Tópicos de Análise de Algoritmos

Parte destes slides são adaptações de slides

do Prof. Paulo Feofiloff e do Prof. José Coelho de Pina.

Aula 4

Multiplicação de inteiros grandes

Secs 5.5 do KT e 28.2 do CLRS.

Multiplicação de inteiros gigantescos

$n :=$ número de algarismos.

Problema: Dados dois números inteiros $X[1\,..\,n]$ e $Y[1\,..\,n]$, calcular o produto $X \cdot Y$.

Entra: Exemplo com $n = 12$

	12											1
X	9	2	3	4	5	5	4	5	6	2	9	8
Y	0	6	3	2	8	4	9	9	3	8	4	4

Multiplicação de inteiros gigantescos

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Entra: Exemplo com $n = 12$

	12											1
X	9	2	3	4	5	5	4	5	6	2	9	8
Y	0	6	3	2	8	4	9	9	3	8	4	4

Sai:

$X \cdot Y$

23																						1
5	8	4	4	0	8	7	2	8	6	7	0	2	7	1	4	1	0	2	9	5	1	2

Algoritmo do ensino fundamental

```
                    * * * * * * * *
                  × * * * * * * * *
                  -----------------
                  * * * * * * * * *
                * * * * * * * * *
              * * * * * * * * *
            * * * * * * * * *
          * * * * * * * * *
        * * * * * * * * *
      * * * * * * * * *
    * * * * * * * * *
----------------------------------
  * * * * * * * * * * * * * * * * *
```

O algoritmo do ensino fundamental é $\Theta(n^2)$.

Da Wiki

In 1960, Kolmogorov organized a seminar on mathematical problems in cybernetics at the Moscow State University, where, among other problems in the complexity of computation, he stated the $\Omega(n^2)$ conjecture, that stated that the traditional algorithm for multiplication of two n-digit numbers was asymptotically optimal, meaning that any algorithm for that task would require $\Omega(n^2)$ elementary operations.

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Kolmogorov was very excited about the discovery; he communicated it at the next meeting of the seminar, which was then terminated. Kolmogorov gave some lectures on the Karatsuba result at conferences all over the world (see, for example, *Proceedings of the International Congress of Mathematicians 1962*, pp. 351–356, and also *6 Lectures delivered at the International Congress of Mathematicians in Stockholm, 1962*) and published the method in 1962, in the *Proceedings of the USSR Academy of Sciences*.

Da Wiki

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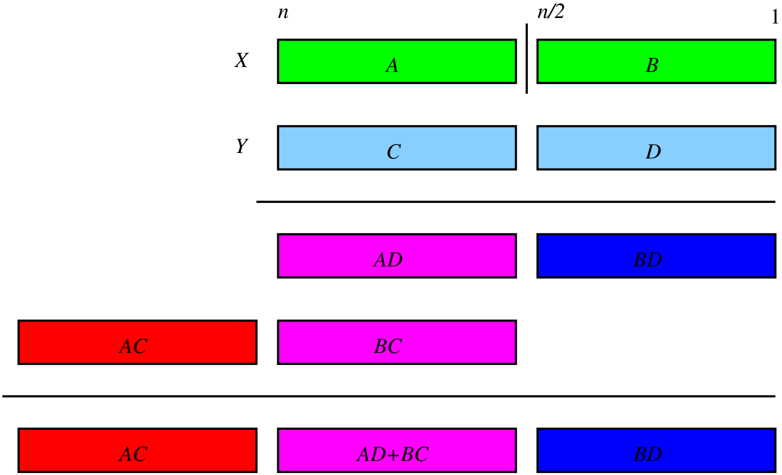
The article had been written by Kolmogorov and contained two results on multiplication, Karatsuba's algorithm and a separate result by Yuri Ofman; it listed "A. Karatsuba and Yu. Ofman" as the authors. Karatsuba only became aware of the paper when he received the reprints from the publisher.

