first show that $\overline{co}^w(A) \in \mathcal{K}_w(U)$. By Lemma 1.1, let $\widetilde{W} \in \mathcal{V}_w(E)$ be such that $A + \widetilde{W} \subset U$. Since U is convex, it is easy to see that $co(A) + \widetilde{W} \subset co(A + \widetilde{W}) \subset U$. Since $\overline{co}^w(A) = \bigcap_{W\in\mathcal{V}_w(E)}(co(A) + W)$, it follows that $\overline{co}^w(A) \subset co(A) + \widetilde{W} \subset U$ and hence $\overline{co}^w(A) \in \mathcal{K}_w(U)$. Now $\widehat{A}_{\mathcal{P}_w(E)} \subset \widehat{A}_{\mathbb{C}\oplus E'} = \overline{co}^w(A) \in \mathcal{K}_w(U)$, where the last equality follows by [10, Proposition 11.1]. Then it follows that U is strongly $\mathcal{P}_{wk}(E)$-convex.

(2) We follow the same arguments of (1), using Lemma 2.2 instead of [10, Proposition 11.1].

(3) It is clear that U is weakly open. Given $A \in \mathcal{K}_w(U)$, we will show that $\sup_A |P| < 1$. Suppose that $\sup_A |P| = 1$. Then there exists a sequence (x_n) in A such that $|P(x_n)| \to 1$. Since A is w-compact, there exists (x_{n_k}) a subsequence of (x_n) such that $x_{n_k} \xrightarrow{w} x \in A \subset U$. Hence $|P(x_{n_k})| \to |P(x)| = 1$, that is, $x \notin U$, which is a contradiction. Now let $y \in \widehat{A}_{\mathcal{P}_w(E)}$. Then $|P(y)| \leqslant \sup_A |P| < 1$, proving that $\widehat{A}_{\mathcal{P}_w(E)} \subset U$. Now U is strongly $\mathcal{P}_{wk}(E)$-convex by Lemma 2.1. □

If U is weakly open and $\mathcal{P}_{wk}(E)$-convex then U is polynomially convex. Indeed, if K is a compact subset of U, then $K \in \mathcal{K}_w(U)$. Now, since $\mathcal{P}_w(E) \subset \mathcal{P}(E)$, we have that $\widehat{K}_{\mathcal{P}(E)} \subset \widehat{K}_{\mathcal{P}_w(E)}$, and then $\widehat{K}_{\mathcal{P}(E)} \cap U \subset \widehat{K}_{\mathcal{P}_w(E)} \cap U \in \mathcal{K}_w(U) \subset \mathcal{B}(U)$, where $\mathcal{B}(U)$ denotes the set of all U-bounded subsets of U. We recall [15, Definitions 1] that an open subset U of a Banach space E is *$\mathcal{P}_b(E)$-convex* if $\widehat{A}_{\mathcal{P}(E)} \cap U \in \mathcal{B}(U)$ for all $A \in \mathcal{B}(U)$; and we say that U is *strongly $\mathcal{P}_b(E)$-convex* if $\widehat{A}_{\mathcal{P}(E)} \subset U$ and $\widehat{A}_{\mathcal{P}(E)} \in \mathcal{B}(U)$ for all $A \in \mathcal{B}(U)$. However, in [15, Lemma 3], we show that the last condition $\widehat{A}_{\mathcal{P}(E)} \in \mathcal{B}(U)$ is superfluous. If U is balanced, then both notions coincide [15, Proposition 6]. In the case of $\mathcal{P}_{wk}(E)$-convex sets, we have a similar result, that will be proved in Theorem 2.5. To show this theorem, we need the following result.

