Multicuts in Unweighted Digraphs with Bounded Degree and Bounded Tree-Width

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Abstract

The Directed Multicut problem can be defined as: given a digraph D and a collection of k pairs of distinct vertices (s_i, t_i) of D, find a minimum set of edges whose removal ensures that none of the strongly connected components includes one of the k pairs of vertices. We restrict our attention to bounded-degree and bounded-tree-width digraphs (the directed version of tree-width) and present some bad and good news. Bad news: Directed Multicut is still NP-hard in unweighted digraphs with tree-width one and bounded degree. Good news: there is a polynomial-time approximation scheme (PTAS) for Directed Multicut in unweighted digraphs with bounded tree-width and bounded degree, that is, for any $\epsilon > 0$, there is a polynomial-time $(1+\epsilon)$-approximation algorithm.

1 Introduction

We consider the **Directed Multicut** problem, introduced by Klein, Plotkin, Rao and Tardos [3]: given a digraph D and a collection of k pairs of distinct vertices (s_i, t_i) of D, find a minimum set of edges whose removal ensures that none of the strongly connected components includes one of the k pairs of vertices. In other words, for all i, either all paths from s_i to t_i *or* all paths from t_i to s_i must be broken.

A particular case of this problem is the **Feedback Arc Set** problem: given a digraph D, find a minimum set of arcs of D whose removal results in an acyclic digraph. This problem is known to be NP-hard [2], therefore so it is the Directed Multicut. We thus search for efficient approximation algorithms. The *performance ratio of an approximation algorithm* A for a minimization problem is the supremum, over all possible instances I, of the ratio between the size of the output of A when running on I and the size of an optimal solution for I. We say A is an *α-approximation algorithm* if its performance ratio is at most α. The smaller the performance ratio, the better.

Klein et al. [3] presented a $O(\log^2 k)$-approximation algorithm for edge-weighted Directed Multicut in general digraphs. In this paper we extend the results in [1] to digraphs. We give a PTAS for Directed Multicut in unweighted digraphs with bounded degree and bounded tree-width. It is worth mentioning that we mean here the concept of tree-width in digraphs, introduced by Johnson, Robertson, Seymour, and Thomas [4], and which differs from the one in undirected graphs. For example, the class of digraphs of tree-width zero consists of all acyclic digraphs. We will present the formal definition of tree-width in the next section.

As in [1], we find useful a variation of the Directed Multicut problem. The **Directed Unrestricted Vertex Multicut** problem is: given a digraph D and a collection of k pairs of vertices (s_i, t_i) of D called *terminals*, find a minimum set of vertices whose removal ensures that none of the strongly connected components includes one of the k pairs of vertices. (Note that terminals might be removed.)

We prove that Directed Multicut is NP-hard in digraphs of tree-width one and bounded degree and that Directed Unrestricted Vertex Multicut is NP-hard in tree-width two and bounded-degree digraphs.

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Our PTAS for Directed Multicut in unweighted digraphs with bounded degree and bounded tree-width follows very much the idea of a PTAS for the undirected case [1]. We first give a straightforward PTAS for Directed Unrestricted Vertex Multicut in digraphs of bounded tree-width. Then we present an approximation-ratio preserving reduction from Directed Multicut to Directed Unrestricted Vertex Multicut. If the Directed Multicut instance digraph has bounded degree and bounded tree-width, the Directed Unrestricted Vertex Multicut instance obtained by the reduction has bounded tree-width. (This reduction is very different from the undirected reduction in [1].) Combining the reduction with the PTAS for Directed Unrestricted Vertex Multicut in digraphs of bounded tree-width, we obtain a PTAS for Directed Multicut in digraphs with bounded degree and bounded tree-width.

2 Tree-Width and Multicuts in Digraphs

Johnson et al. [4] presented the following definition of tree-width in digraphs.

An *arborescence* is a digraph R such that R has a vertex r_0, called the root of R, with the property that for every vertex $r \in V(R)$ there is a unique directed walk from r_0 to r. Thus every arborescence arises from a tree by selecting a root, and directing all edges away from the root. If $r, r' \in V(R)$, we write $r' > r$ if $r' \neq r$ and there exists a directed walk in R with initial vertex r and terminal vertex r'. If $e \in E(R)$, we write $r' > e$ if either $r' = r$ or $r' > r$, where r is the head of e. We also write $e \sim r$ when e is incident with r.