Divisão e conquista



$$X \cdot Y = A \cdot C \times 10^{2\lceil n/2 \rceil} + (A \cdot D + B \cdot C) \times 10^{\lceil n/2 \rceil} + B \cdot D$$

Exemplo

	4			1
X	3	1	4	1

	4			1
Y	5	9	3	6

Exemplo

X

4			1
3	1	4	1

Y

4			1
5	9	3	6

A

3	1

B

4	1

C

5	9

D

3	6

Exemplo

	4			1
X	3	1	4	1

	4			1
Y	5	9	3	6

A	3	1	B	4	1	C	5	9	D	3	6

$$X \cdot Y = A \cdot C \times 10^4 + (A \cdot D + B \cdot C) \times 10^2 + B \cdot D$$

$A \cdot C = 1829$ $\qquad (A \cdot D + B \cdot C) = 1116 + 2419 = 3535$

$B \cdot D = 1476$

$A \cdot C$	1	8	2	9	0	0	0	0
$(A \cdot D + B \cdot C)$			3	5	3	5	0	0
$B \cdot D$					1	4	7	6
$X \cdot Y =$	1	8	6	4	4	9	7	6

Algoritmo de Multi-DC

Algoritmo recebe inteiros $X[1\,.\,.\,n]$ e $Y[1\,.\,.\,n]$ e devolve $X \cdot Y$.

MULT (X, Y, n)

se $n = 1$ **devolva** $X \cdot Y$
$q \leftarrow \lceil n/2 \rceil$
$A \leftarrow X[q+1\,.\,.\,n]$ $\qquad B \leftarrow X[1\,.\,.\,q]$
$C \leftarrow Y[q+1\,.\,.\,n]$ $\qquad D \leftarrow Y[1\,.\,.\,q]$
$E \leftarrow$ MULT$(A, C, \lfloor n/2 \rfloor)$
$F \leftarrow$ MULT$(B, D, \lceil n/2 \rceil)$
$G \leftarrow$ MULT$(A, D, \lceil n/2 \rceil)$
$H \leftarrow$ MULT$(B, C, \lceil n/2 \rceil)$
$R \leftarrow E \times 10^{2\lceil n/2 \rceil} + (G + H) \times 10^{\lceil n/2 \rceil} + F$
devolva R

$T(n) =$ consumo de tempo do algoritmo para multiplicar dois inteiros com n algarismos.

Consumo de tempo

linha		todas as execuções da linha
1	$=$	$\Theta(1)$
2	$=$	$\Theta(1)$
3	$=$	$\Theta(n)$
4	$=$	$\Theta(n)$
5	$=$	$T(\lfloor n/2 \rfloor)$
6	$=$	$T(\lceil n/2 \rceil)$
7	$=$	$T(\lceil n/2 \rceil)$
8	$=$	$T(\lceil n/2 \rceil)$
9	$=$	$\Theta(n)$
10	$=$	$\Theta(n)$
total	$=$	$T(\lfloor n/2 \rfloor) + 3\,T(\lceil n/2 \rceil) + \Theta(n)$

Consumo de tempo

Sabemos que

$$\begin{aligned} T(1) &= \Theta(1) \\ T(n) &= T(\lfloor n/2 \rfloor) + 3\,T(\lceil n/2 \rceil) + \Theta(n) \quad \text{para } n = 2, 3, 4, \ldots \end{aligned}$$

está na mesma classe Θ que a solução de

$$T'(n) = 4\,T'(n/2) + n$$

n	1	2	4	8	16	32	64	128	256	512
$T'(n)$	1	6	28	120	496	2016	8128	32640	130816	523776

Conclusões

$T'(n)$ é $\Theta(n^2)$.

$T(n)$ é $\Theta(n^2)$.

O consumo de tempo do algoritmo MULT é $\Theta(n^2)$.

Tanto trabalho por nada . . .

Será?!?

Pensar pequeno

Olhar para números com 2 algarismos (n=2).

Suponha $X = a\,b$ e $Y = c\,d$.
Se cada multiplicação custa R$ 1,00 e
cada soma custa R$ 0,01, quanto custa $X \cdot Y$?

Pensar pequeno

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Suponha $X = a\,b$ e $Y = c\,d$.
Se cada multiplicação custa R$ 1,00 e
cada soma custa R$ 0,01, quanto custa $X \cdot Y$?