Theorem 2.4. *Let E be a Banach space.*

(1) *Let $A \subset E$ be a weakly compact subset of E and let U be a weakly open subset of E such that $\widehat{A}_{\mathcal{P}_f(E)} \subset U$. Then there exists a weakly open set $\widetilde{U}$ which is strongly $\mathcal{P}_{wk}(E)$-convex, such that $\widehat{A}_{\mathcal{P}_f(E)} \subset \widetilde{U} \subset U$.*
(2) *Let $B \subset E'$ be a weak-star compact subset of E' and let V be a weak-star open subset of E' such that $\widehat{B}_{\mathcal{P}_{f^*}(E')} \subset V$. Then there exists a weak-star open set $\widetilde{V}$ which is strongly $\mathcal{P}_{w^*k}(E')$-convex, such that $\widehat{B}_{\mathcal{P}_{f^*}(E')} \subset \widetilde{V} \subset V$.*

Proof. (1) We are inspired by ideas of [10, Lemma 24.7 and Theorem 28.2]. Since A is weakly compact, it follows that $C = \overline{co}^w(A)$ is weakly compact. If $C \subset U$, then we take $\widetilde{U} = C + W$, where $W \in \mathcal{V}_w(E)$ is convex and such that $C + W \subset U$ (Lemma 1.1). Since $\mathbb{C} \oplus E' \subset \mathcal{P}_f(E)$, we have that $\widehat{A}_{\mathcal{P}_f(E)} \subset \widehat{A}_{\mathbb{C}\oplus E'} = C$, where the last equality follows by [10, Proposition 11.1]. Now $\widetilde{U}$ is strongly $\mathcal{P}_{wk}(E)$-convex by Example 2.3, and hence $\widetilde{U}$ is the desired set. If C is not contained in U, then for each $y \in C \setminus U$ there is $P \in \mathcal{P}_f(E)$ such that $\sup_A |P| < 1 < |P(y)|$. Since $C \setminus U$ is weakly compact, we can find polynomials $P_1, \ldots, P_k \in \mathcal{P}_f(E)$ such that $C \setminus U \subset \bigcup_{j=1}^k \{x \in E\colon |P_j(x)| > 1\}$. Hence

$$C \cap \bigl\{x \in E\colon \bigl|P_j(x)\bigr| \leqslant 1, \text{ for } j = 1, \ldots, k\bigr\} \subset U. \tag{3}$$

We claim that there exists $W \in \mathcal{V}_w(E)$ such that $(C + W) \cap \{x \in E\colon |P_j(x)| < 1$, for $j = 1, \ldots, k\} \subset U$. If this is not the case, then for each $W \in \mathcal{V}_w(E)$ there exists $z_W = x_W + y_W$, with $x_W \in C$, $y_W \in W$, and $|P_j(z_W)| < 1$, for $j = 1, \ldots, k$; such that $z_W \notin U$. Since C is weakly compact, without loss of generality, there exists $x \in C$ such that $x_W \xrightarrow{w} x$, and hence $z_W \xrightarrow{w} x \in C$. Since $P_j(z_W) \to P_j(x)$, for $j = 1, \ldots, k$, it follows that $|P_j(x)| \leqslant 1$, $j = 1, \ldots, k$, which means that $x \in U$, by (3). Let $\widetilde{W} \in \mathcal{V}_w(E)$ be such that $x + \widetilde{W} \subset U$. For this $\widetilde{W}$, there exists $W_0 \in \mathcal{V}_w(E)$ such that $z_{W_0} \in x + \widetilde{W} \subset U$, which is a contradiction. Now $\widetilde{U} = (C + W) \cap \{x \in E\colon |P_j(x)| < 1$, for $j = 1, \ldots, k\}$ is strongly $\mathcal{P}_{wk}(E)$-convex, because it is a finite intersection of strongly $\mathcal{P}_{wk}(E)$-convex sets (see Example 2.3). Finally, it is clear that $\widehat{A}_{\mathcal{P}_f(E)} \subset \widetilde{U} \subset U$.

