1 Project description: Asymptotic and Algorithmic Graph Theory

1.1 Justification

We hope that within the framework of the proposed exchange project the existing collaboration between the groups at the Humboldt-University Berlin and the University of São Paulo may be continued and strengthened.

Past successes have included several joint publications on Ramsey theory [5, 6, 8]. Areas covered by the new project will be asymptotic graph theory, property testing, and the theory of approximation algorithms. The link between these areas is provided by the methods which we hope to use in order to find solutions. As an example we mention here the Regularity Lemma of Szemerédi, which has proved to be an invaluable tool in all three of the areas we propose to investigate.

Both groups (in particular the three research students involved), stand to gain a great deal from this project, which we hope will give the groups an opportunity to share their expertise and to make progress on the open problems listed below.

In the following sections we will outline the current state of research on these problems, the past contributions made by the two groups, and our common goals.

1.2 Graphical Evolution

The standard model for the evolution of random graphs was introduced by Erdős and Rényi: We look at the properties of "almost all" graphs on m edges, where $0 \le m \le \binom{n}{2}$. Much work has been done on this model with the result that it is now very well understood. But in practice one is often only interested in some special graph classes: Hence both from a theoretical as well as practical point of view it would be highly desirable to study the evolution of graphs under given side constraints. We make this precise in the next few sections.

1.2.1 Constrained evolution I: Forbidding a fixed subgraph

Consider the class of all n-vertex triangle-free graphs on $m = m(n)$ edges. We would like to know what a typical graph of this class looks like, in particular, we would like to know its chromatic number. Prömel and Steger [13] gave a solution for a large range of m, but a significant gap remains.

Problem 1 *Let $n^{3/2} \ll m \ll n^{7/4} \log n$. Is almost every n-vertex triangle-free graph with m edges bipartite?*

Recently, Łuczak [38] made some progress on this question by showing (with help of the Regularity Lemma of Szemerédi, see below) that in this range, a typical graph is "almost" bipartite. We hope that these recent advances might lead to a complete solution of the above problem. Furthermore, the problem of forbidding other subgraphs remains wide open. For some related results see Prömel and Steger [12, 14].

1.2.2 Constrained evolution II: Extremal graph processes

The problem is similar to the above but the approach is now slightly different: Fix some antimonotone (hereditary) property P. We start with an empty graph on n vertices, and add edges one by one at random, but always subject to the condition that the graph we obtain must have P. We stop when no more edges can be added and call the resulting graph G_P. How many edges does G_P have? A rather surprising result by Erdős et al. [29] says that if P is the property of being triangle-free then G_P has approximately $n^{3/2}$ edges, while if P is the property of being bipartite (thus we are forbidding *any* odd cycle), then G_P has approximately n^2 edges.

Problem 2 *What happens if P is the property of containing no odd cycles of length at most k, for $3 < k < n$?*

1.2.3 Constrained evolution III: Partially ordered sets

Partial orders (also called posets) are structures which arise frequently both in theory and practice. One natural interpretation of poset evolution is that of a sorting process: Initially, we are given a set with no information about its order, and we gradually gain information about relations between the elements. No universal model for the evolution of random posets has emerged so far. However, there are several models which we have successfully investigated:

Suppose we fix the class of posets on n points and with m relations. Then results obtained by Prömel, Steger, and Taraz [17] show how the typical structure of a poset depends on m. Closely related results can be found in Brightwell, Prómel and Steger [1].

Another approach is to fix a "backbone structure" and look at random subsets. For example, select each subset of $[n] = \{1, \dots, n\}$ with probability p and order the selected subsets by inclusion. Perhaps the first question concerning this rather natural model was posed by Erdős. In modern language, Erdős' question asked for the determination of the threshold function for the occurence of a comparable pair (that is, the emergence of an edge in the comparability graph). This was answered by Rényi in the early 60s [39]. Quite surprisingly, this model was not considered for a long time, but recently Kreuter [9] took up the investigation of the emergence of small substructures in such random posets. Other recent results concerning this model are due to Kohayakawa and Kreuter [4, 7] and to Osthus [10]. It is probably fair to say that we have a global picture of how this structure evolves with p, but many unsolved problems remain.