Eis $X \cdot Y$ por R$ 4,03:

X		a	b
Y		c	d
		ad	bd
	ac	bc	
$X \cdot Y$	ac	$ad + bc$	bd

$$X \cdot Y = ac \times 10^2 + (ad + bc) \times 10^1 + bd$$

Pensar pequeno

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Solução mais barata?

Pensar pequeno

Olhar para números com 2 algarismos (n=2).

Suponha $X = a\,b$ e $Y = c\,d$.
Se cada multiplicação custa R$ 1,00 e
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X		a	b
Y		c	d
		ad	bd
	ac	bc	
$X \cdot Y$	ac	$ad + bc$	bd

$X \cdot Y = ac \times 10^2 + (ad + bc) \times 10^1 + bd$

Solução mais barata?
Gauss faz por R$ 3,06!

$X \cdot Y$ por apenas R$ 3,06

X		a	b
Y		c	d
		ad	bd
	ac	bc	
$X \cdot Y$	ac	$ad + bc$	bd

$X \cdot Y$ por apenas R$ 3,06

X		a	b
Y		c	d
		ad	bd
	ac	bc	
$X \cdot Y$	ac	$ad + bc$	bd

$$(a+b)(c+d) = ac + ad + bc + bd \Rightarrow$$
$$ad + bc = (a+b)(c+d) - ac - bd$$

$g = (a+b)(c+d)$ $\quad e = ac$ $\quad f = bd$ $\quad h = g - e - f$

$$X \cdot Y \ \text{(por R\$ 3,06)} = e \times 10^2 + h \times 10^1 + f$$

Exemplo

$$X = 2133 \quad Y = 2312 \quad X \cdot Y = ?$$
$$ac = ? \quad bd = ? \quad (a+b)(c+d) = ?$$

Exemplo

$$\begin{array}{llllll} X = & 2133 & Y = & 2312 & X \cdot Y = & ? \\ ac = & ? & bd = & ? & (a+b)(c+d) = & ? \end{array}$$

$$\begin{array}{llllll} X = & 21 & Y = & 23 & X \cdot Y = & ? \\ ac = & ? & bd = & ? & (a+b)(c+d) = & ? \end{array}$$

Exemplo

$$X = 2133 \quad Y = 2312 \quad X \cdot Y = ?$$
$$ac = ? \quad bd = ? \quad (a+b)(c+d) = ?$$

$$X = 21 \quad Y = 23 \quad X \cdot Y = ?$$
$$ac = ? \quad bd = ? \quad (a+b)(c+d) = ?$$

$$X = 2 \quad Y = 2 \quad X \cdot Y = 4$$

Exemplo

$$\begin{aligned} X &= 2133 & Y &= 2312 & X \cdot Y &= ? \\ ac &= ? & bd &= ? & (a+b)(c+d) &= ? \end{aligned}$$

$$\begin{aligned} X &= 21 & Y &= 23 & X \cdot Y &= ? \\ ac &= 4 & bd &= ? & (a+b)(c+d) &= ? \end{aligned}$$

Exemplo

$$\begin{aligned} X &= 2133 & Y &= 2312 & X \cdot Y &= ? \\ ac &= ? & bd &= ? & (a+b)(c+d) &= ? \end{aligned}$$

$$\begin{aligned} X &= 21 & Y &= 23 & X \cdot Y &= ? \\ ac &= 4 & bd &= ? & (a+b)(c+d) &= ? \end{aligned}$$

$$X = 1 \quad Y = 3 \quad X \cdot Y = 3$$

Exemplo

$$\begin{aligned} X &= 2133 & Y &= 2312 & X \cdot Y &= ? \\ ac &= ? & bd &= ? & (a+b)(c+d) &= ? \end{aligned}$$

$$\begin{aligned} X &= 21 & Y &= 23 & X \cdot Y &= ? \\ ac &= 4 & bd &= 3 & (a+b)(c+d) &= ? \end{aligned}$$