(2) We follow the same approach of (1), using Lemma 2.2 instead of [10, Proposition 11.1]. □

Theorem 2.5. *Let E be a Banach space with a shrinking Schauder basis, let U be a weakly open subset of E and let V be a weak-star open subset of E'. Then*

(1) *U is $\mathcal{P}_{wk}(E)$-convex if and only if U is strongly $\mathcal{P}_{wk}(E)$-convex.*
(2) *V is $\mathcal{P}_{w^*k}(E')$-convex if and only if V is strongly $\mathcal{P}_{w^*k}(E')$-convex.*

Proof. To show the nontrivial implication, let $A \in \mathcal{K}_w(U)$. By Lemma 2.1, it suffices to show that $\widehat{A}_{\mathcal{P}_w(E)} \subset U$. Let us write $\widehat{A}_{\mathcal{P}_w(E)} = (\widehat{A}_{\mathcal{P}_w(E)} \cap U) \cup (\widehat{A}_{\mathcal{P}_w(E)} \setminus U)$. Since U is $\mathcal{P}_{wk}(E)$-convex, we have that $\widehat{A}_{\mathcal{P}_w(E)} \cap U \in \mathcal{K}_w(U)$ and then by Lemma 1.1 there exists $\widetilde{W} \in \mathcal{V}_w(E)$ such that $\widehat{A}_{\mathcal{P}_w(E)} \cap U + \widetilde{W} \subset U$, which implies that $(\widehat{A}_{\mathcal{P}_w(E)} \cap U + \widetilde{W}) \cap (\widehat{A}_{\mathcal{P}_w(E)} \setminus U) = \emptyset$. Let $W \in \mathcal{V}_w(E)$ such that $W + W \subset \widetilde{W}$. It follows by [15, Lemma 5] that $(A_0 + W) \cap (A_1 + W) = \emptyset$, where $A_0 = (\widehat{A}_{\mathcal{P}_w(E)}) \cap U$ and $A_1 = \widehat{A}_{\mathcal{P}_w(E)} \setminus U$. If we denote $U' = (A_0 + W) \cup (A_1 + W)$, then it is easy to see that $U' = \widehat{A}_{\mathcal{P}_w(E)} + W = \widehat{A}_{\mathcal{P}_f(E)} + W$, where the last equality follows by Corollary 1.5. Let us define $f \in \mathcal{H}_{wuk}(U')$ by $f = 0$ in $A_0 + V$ and $f = 1$ in $A_1 + V$. Now U' is a weakly open subset of E that contains $\widehat{A}_{\mathcal{P}_f(E)}$. By Theorem 2.4, there exists a weakly open set $\widetilde{U}$ which is strongly $\mathcal{P}_{wk}(E)$-convex, such that $\widehat{A}_{\mathcal{P}_f(E)} \subset \widetilde{U} \subset U'$. Since $\widehat{A}_{\mathcal{P}_f(E)}$ is weakly compact, we have that $\widehat{A}_{\mathcal{P}_f(E)} \in \mathcal{K}_w(\widetilde{U})$. Since $\widetilde{U}$ is strongly $\mathcal{P}_{wk}(E)$-convex and $f|_{\widetilde{U}} \in \mathcal{H}_{wuk}(\widetilde{U})$, we can apply Theorem 1.8 and find a polynomial $P \in \mathcal{P}_f(E)$ such that $\sup_{\widehat{A}_{\mathcal{P}_f(E)}} |f|_{\widetilde{U}} - P| < 1/2$. Since $A \subset A_0$, we have that $\sup_A |P| < 1/2$ and hence $\sup_{\widehat{A}_{\mathcal{P}_f(E)}} |P| < 1/2$. Now let $y \in A_1 \subset \widetilde{U}$. Then $1/2 > |P(y) - f|_{\widetilde{U}}(y)| = |P(y) - 1| = |1 - P(y)| \geqslant 1 - |P(y)|$. Hence it follows that $|P(y)| > 1/2$, which is a contradiction. □

Next we will study the spectrum of $\mathcal{H}_{wuk}(U)$ when E is reflexive. Since in this case the algebras $\mathcal{H}_{wuk}(U)$ and $\mathcal{H}_{w^*uk}(V)$ are of the same type, it suffices to deal with $\mathcal{H}_{wuk}(U)$. Let E be a Banach space and U be an open subset of E. The *spectrum of* $\mathcal{H}_{wuk}(U)$ is the set of all continuous complex homomorphisms $T : \mathcal{H}_{wuk}(U) \to \mathbb{C}$, and is denoted by $\mathcal{S}_{wuk}(U)$. Let $z \in U$. Then $\delta_z : \mathcal{H}_{wuk}(U) \to \mathbb{C}$ defined by $\delta_z(f) = f(z)$, for all $f \in \mathcal{H}_{wuk}(U)$ is called *evaluation at* z. It is easy to see that $\delta_z \in \mathcal{S}_{wuk}(U)$, for every $z \in U$, and in this sense we say that U *is contained in* $\mathcal{S}_{wuk}(U)$. In the next theorem we show that under certain conditions on E and U all the elements of $\mathcal{S}_{wuk}(U)$ are evaluations at some point of U, and in this sense we say that $\mathcal{S}_{wuk}(U)$ *is identified with* U.