More generally, the objective is to study other posets and see if a general pattern emerges. In particular we would like to know how well our methods adapt to other models.

1.3 Combinatorial property testing and Regularity

The object of combinatorial property testing is to give efficient algorithms which determine whether a given object has a given property or is "far" from having this property. As demonstrated by Goldreich and his co-authors [32, 33, 34, 35], this approach has been particularly successful in the case of dense graphs and of some success in the case of sparser graphs.

An extremely useful tool in this area is the Regularity Lemma of Szemerédi. For instance, it can be used to prove the correctness of a simple algorithm which tests whether a given graph is close to being H-free, for some fixed subgraph H. Unfortunately, the large constants involved in the proof of the Lemma imply that the running time of the tester is prohibitively large.

Problem 3 *Is it possible to develop a (fast) tester for H-freeness without recourse to the Regularity Lemma?*

The problem above may indeed be deep. For instance, even the case in which H is a triangle is of interest, and a satisfactory answer for this case would have consequences concerning the estimates involved in the Ruzsa–Szemerédi theorem on triple-systems [42], or, equivalently, on Roth's theorem [41] on 3-term arithmetic progressions in subsets of integers.

In any case, we observe that a result first proved making heavy use of Szemerédi's Regularity Lemma, namely, the main result of Rödl and Duke [40], was independently proved by Goldreich, Goldwasser, and Ron [32], from first principles.

1.3.1 Regularity Lemma

As to the the Regularity Lemma itself, for several reasons it seems a worthwhile aim to develop a version for hypergraphs that guarantees a similarly powerful "amplification"—none of the existing variants (due to Frankl and Rödl [30], Chung [24], Frieze and Kannan [31]) seems to suffice in this respect. Not only would such a version help to generalize results in the area of property testing from graphs to hypergraphs, but it would also be interesting in its own right: Frankl and Rödl conjectured that it would help to yield a new and much simpler proof of the famous theorem of Szemerédi on arithmetic progressions.

We believe that a key notion in developing the "correct" concept of regularity for hypergraphs was introduced by Chung and Graham [25]. Indeed, under this project, we shall investigate how the notion of *deviation* introduced in [25] may be related to the notion of ε-regularity. It is well known that these notions are in fact closely tied together in the particular case of graphs. The case of hypergraphs must be explored—perhaps we shall arrive at the "correct" notion of ε-regularity for hypergraphs alluded to above.

1.3.2 Sparse case

The difficulties involved in developing property testers for sparser graphs [34, 35] have a similar flavour to the ones one encounters when trying to develop a satisfactory version of Szemerédi's Lemma for sparse graphs. Despite these difficulties, some progress has been made by Goldreich and Ron [34, 35] on the property testing setting, and by Kohayakawa and Rödl on the regularity setting (see, for instance, [37]). Developing a suitable version of Szemerédi's Regularity

Lemma for sparse graphs and, more generally, for sparse hypergraphs remains a most challenging problem.

1.4 Approximation Algorithms

As is well known, the area of approximation algorithms is currently under vigorous investigation. Research may be focussed on obtaining inapproximability results (for an overview see Hougardy, Prömel and Steger [3] and Mayr, Prömel, and Steger, Eds. [11]), or on developing approximation algorithms or polynomial approximation schemes for NP-hard problems.

As an example of the latter kind we mention the recent results of Hougardy and Prömel [2] and Prömel and Steger [15, 16] on approximating the mininum Steiner tree in graphs.

Two further problems under investigation are: (a) the *largest planar subgraph* problem and (b) the *minimum spanning k-edge-connected subgraph* problem.

For (a), Fernandes has obtained several improvements on previous algorithms, and the currently best approximation ratios are achieved by her methods [19, 20, 21, 22]. For (b), Cheriyan and Thurimella [23] and Fernandes [26, 27] independently obtained the current best results. However, here it should be mentioned that the techniques of Cheriyan and Thurimella seem to be more promising for further applications. Indeed, in this project we hope to combine the ideas in [23] and in [26, 27] to approach other algorithmic problems related to connectivity.

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