Exemplo

$$X = 2133 \quad Y = 2312 \qquad X \cdot Y = ?$$
$$ac = ? \quad bd = ? \qquad (a+b)(c+d) = ?$$

$$X = 21 \quad Y = 23 \qquad X \cdot Y = ?$$
$$ac = 4 \quad bd = 3 \qquad (a+b)(c+d) = ?$$

$$X = 3 \quad Y = 5 \quad X \cdot Y = 15$$

Exemplo

$$X = 2133 \quad Y = 2312 \quad X \cdot Y = ?$$
$$ac = ? \quad bd = ? \quad (a+b)(c+d) = ?$$

$$X = 21 \quad Y = 23 \quad X \cdot Y = 483$$
$$ac = 4 \quad bd = 3 \quad (a+b)(c+d) = 15$$

Exemplo

$$X = 2133 \quad Y = 2312 \qquad X \cdot Y = ?$$

$$ac = 483 \quad bd = ? \qquad (a+b)(c+d) = ?$$

Exemplo

$$\begin{aligned} X &= 2133 & Y &= 2312 & X \cdot Y &= ? \\ ac &= 483 & bd &= ? & (a+b)(c+d) &= ? \end{aligned}$$

$$\begin{aligned} X &= 33 & Y &= 12 & X \cdot Y &= ? \\ ac &= ? & bd &= ? & (a+b)(c+d) &= ? \end{aligned}$$

Exemplo

$$\begin{array}{llllll} X = & 2133 & Y = & 2312 & X \cdot Y = & ? \\ ac = & 483 & bd = & ? & (a+b)(c+d) = & ? \end{array}$$

$$\begin{array}{llllll} X = & 33 & Y = & 12 & X \cdot Y = & 396 \\ ac = & 3 & bd = & 6 & (a+b)(c+d) = & 18 \end{array}$$

Exemplo

$$\begin{array}{llllll} X = & 2133 & Y = & 2312 & X \cdot Y = & ? \\ ac = & 483 & bd = & 396 & (a+b)(c+d) = & ? \end{array}$$

Exemplo

$$X = 2133 \quad Y = 2312 \quad X \cdot Y = ?$$
$$ac = 483 \quad bd = 396 \quad (a+b)(c+d) = ?$$

$$X = 54 \quad Y = 35 \quad X \cdot Y = ?$$
$$ac = ? \quad bd = ? \quad (a+b)(c+d) = ?$$

Exemplo

$$\begin{array}{llllll} X = & 2133 & Y = & 2312 & X \cdot Y = & ? \\ ac = & 483 & bd = & 396 & (a+b)(c+d) = & ? \end{array}$$

$$\begin{array}{llllll} X = & 54 & Y = & 35 & X \cdot Y = & 1890 \\ ac = & 15 & bd = & 20 & (a+b)(c+d) = & 72 \end{array}$$

Exemplo

$$\begin{aligned} X &= 2133 & Y &= 2312 & X \cdot Y &= ? \\ ac &= 483 & bd &= 396 & (a+b)(c+d) &= 1890 \end{aligned}$$

Exemplo

$$
\begin{array}{llllll}
X = & 2133 & Y = & 2312 & X \cdot Y = & 4931496 \\
ac = & 483 & bd = & 396 & (a+b)(c+d) = & 1890
\end{array}
$$

Algoritmo de Karatsuba e Ofman

Algoritmo recebe inteiros $X[1\,..\,n]$ e $Y[1\,..\,n]$ e devolve $X \cdot Y$.

KARATSUBA (X, Y, n)
- **se** $n \leq 3$ **devolva** $X \cdot Y$
- $q \leftarrow \lceil n/2 \rceil$
- $A \leftarrow X[q+1\,..\,n]$ $\quad$ $B \leftarrow X[1\,..\,q]$
- $C \leftarrow Y[q+1\,..\,n]$ $\quad$ $D \leftarrow Y[1\,..\,q]$
- $E \leftarrow$ KARATSUBA$(A, C, \lfloor n/2 \rfloor)$
- $F \leftarrow$ KARATSUBA$(B, D, \lceil n/2 \rceil)$
- $G \leftarrow$ KARATSUBA$(A+B, C+D, \lceil n/2 \rceil + 1)$
- $H \leftarrow G - F - E$
- $R \leftarrow E \times 10^{2\lceil n/2 \rceil} + H \times 10^{\lceil n/2 \rceil} + F$
- **devolva** R

$T(n)$ = consumo de tempo do algoritmo para multiplicar dois inteiros com n algarismos.