Theorem 2.6. *Let E be a reflexive Banach space with a Schauder basis and let U be a $\mathcal{P}_{wk}(E)$-convex and weakly open subset of E. Then the spectrum of $\mathcal{H}_{wuk}(U)$ is identified with U.*

Proof. We follow ideas of [11, Theorem 1.1]. Let $T \in \mathcal{S}_{wuk}(U)$. Since T is continuous, there exists $A \in \mathcal{K}_w(U)$ and $C > 0$ such that $|T(f)| \leqslant C \sup_A |f|$, for all $f \in \mathcal{H}_{wuk}(U)$. Since T is multiplicative, by a classical argument, we may assume that $C = 1$. Let $r > 0$ be such that $A \subset B(0, r)$. In particular, we have that $|T(f)| \leqslant \sup_A |f| \leqslant \sup_{B(0,r)} |f|$, for all $f \in E'$. Hence we have that $T \in E'' = E$, so there exists a unique $a \in E$ such that $T(f) = f(a)$, for all $f \in E'$, and hence $T(P) = P(a)$, for all $P \in \mathcal{P}_f(E)$. Then it follows that $|P(a)| = |T(P)| \leqslant \sup_A |P|$, for all $P \in \mathcal{P}_f(E)$, which implies that $a \in \widehat{A}_{\mathcal{P}_f(E)} = \widehat{A}_{\mathcal{P}_w(E)}$, where the last equality follows by Corollary 1.5. Now by Theorem 2.5, we have that U is strongly $\mathcal{P}_{wk}(E)$-convex, and hence $a \in U$. Then we apply Theorem 1.8 and get that $T(f) = f(a)$, for all $f \in \mathcal{H}_{wuk}(U)$. □

Example 2.7. Let E be a reflexive Banach space and let U be a convex and weakly open subset of E. By Example 1.2 we have that $\mathcal{H}_{wuk}(U) = \mathcal{H}_{wu}(U)$. Since U is convex, it follows by Example 2.3 that U is strongly $\mathcal{P}_{wk}(E)$-convex. If we assume that E has a Schauder basis, we have by Theorem 1.8 that $\mathcal{P}_f(E)$ is dense in $\mathcal{H}_{wu}(U)$. Also, it follows by Theorem 2.6 that $\mathcal{S}_{wu}(U) = U$. In [6,7], it is shown that if E is a Banach space such that E' has the approximation property, and $U \subset E$ is a convex and balanced open set, then $\mathcal{S}_{wu}(U) = \operatorname{int}(\overline{U}^{w^*})$, where the interior is taken in the norm of E''. In particular, if E is reflexive with a Schauder basis, then

$\mathcal{S}_{wu}(U) = U$. So, for the reflexive case, we are able to omit the hypothesis of U being balanced, but we have to assume that U is weakly open.

Example 2.8. Let E be a reflexive Banach space such that $\mathcal{P}(E) = \mathcal{P}_w(E)$. Let U be a weakly open subset of E, which is strongly $\mathcal{P}_b(E)$-convex (and hence strongly $\mathcal{P}_{wk}(E)$-convex). We have that $\overline{U_n}^w \subset \widehat{(U_n)}_{\mathcal{P}_w(E)} = \widehat{(U_n)}_{\mathcal{P}(E)} \subset U$. Since E is reflexive, we have that $\overline{U_n}^w$ is weakly compact, and hence $\overline{U_n}^w \in \mathcal{K}_w(U)$. Consequently $\mathcal{H}_{wuk}(U) = \mathcal{H}_{wu}(U)$. If, in addition, we assume that E has a Schauder basis, then by Example 2.7 we have that $\mathcal{P}_f(E)$ is dense in $\mathcal{H}_{wu}(U)$ and $\mathcal{S}_{wu}(U) = U$. An example of a Banach space with all the required properties is Tsirelson's space [13]. In [15, Theorem 11 and Proposition 6] it is shown that if E is a reflexive Banach space such that $\mathcal{P}(E) = \mathcal{P}_w(E)$, and $U \subset E$ is balanced and $\mathcal{P}_b(E)$-convex, then $\mathcal{S}_{wu}(U) = U$. As observed before, every balanced $\mathcal{P}_b(E)$-convex open set is strongly $\mathcal{P}_b(E)$-convex. So for the particular case when E is Tsirelson's space, we also improve results from [15].

