

Single Valued Combinatorial Auctions with Budgets

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“... any color that he [the customer] wants so long as it is black.”

— Henry Ford, My Life and Work (1922)

“... Illogical approach to advertising budgets ...”

— Michael Schudson, Advertising, The Uneasy Persuasion: Its Dubious Impact on American Society (1984)

ABSTRACT

We consider budget constrained combinatorial auctions where bidder i has a private value v_i for each of the items in some set S_i , and a budget constraint b_i on his total payment. The value to agent i of a set of items R is $|R \cap S_i| \cdot v_i$ when his payment does not exceed his budget. Such auctions capture adword auctions, where advertisers offer a bid for those adwords that (hopefully) reach their target audience, and advertisers have budgets as well. It is known that even if all items are identical and all budgets are public it is not possible to be truthful and maximize social welfare. Our main result is a novel auction that runs in polynomial time, is incentive compatible, and ensures efficiency by Pareto-optimality. The auction is incentive compatible with respect to the private valuations, v_i , whereas the budgets, b_i , and the sets of interest, S_i , are assumed to be public knowledge. This extends the result of Dobzinski et al. [7, 8] for auctions of multiple *identical* items and public budgets to single-valued *combinatorial* auctions with public budgets.

Categories and Subject Descriptors

F.2.m [ANALYSIS OF ALGORITHMS AND PROBLEM COMPLEXITY]: Miscellaneous

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General Terms

Algorithms, Economics, Theory

Keywords

Multi-Unit Auctions, Combinatorial auctions, Truthfulness, Pareto-optimality

1. INTRODUCTION

In recent years ad auctions have attracted considerable attention from computer scientists. In ad auctions advertisers bid on given set of keywords. Their valuation for keywords depends on the click-through-rate and can be included into a generalized second price auction, which is used by firms such as Google, Yahoo, and Microsoft [9]. Such an auction can be used if keywords are linearly ordered by their valuations.

In contrast, we consider every potential advertising slot as an different “item”. We assume that advertisers have the same valuation for each item they are interested to get. We consider the problem of agents with additive valuations that have single (private) value for any of an agent-specific set of items. E.g., adwords “Corn” or “Gold” are worth 75 cents per impression to agent A , adwords “Gold” or “Silver” or, sometimes, “Lead”, (“Lead only if between 5-9PM EST), are each worth 60 cents to agent B , and so forth. *I.e.*, the sets of items the two agents are interested in are potentially different, and may overlap. Each agent has a private value, the same value for each of the items in the agent’s set of interest. In addition, every agent has an agent-specific budget, e.g., agent A has a budget of \$55, agent B has a budget of \$7.

The Vickrey multiunit auction [15] was designed to deal with a setting where all items are different and agents can have arbitrary valuation functions. However, it cannot be used when agents have budgets. In a seminal paper on multiunit auctions with budgets by Dobzinski, Lavi and Nisan [7, 8], they consider the case when all units are identical, *i.e.*, all agents are interested in getting all items.

Dobzinski *et al.* give an incentive-compatible auction (with respect to valuation) that produces a Pareto-optimal allocation. This result holds if one assumes that the budgets are public information. They also show that this assumption cannot be dropped, *i.e.*, there is no incentive-compatible auction with respect to both valuation and budgets that produces a Pareto-optimal allocation.

This paper is an extension of [7, 8] to a single valued combinatorial auction. This shows that even in the single-valued

combinatorial setting achieving some notion of efficiency is possible. Pareto-optimality is a basic notion for efficiency of allocation, *i.e.*, it seems to be the least one should aim for. If an allocation were not Pareto-optimal then agents could trade amongst themselves and improve their lot, thus increasing efficiency. However, in many settings we would like to achieve something stronger. We would like to produce an allocation such that no trades are possible because such allocations are less prone to speculations [11]. The main technical part of our approach is to show that in our setting both “no trade” and Pareto-optimality conditions are equivalent. This result is of independent interest.

For a comparison between the different settings see Table 1.

Our work seeks to map out the frontier of the possible. We give an incentive compatible combinatorial auction that produces Pareto-optimal allocation in the case when agents have budgets and their valuations are private. This setting is not entirely general but also gives a non-trivial class of auctions that contains some existing auctions, *e.g.*, Google’s auctions for TV ads [13]. Furthermore, we show that these restrictions are unavoidable.

The Setting

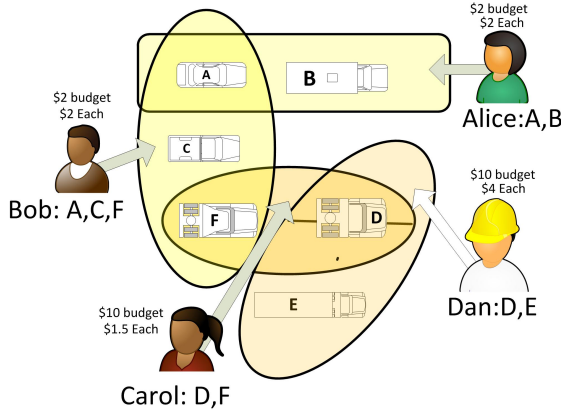


Figure 1: Example with four bidders, each interested in a subset of items, with a value per item and an overall budget.