Consumo de tempo

linha		todas as execuções da linha
1	$=$	$\Theta(1)$
2	$=$	$\Theta(1)$
3	$=$	$\Theta(n)$
4	$=$	$\Theta(n)$
5	$=$	$T(\lfloor n/2 \rfloor)$
6	$=$	$T(\lceil n/2 \rceil)$
7	$=$	$T(\lceil n/2 \rceil + 1) + \Theta(n)$
8	$=$	$\Theta(n)$
9	$=$	$\Theta(n)$
10	$=$	$\Theta(n)$
total	$=$	$T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + T(\lceil n/2 \rceil + 1) + \Theta(n)$

Consumo de tempo

Sabemos que

$$
\begin{aligned}
T(n) &= \Theta(1) \quad \text{para } n = 1,2,3 \\
T(n) &= T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + T(\lceil n/2 \rceil + 1) + \Theta(n) \quad n \geq 4
\end{aligned}
$$

está na mesma classe Θ que a solução de

$$T'(n) = 3T'(n/2) + n$$

n	1	2	4	8	16	32	64	128	256	512
$T'(n)$	1	5	19	65	211	665	2059	6305	19171	58025

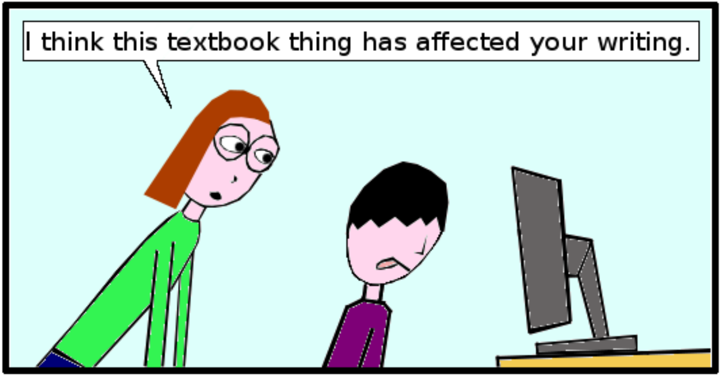
Conclusões

$T'(n)$ é $\Theta(n^{\lg 3})$.

Logo $T(n)$ é $\Theta(n^{\lg 3})$.

O consumo de tempo do algoritmo KARATSUBA é $\Theta(n^{\lg 3})$ $(1{,}584 < \lg 3 < 1{,}585)$.

Karatsuba em tirinhas



Mais conclusões

Consumo de tempo de
algoritmos para multiplicação de inteiros:

Jardim de infância	$\Theta(n\,10^n)$
Ensino fundamental	$\Theta(n^2)$
Karatsuba e Ofman'60	$\mathrm{O}(n^{1.585})$
Toom e Cook'63	$\mathrm{O}(n^{1.465})$
(divisão e conquista; generaliza o acima)	
Schönhage e Strassen'71	$\mathrm{O}(n \lg n \lg\lg n)$
(FFT em aneis de tamanho específico)	
Fürer'07	$\mathrm{O}(n \lg n\, 2^{\mathrm{O}(\log^* n)})$
Harvey e van der Hoeven'19	$\mathrm{O}(n \lg n)$

Ambiente experimental

A plataforma utilizada nos experimentos é um PC rodando Linux Debian ?.? com um processador Pentium II de 233 MHz e 128MB de memória RAM .

Os códigos estão compilados com o gcc versão 2.7.2.1 e opção de compilação -O2.

As implementações comparadas neste experimento são as do algoritmo do ensino fundamental e do algoritmo KARATSUBA.

O programa foi escrito por Carl Burch:

`http://www-2.cs.cmu.edu/~cburch/251/karat/.`

Resultados experimentais

n	Ensino Fund.	KARATSUBA
4	0.005662	0.005815
8	0.010141	0.010600
16	0.020406	0.023643
32	0.051744	0.060335
64	0.155788	0.165563
128	0.532198	0.470810
256	1.941748	1.369863
512	7.352941	4.032258

Tempos em 10^3 segundos.