3. Banach–Stone theorems

Next theorem is a consequence of Theorem 2.6. It says that, under the same hypotheses of Theorem 2.6, every proper finitely generated ideal of $\mathcal{H}_{wuk}(U)$ has a common zero. The proof will be omitted since it follows the same spirit of [11, Theorem 1.5].

Theorem 3.1. *Let E be a reflexive Banach space with a Schauder basis and let U be a $\mathcal{P}_{wk}(E)$-convex and weakly open subset of E. Then given $f_1, \dots, f_n \in \mathcal{H}_{wuk}(U)$ without common zeros, there exists $g_1, \dots, g_n \in \mathcal{H}_{wuk}(U)$ such that $\sum_{i=1}^n f_i g_i = 1$.*

In the spirit of Example 2.7 we have the following corollary for the algebra $\mathcal{H}_{wu}(U)$.

Corollary 3.2. *Let E be a reflexive Banach space with a Schauder basis and let U be a convex and weakly open subset of E. Then given $f_1, \dots, f_n \in \mathcal{H}_{wu}(U)$ without common zeros, there exists $g_1, \dots, g_n \in \mathcal{H}_{wu}(U)$ such that $\sum_{i=1}^n f_i g_i = 1$.*

Let E and F be Banach spaces, and let $U \subset E$ and $V \subset F$ be open subsets. We denote by $\mathcal{H}_{wuk}(V, U)$ the set of all holomorphic mappings $\varphi : V \to U$ such that $\varphi : (V, \sigma(F, F')) \to (U, \sigma(E, E'))$ is uniformly continuous when restricted to each $B \in \mathcal{K}_w(V)$. Let $\varphi \in \mathcal{H}_{wuk}(V, U)$. Then it is easy to see that $C_\varphi : \mathcal{H}_{wuk}(U) \to \mathcal{H}_{wuk}(V)$ defined by $C_\varphi(f) = f \circ \varphi$, for all $f \in \mathcal{H}_{wuk}(U)$, is a continuous algebra-homomorphism. An homomorphism of such type is called *composition operator*. In the next theorem, we show that under the same conditions of Theorem 2.6, every continuous algebra-homomorphism from $\mathcal{H}_{wuk}(U)$ into $\mathcal{H}_{wuk}(V)$ is a composition operator.

Theorem 3.3. *Let E and F be Banach spaces, with E reflexive with a Schauder basis. Let $U \subset E$ be $\mathcal{P}_{wk}(E)$-convex and weakly open, and let $V \subset F$ be an open subset. Then every continuous algebra-homomorphism $T : \mathcal{H}_{wuk}(U) \to \mathcal{H}_{wuk}(V)$ is a composition operator.*

Proof. We follow ideas of [14, Theorems 2 and 10]. We must find a mapping $\varphi \in \mathcal{H}_{wuk}(V, U)$ such that $T = C_\varphi$. Let $w \in V$ and note that $\delta_w \circ T \in \mathcal{S}_{wuk}(U)$. By Theorem 2.6, there exists a unique $z \in U$ such that $\delta_w \circ T = \delta_z$. If we define $\varphi(w) = z$, then it follows that $T(f) = f \circ \varphi$, for all $f \in \mathcal{H}_{wuk}(U)$. In particular, $f \circ \varphi$ is holomorphic, for all $f \in E'$, and hence by [10, Theorem 8.12] we have that φ is a holomorphic mapping. It remains to show that $\varphi : (V, \sigma(F, F')) \to (U, \sigma(E, E'))$ is uniformly continuous when restricted to each $B \in \mathcal{K}_w(V)$. So let $B \in \mathcal{K}_w(V)$, $f \in E'$ and $\varepsilon > 0$. Since $f \circ \varphi \in \mathcal{H}_{wuk}(V)$, there exists $W \in \mathcal{V}_w(F)$ such that if $x, y \in B$, and $x - y \in W$ then $|f \circ \varphi(x) - f \circ \varphi(y)| < \varepsilon$. This shows that $\varphi \in \mathcal{H}_{wuk}(V, U)$. □