Let D be a digraph and let $Z \subseteq V(D)$. The digraph obtained from D by deleting Z will be denoted by $D \setminus Z$. We say that a set $S \subseteq V(D)$ is *Z-normal* if there is no directed walk in $D \setminus Z$ with first and last vertex in S that uses a vertex of $D \setminus (Z \cup S)$.

An *arboreal decomposition of a digraph* D is a triple (R, X, W), where R is an arborescence, and $X = (X_e \subseteq V(D) : e \in E(R))$ and $W = (W_r \subseteq V(D) : r \in V(R))$ satisfy

(1) $(W_r : r \in V(R))$ is a partition of $V(D)$, and

(2) if $e \in E(R)$, then $\cup\{W_r : r \in V(R), r > e\}$ is X_e-normal.

The *width of* (R, X, W) is the least integer w such that $|\cup_{e \sim r} X_e \cup W_r| \leq w + 1$ for all $r \in V(R)$. The *tree-width of* D, denoted by $dtw(D)$, is the least integer w such that D has an arboreal decomposition of width w.

Next we describe a PTAS for Unrestricted Vertex Multicut in digraphs with bounded tree-width. This PTAS is very similar to the one for undirected graphs in [1]. Recall that a PTAS consists of, for each $\epsilon > 0$, a polynomial-time algorithm for the problem with a performance ratio of at most $1 + \epsilon$. Let us describe such an algorithm.

The input of the algorithm is a digraph D, an arboreal decomposition $\Theta = (R, X, W)$ of D, and a set $\mathcal{C}$ of pairs of vertices of D. Given a subdigraph D' of D, denote by $\mathcal{C}(D')$ the set of pairs in $\mathcal{C}$ whose two vertices are in D', and by $D \setminus D'$ the subdigraph of D induced by $V(D) \setminus V(D')$. For the description of the algorithm, all the instances we mention are on a subdigraph D' of D and the set of pairs to be disconnected is $\mathcal{C}(D')$. So we will drop $\mathcal{C}(D')$ of the notation and refer to an instance only by the digraph D'. Denote by *opt*(D') the size (i.e., the number of vertices) of an optimal solution for D'.

Consider R rooted at r_0 and an arbitrary ordering of the children of each vertex of R. For a vertex u of R, let $R(u)$ be the subarborecence of R rooted at u. Let $D(u)$ be the subdigraph of D induced by the set $Z = \cup\{W_{r'} : r' \in V(R), r' \geq u\}$. Let $t = \lceil (dtw(D) + 1)/\epsilon \rceil$.

In this abstract, we give only a general description of the algorithm: label the vertices of R in postorder. Find the lowest labeled vertex r such that an optimal solution for $D(r)$ has at least t vertices. If there is no such vertex, let r be the root of R. Let e be the edge of R entering r (if r is the root, let e undefined). Using the already computed optimal solutions for the children of r, find an approximate solution S_r for $D(r)$ such that $|S_r| \leq (1+\epsilon)opt(D(r))$ and $X_e \cup W_r \subseteq S_r$. If r is the root of R, then output S_r. Otherwise, let $D' = D \setminus (D(r) \cup X_e)$ and let $\Theta' = (R', X', W')$ be the arboreal decomposition of D' where $R' = R \setminus R(r)$ and $X'_f = X_f \setminus (X_e \cup V(D(r)))$, for all $f \in E(R')$, and $W'_s = W_s \setminus X_e$, for all $s \in V(R')$. Recursively get a solution S' for D'. Output $S = S' \cup S_r$.

The set S produced is a solution for the Directed Unrestricted Vertex Multicut in D. Indeed if C is a directed circuit either entirely in $D(r) \cup X_e$ or entirely in D' and containing s and t for some $(s,t) \in \mathcal{C}$, then $V(C) \cap S \neq \emptyset$ (C either intersects S_r or S'). If C is a directed circuit with vertices in $D(r)$ (that is, in Z) and in D', containing s and t for some $(s,t) \in \mathcal{C}$, then C has a vertex in $X_e \subseteq S_r \subseteq S$ because Z is X_e-normal. Besides, the set S has size at most $(1+\epsilon)\,opt(D)$.