Multiplicação de matrizes

Problema: Dadas duas matrizes $X[1\,..\,n, 1\,..\,n]$ e $Y[1\,..\,n, 1\,..\,n]$ calcular o produto $X \cdot Y$.

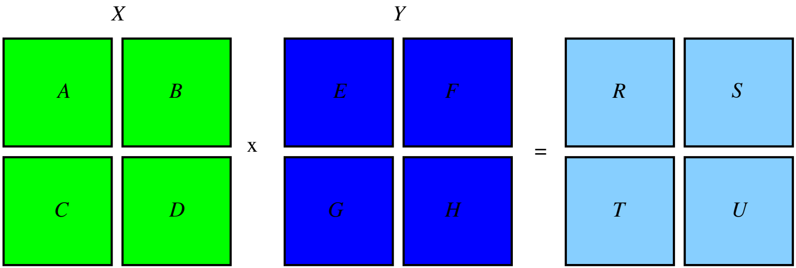
O algoritmo tradicional de multiplicação de matrizes consome tempo $\Theta(n^3)$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$$

$$\begin{aligned} r &= ae + bg \\ s &= af + bh \\ t &= ce + dg \\ u &= cf + dh \end{aligned} \tag{1}$$

Solução custa R$ 8,04

Divisão e conquista



$$
\begin{aligned}
R &= AE + BG \\
S &= AF + BH \\
T &= CE + DG \\
U &= CF + DH
\end{aligned} \tag{2}
$$

Algoritmo de multiplicação de matrizes

Algoritmo recebe inteiros $X[1\,..\,n]$ e $Y[1\,..\,n]$ e devolve $X \cdot Y$.

MULTI-M (X, Y, n)
se $n = 1$ **devolva** $X \cdot Y$
$(A, B, C, D) \leftarrow$ PARTICIONE(X, n)
$(E, F, G, H) \leftarrow$ PARTICIONE(Y, n)
$R \leftarrow$ MULTI-M$(A, E, n/2)$ + MULTI-M$(B, G, n/2)$
$S \leftarrow$ MULTI-M$(A, F, n/2)$ + MULTI-M$(B, H, n/2)$
$T \leftarrow$ MULTI-M$(C, E, n/2)$ + MULTI-M$(D, G, n/2)$
$U \leftarrow$ MULTI-M$(C, F, n/2)$ + MULTI-M$(D, H, n/2)$
$P \leftarrow$ CONSTRÓI-MAT(R, S, T, U)
devolva P

$T(n) =$ consumo de tempo do algoritmo para
multiplicar duas matrizes de n linhas e n colunas.

Consumo de tempo

linha		todas as execuções da linha
1	$=$	$\Theta(1)$
2	$=$	$\Theta(n^2)$
3	$=$	$\Theta(n^2)$
4	$=$	$T(n/2) + T(n/2)$
5	$=$	$T(n/2) + T(n/2)$
6	$=$	$T(n/2) + T(n/2)$
7	$=$	$T(n/2) + T(n/2)$
8	$=$	$\Theta(n^2)$
9	$=$	$\Theta(n^2)$
total	$=$	$8\,T(n/2) + \Theta(n^2)$

Consumo de tempo

As dicas no nosso estudo de recorrências sugere que a solução da recorrência

$$\begin{aligned} T(1) &= \Theta(1) \\ T(n) &= 8\,T(n/2) + \Theta(n^2) \quad \text{para } n = 2, 3, 4, \ldots \end{aligned}$$

está na mesma classe Θ que a solução de

$$T'(n) = 8\,T'(n/2) + n^2$$

n	1	2	4	8	16	32	64	128	256
$T'(n)$	1	12	112	960	7936	64512	520192	4177920	33488896

Conclusões

$T'(n)$ é $\Theta(n^3)$.

Logo $T(n)$ é $\Theta(n^3)$.

O consumo de tempo do algoritmo MULTI-M é $\Theta(n^3)$.

