COMPUTING THE BINOMIAL COEFFICIENTS

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There is a simple and overflow-robust formula for computing binomial coefficients that I learned many, o so many years ago, but that I keep forgetting so often that I finally decided to put it on paper.

One of the problems of the computation of binomial coefficient is that if you are too naïve about it, you will end up with overflows. For instance, you want to compute

$$\binom{1000}{999} \tag{1}$$

The value of this coefficient is 1000, which is well within the representation capabilities of a computer, but if you try to compute it the naïve way, according to the definition

$$\binom{1000}{999} = \frac{1000!}{999!\,(1000-999)!} \tag{2}$$

you will have to start by computing $1000!$, which, as far as I know, no computer today can represent. However, write the generic coefficient as

$$\begin{aligned}\binom{n}{k} = \frac{n!}{n!\,(n-k)!} &= \frac{1\cdot 2\cdots n}{(1\cdot 2\cdots k)(1\cdot 2\cdots (n-k))} \\ &= \frac{(k+1)\cdot(k+2)\cdots n}{1\cdot 2\cdots (n-k)}\end{aligned} \tag{3}$$

where we have used the $k!$ to eliminate the first k factors of $n!$. This we can write as

$$\frac{(k+1)\cdot(k+2)\cdots n}{1\cdot 2\cdots (n-k)} = \frac{(k+1)\cdot(k+2)\cdots k+n-k}{1\cdot 2\cdots (n-k)} = \frac{k+1}{1}\cdot\frac{k+2}{2}\cdots\frac{k+(n-k)}{n-k} \tag{4}$$

that is,

$$\binom{n}{k} = \prod_{i=1}^{n-k}\frac{k+i}{i} == \prod_{i=1}^{n-k} 1+\frac{k}{i} \tag{5}$$

The nice property of this formulation is that it is a product of factors all greater than 1, which guarantees that the result will be monotonically increasing until it reaches the final result. This guarantees that, provided the final result does not cause an overflow, there will be no overflows along the way.

To have a more efficient computation, we can notice that the binomial formula is symmetric in k and $n-k$ so that we can write a similar formula in $n-k$:

$$\binom{n}{k} = \prod_{i=1}^{k} 1+\frac{n-k}{i} \tag{6}$$

It is convenient, of course, to use always the product with the fewer factors, so we can write the procedure as

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binomial(n, k) =
     c = 1;
     if n-k < k then
       for i=1 to n-k do
         r ← r*(1 + k/i);
       od
     else
       for i=1 to k do
         r ← r*(1 + (n-k)/i);
       od
     fi
     return r;
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As a final observation, Dijkstra observed that it would be convenient to write the coefficients in a form that reflects their symmetry in two arguments, writing

$$C(n,k) = \frac{(n+k)!}{n!\,k!} \tag{7}$$

With this definition we have $\left(\begin{array}{c} n \\ k \end{array}\right) = C(n-k,k)$ and, for example, the "theorem"

$$\left(\begin{array}{c} n \\ k \end{array}\right) = \left(\begin{array}{c} n \\ n-k \end{array}\right) \tag{8}$$

becomes trivial. In this case our two formulas become

$$C(n,k) = \prod_{i=1}^{k} 1 + \frac{n}{i} = \prod_{i=1}^{n} 1 + \frac{k}{i} \tag{9}$$

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