Corollary 3.4. *Let E and F be Banach spaces, with E reflexive with a Schauder basis. Let $U \subset E$ be convex and weakly open, and let $V \subset F$ be an open subset. Then every continuous algebra-homomorphism $T : \mathcal{H}_{wu}(U) \to \mathcal{H}_{wu}(V)$ is a composition operator.*

Similar results as Corollary 3.4 appear in [7, Theorem 15 and Corollary 16], for absolutely convex open subsets of Banach spaces whose dual has the approximation property.

In [5], S. Banach proved that two compact metric spaces X and Y are homeomorphic if and only if the Banach algebras $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ are isometrically isomorphic. M.H. Stone, in [12], generalized this result to arbitrary compact Hausdorff topological spaces, the well-known Banach–Stone theorem. In the next theorem we establish a similar result for the algebras $\mathcal{H}_{wuk}(U)$ and $\mathcal{H}_{wuk}(V)$.

Theorem 3.5. *Let E and F be reflexive Banach spaces, both with Schauder bases. Let $U \subset E$ and $V \subset F$ be weakly open sets, such that U is $\mathcal{P}_{wk}(E)$-convex and V is $\mathcal{P}_{wk}(F)$-convex. Then the following conditions are equivalent.*

(1) *There exists a bijective mapping $\varphi : V \to U$ such that $\varphi \in \mathcal{H}_{wuk}(V, U)$ and $\varphi^{-1} \in \mathcal{H}_{wuk}(U, V)$.*
(2) *The algebras $\mathcal{H}_{wuk}(U)$ and $\mathcal{H}_{wuk}(V)$ are topologically isomorphic.*

Proof. We follow ideas of [14, Theorem 12].

(1) $\Rightarrow$ (2) Let us consider the composition operator $C_\varphi : \mathcal{H}_{wuk}(U) \to \mathcal{H}_{wuk}(V)$. Then it is easy to see that C_φ is bijective and that $(C_\varphi)^{-1} = C_{\varphi^{-1}}$.

(2) $\Rightarrow$ (1) Let $T : \mathcal{H}_{wuk}(U) \to \mathcal{H}_{wuk}(V)$ be a topological isomorphism. By Theorem 3.3, there exists $\varphi \in \mathcal{H}_{wuk}(V, U)$ such that $T = C_\varphi$ and $\psi \in \mathcal{H}_{wuk}(U, V)$ such that $T^{-1} = C_\psi$. Then it is not difficult to see that $\psi = \varphi^{-1}$, and this completes the proof. □

Corollary 3.6. *Let E and F be reflexive Banach spaces, both with Schauder bases. Let $U \subset E$ and $V \subset F$ be convex and weakly open sets. Then the following conditions are equivalent.*

(1) *There exists a bijective mapping $\varphi : V \to U$ such that $\varphi \in \mathcal{H}_{wu}(V, U)$ and $\varphi^{-1} \in \mathcal{H}_{wu}(U, V)$.*
(2) *The algebras $\mathcal{H}_{wu}(U)$ and $\mathcal{H}_{wu}(V)$ are topologically isomorphic.*

Corollary 3.4 improves [11, Theorem 1.6] and Corollary 3.6 improves [14, Corollary 14].

Acknowledgments

I am indebted to Prof. J. Mujica for his helpful suggestions concerning this work, and also for his constant encouragement. I wish also to thank Prof. M. Matos for suggesting the study of the algebra $\mathcal{H}_{w^*uk}(V)$.

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