3 Reduction from Directed Multicut to Unrestricted Vertex Multicut

Now we present an approximation-ratio preserving reduction from Directed Multicut to Directed Unrestricted Vertex Multicut. The reduction is very different from the undirected reduction in [1]. Consider an instance of Directed Multicut, that is, a digraph $D = (V, E)$ and a set $\mathcal{C}$ of pairs of distinct vertices of D. Let us describe the corresponding instance of Directed Unrestricted Vertex Multicut.

The input graph for Directed Unrestricted Vertex Multicut is the following digraph D'. (To avoid confusion, we will refer to vertices of D' as nodes and to edges of D' as arcs.) The set of nodes of D' is E plus m copies $u_1, \ldots, u_m$ of each vertex $u \in V$, where m is the number of edges in D. There is an arc in D' from a copy of $u \in V$ to $e \in E$ if u is the tail of e, and there is an arc in D' from $e \in E$ to a copy of $u \in V$ if u is the head of e. These are all the arcs in D'. Note that D' is a bipartite digraph (edges of D in one side and copies of vertices of D in the other).

Now let us describe the set of pairs of vertices of D'. For each pair (s,t) in $\mathcal{C}$, we have in $\mathcal{C}'$ pairs (s_i, t_j) for all $1 \leq i, j \leq m$. Clearly D' can be obtained from D in polynomial time. Note that $\mathcal{C}'$ has km^2 pairs. Also $\mathcal{C}'$ can be obtained from D and $\mathcal{C}$ in polynomial time.

Lemma 3.1 *For any solution S' for the Directed Unrestricted Vertex Multicut in D' there is a solution $S \subseteq E(D)$ such that $|S| \leq |S'|$.*

Proof. We may assume S' is a minimal solution for the Directed Unrestricted Vertex Multicut in D'. If $S' \subseteq E(D)$ then we take $S := S'$. If not, let $u \in V(D)$ and $1 \leq i \leq m$ be such that $u_i \in S'$. Then $S' - \{u_i\}$ is not a solution in D', which implies that there is a pair $(s,t) \in \mathcal{C}'$ and a directed circuit Q in $D' \setminus (S' - \{u_i\})$ containing s and t (because s and t are in the same strongly connected component). If $u_j \notin S'$ for some $1 \leq j \leq m$, the circuit obtained from Q by substituting u_i by u_j is a directed circuit in $D' \setminus S'$ and contains a pair in $\mathcal{C}'$, a contradiction. Thus $u_j \in S'$ for all $1 \leq j \leq m$. In this case, $|S'| \geq m = |E(D)|$ and we take $S := E(D)$. In both cases S is a solution in D' and $|S| \leq |S'|$, which completes the proof of the lemma. ∎

Lemma 3.1 together with the following theorem, whose proof is similar to the proof of Theorem 4 in [1], completes the reduction.

Theorem 3.2 *$S \subseteq E(D)$ is a solution for Directed Multicut in D if and only if S is a solution for Directed Unrestricted Vertex Multicut in D'.*

The next lemma shows that, if the instance of Directed Multicut is a digraph with bounded degree and bounded tree-width, then the corresponding instance of Directed Unrestricted Vertex Multicut has bounded tree-width. For this, we need some notation. For a vertex v in a digraph D, we denote by $\delta^+_D(v)$ the set of edges in D with v as tail. For a set S of vertices of D, $\delta^+_D(S)$ is the set of edges with tail in S and head not in S.

Lemma 3.3 *If D has bounded degree and bounded tree-width, then D' has bounded tree-width.*

Proof. Let D be a digraph with bounded degree and bounded tree-width and D' be obtained from the reduction described above. Let us present an arboreal decomposition of D' whose tree-width is at most

$(dtw(D) + 1)(\Delta + 1)$, where Δ is the maximum outdegree of D.

Let $\Theta = (R, X, W)$ be an arboreal decomposition of D of width $dtw(D)$. Let R' be an arborescence obtained from R by substituting each edge $e \in E(R)$ from a vertex r' to a vertex r in R by a directed path $P_e = \langle u_0, \ldots, u_m \rangle$ of length m, starting at $u_0 = r'$ and ending at $u_m = r$, where $u_1, \ldots, u_{m-1}$ are new vertices. Let f be the edge in $E(P_e)$ from u_{i-1} to u_i, $1 \leq i \leq m$. If $i = 1$, then let $Y_f := \delta^+_D(X_e)$, else let $Y_f := \delta^+_D(X_e \cup W_r)$. For $i = 1, \ldots, m$, let $W^i_r \subseteq V(D')$ be the set containing the i^{th} copy in D' of each vertex in W_r. Set $Z_{u_1} := W^1_r \cup \delta^+_D(W_r)$ and, for $i = 2, \ldots, m$, set $Z_{u_i} := W^i_r$. Let us prove that $\Theta' = (R', Y, Z)$ is an arboreal decomposition of D'.