Strassen: $X \cdot Y$ por apenas R$ 7,18

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$$

Strassen: $X \cdot Y$ por apenas R$ 7,18

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$$

$$p_1 = a(f - h) = af - ah$$
$$p_2 = (a + b)h = ah + bh$$
$$p_3 = (c + d)e = ce + de$$
$$p_4 = d(g - e) = dg - de$$
$$p_5 = (a + d)(e + h) = ae + ah + de + dh$$
$$p_6 = (b - d)(g + h) = bg + bh - dg - dh$$
$$p_7 = (a - c)(e + f) = ae + af - ce - cf$$

Strassen: $X \cdot Y$ por apenas R$ 7,18

$$
\begin{aligned}
p_1 &= a(f - h) = af - ah \\
p_2 &= (a + b)h = ah + bh \\
p_3 &= (c + d)e = ce + de \\
p_4 &= d(g - e) = dg - de \\
p_5 &= (a + d)(e + h) = ae + ah + de + dh \\
p_6 &= (b - d)(g + h) = bg + bh - dg - dh \\
p_7 &= (a - c)(e + f) = ae + af - ce - cfd
\end{aligned}
$$

$$
\begin{aligned}
r &= p_5 + p_4 - p_2 + p_6 = ae + bg \\
s &= p_1 + p_2 = af + bh \\
t &= p_3 + p_4 = ce + dg \\
u &= p_5 + p_1 - p_3 - p_7 = cf + dh
\end{aligned}
$$

Algoritmo de Strassen

```
STRASSEN (X, Y, n)
    se n = 1 devolva X · Y
    (A, B, C, D) ← PARTICIONE(X, n)
    (E, F, G, H) ← PARTICIONE(Y, n)
    P1 ← STRASSEN(A, F − H, n/2)
    P2 ← STRASSEN(A + B, H, n/2)
    P3 ← STRASSEN(C + D, E, n/2)
    P4 ← STRASSEN(D, G − E, n/2)
    P5 ← STRASSEN(A + D, E + H, n/2)
    P6 ← STRASSEN(B − D, G + H, n/2)
    P7 ← STRASSEN(A − C, E + F, n/2)
    R ← P5 + P4 − P2 + P6
    S ← P1 + P2
    T ← P3 + P4
    U ← P5 + P1 − P3 − P7
    devolva P ← CONSTRÓI-MAT(R, S, T, U)
```

Consumo de tempo

linha		todas as execuções da linha
1	$=$	$\Theta(1)$
2-3	$=$	$\Theta(n^2)$
4-10	$=$	$7\,T(n/2) + \Theta(n^2)$
11-14	$=$	$\Theta(n^2)$
15	$=$	$\Theta(n^2)$
total	$=$	$7\,T(n/2) + \Theta(n^2)$

Consumo de tempo

As dicas no nosso estudo de recorrências sugeri que a solução da recorrência

$$\begin{aligned} T(1) &= \Theta(1) \\ T(n) &= 7\,T(n/2) + \Theta(n^2) \quad \text{para } n = 2, 3, 4, \ldots \end{aligned}$$

está na mesma classe Θ que a solução de

$$T'(n) = 7\,T'(n/2) + n^2$$

n	1	2	4	8	16	32	64	128	256
$T'(n)$	1	11	93	715	5261	37851	269053	1899755	13363821

Conclusões

$T'(n)$ é $\Theta(n^{\lg 7})$.

Logo $T(n)$ é $\Theta(n^{\lg 7})$.

O consumo de tempo do algoritmo STRASSEN é $\Theta(n^{\lg 7})$ ($2{,}80 < \lg 7 < 2{,}81$).

Mais conclusões

Consumo de tempo de algoritmos para multiplicação de matrizes:

Ensino fundamental	$\Theta(n^3)$
Strassen (1969)	$\mathrm{O}(n^{2.81})$
$\vdots$	$\vdots$
Coppersmith e Winograd (1987)	$\mathrm{O}(n^{2.3755})$
Stothers (2010)	$\mathrm{O}(n^{2.3736})$
$\vdots$	$\vdots$
Alman e Williams (2020)	$\mathrm{O}(n^{2.3729})$
Duan, Wu e Zhou (2022)	$\mathrm{O}(n^{2.3719})$