First note that $(Z_u : u \in V(R'))$ is a partition of $V(D')$. Now let f be an edge of R'. We need to show that $Z' := \cup\{Z_u : u \in V(R'), u > f\}$ is Y_f-normal. Let $e \in E(R)$ be such that $f \in E(P_e)$. Let $x, y \in Z'$ be such that there is a directed path Q' in D' from x to y which uses a node in $V(D') - Z'$. Choose $x, y \in Z'$ so that Q' has minimum length. We need to show that $V(Q') \cap Y_f \neq \emptyset$. We analyze two cases.

If $x \notin E(D)$ then let $x' \in V(D')$ be the node following x in Q'. Note that $x' \in E(D) - Z'$, because of the bipartition of D' and the choice of x and y. This implies that f is not the first edge of the path P_e. Thus x is a copy of a vertex in W_r and $x' \in \delta^+_D(W_r) \subseteq Y_f$.

If $x \in E(D)$ then let Q be the path in D which starts at the tail of x and visits the vertices of D in the same order in which they (their copies in fact) appear in Q'. Let $Z \subseteq V(D)$ be the set of vertices with at least one copy in Z'. Note that $Z = \cup\{W_r : r \in V(R), r > e\}$. Both endpoints of Q are in Z. Indeed, the tail of any edge of D in Z' is also in Z' (because, by construction, if $a \in E(D) \cap Z_u$ then one copy of its tail is also in Z_u) so the first vertex of Q is in Z. Also the last vertex of Q' is not in $E(D)$ (for the same reason), and this implies that the last vertex of Q is in Z. Thus Q is a path in D between two vertices of Z. Since Θ is an arboreal decomposition of D, there is a vertex of X_e in Q. But this implies Q' has a node in Y_f, because $Y_f \supseteq \delta^+_D(X_e)$. This completes the proof that Θ' is an arboreal decomposition of D'.

Now we just need to verify that the width of Θ' is at most $(dtw(D) + 1)(\Delta + 1)$. For that, let $r' \in V(R')$. If r' is a vertex of R, then let $r = r'$, otherwise, r' was added when an edge e of R was substituted by a path between its endpoints. Let r be the tail of e. Note that $|\cup_{f \sim r'} Y_f \cup Z_{r'}| \leq |(\cup_{e \sim r} X_e \cup W_r) \cup \delta^+_D(\cup_{e \sim r} X_e \cup W_r)| \leq (dtw(D) + 1)(\Delta + 1)$. ■

Combining the reduction described above and the PTAS from the previous section, we obtain a PTAS for Directed Unweighted Multicut in digraphs with bounded degree and bounded tree-width.

4 Complexity results

A *bidirected tree* is a digraph which has as underlying graph a tree and two opposite directed edges for each edge in its underlying graph. Similarly, a *bidirected series-parallel digraph* has as underlying graph a series-parallel graph, and two opposite directed edges for each edge in its underlying graph. Bidirected trees have tree-width one and bidirected series-parallel digraphs have tree-width two, by Proposition 2.1 in [4]. By a simple modification of the proofs for the undirected case [1, Theorems 7 and 10], we can prove the following.

Theorem 4.1 *Directed Multicut and Directed Unrestricted Vertex Multicut are NP-hard respectively in bidirected trees and bidirected series-parallel digraphs with maximum in and out degree three.*

The modification consists of using, in each of the reductions, a digraph whose underlying graph is exactly the one described in the corresponding undirected case. Besides, the digraph has two opposite directed edges for each edge in its underlying graph. Note that any directed path in this digraph corresponds to a path in its underlying graph and *vice-versa*. For the edge version, we also note that any feasible solution can be converted in a feasible solution with edges directed away from the root, without increasing the size of the solution, by simply replacing any edge directed towards the root by the parallel edge directed away from the root. The reductions work exactly as in the undirected case, so we omit them in this abstract.

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