In this paper we study combinatorial auctions of the following **general form**:

- Every agent (bidder) $1 \leq a \leq n$ has a *publicly known* budget, $b_a \geq 0$, and an unknown (private) valuation $v_a > 0$;
- Every agent a is “interested” in some *publicly known* set of items, S_a . We assume that there is at least one agent interested in every item. Agent a is allocated some (possibly empty) subset of S_a . An example is given in Figure 1.
- The auction produces an allocation (M, P) .
 1. $M \subseteq \{1, \dots, n\} \times \{1, \dots, m\}$ is a (partial) assignment of agents to items: items are assigned to some agent, there may be agents that have no items assigned to them.

2. $P \in \mathcal{R}^n$ is a vectors of payments made by the agents.

- For agent $1 \leq a \leq n$, let M_a be the number of items sold to agent a over the course of the auction and P_a be the total payment made by agent a during the course of the auction. The allocation (M, P) must obey the following conditions:

1. The payment by agent a , P_a , cannot exceed the budget b_a .
2. The utility for agent $1 \leq a \leq n$ is $u_a = M_a v_a - P_a$.
3. The utility for the auctioneer is $\sum_{j=1}^n P_j$.
4. Bidder-rationality: for all agents $1 \leq a \leq n$, $u_a \geq 0$.
5. Auctioneer-rationality: the utility of the auctioneer, $\sum_{j=1}^n P_j \geq 0$.

Note¹ that auctioneer-rationality is implied by *no positive transfers*: $P_a \geq 0$ for all $1 \leq a \leq n$.

Given valuations, v_a , budgets, b_a , and sets of interest, S_a , we define (M, P) to be *Pareto-optimal* if there is no other allocation (M', P') such that²

1. The utility of a bidder in (M, P) is no less than her utility in (M', P') ; and
2. The utility of the auctioneer in (M, P) is no less than the utility in (M', P') ; and
3. At least one bidder or the auctioneer is better off in (M', P') compared with (M, P) .

An auction is said to be incentive compatible if it is a dominant strategy for all bidders to reveal their true valuation. An auction is said to be Pareto-optimal if the allocation it produces is Pareto-optimal. An auction is said to make no positive transfers if the allocation it produces has no positive transfers.

Related work.

The Dobzinski et al. [7, 8] multi-unit auction with budgets is based on the clinching auction by Ausubel [3]. Our setting is an extension of their work to single valued combinatorial auctions with budgets.

One possible justification for such auctions is the paper of Nisan et al. [13] describing Google’s auction for TV Ads. Google’s auction allows bidders to select shows, times, and days they wish to advertise on; and then give a per-ad impression bid and a total budget. The auction is not incentive compatible (even if budgets are assumed to be public). The Google TV ad interface allows agents to describe a set of interest, S_i , and a (single) per-impression bid. *I.e.*, if all slots have the same number of impressions then this fits our setting.

Ascending clinching auctions are used in the FCC spectrum auctions [12, 4, 3]. We believe that studying properties of such auctions, is not only of theoretical interest, but could help to design better auctions in practice.

¹In [8] the authors refer to what we call auctioneer rationality by the term “weakly no positive transfers”.

²Note that no restrictions are placed on the matching M' or on the payments P' .

Setting	Valuation	Auction	Private Data	Public Data	Properties
Multiunit Value function (f_a) No Budgets	$f_a(\# \text{ items})$	Vickrey [15] Multiunit	f_a	—	Maximizes Social Welfare
Multiunit Additive Value v_a Finite Budgets (b_a)	$v_a \cdot \# \text{ items}$	DLN [7]	v_a	b_a	Pareto Optimal
Combinatorial (S_a) Additive Value (v_a) Finite Budgets (b_a)	$v_a \cdot \#(\text{items} \in S_a)$	This Paper	v_a	b_a, S_a	Pareto Optimal

Table 1: Incentive compatible Multi unit and Single Valued Auctions

Bhattacharya et al. [5] showed that for one infinitely divisible good, a bidder cannot improve her utility by under reporting her budget. This leads to a randomized truthful in expectation algorithm (of one infinitely divisible good) with private budgets and private valuations. In [6] Bhattacharya et al. also consider revenue optimal Bayesian incentive compatible auctions with budgets.

Aggarwal et al. [1] (and independently Ashlagi et al. [2]) considered the case where bidders seek at most one item — not quite relevant for ad auctions. In this setting they give an incentive compatible auction, with respect to both valuation and budgets.

Our auction is simultaneously a specialization and a generalization of the Ausubel-Milgrom ascending package auction [4]. The Ausubel-Milgrom ascending package auction is a non-transferable utility core allocation, even when budgets are present. *I.e.*, no coalition can perform a “trade”, preserving budgets, where the coalition members are better off. However, the combinatorial complexity of the set of possible packages makes this a non-polynomial time algorithm, and the incentive compatibility (with respect to valuation) was only known if all items are identical (the multi-unit case) and the budgets are public. Hatfield and Milgrom [10] present a unified way of viewing the Ausubel-Milgrom auction as well as various other matching problems (not quasi-linear).

Our Results.

In this paper we give a combinatorial auction with the properties described above (“The Setting”). *I.e.*, our auction is incentive compatible with respect to private valuations, individually rational, and the auctioneer makes no positive transfers. We describe the auction as an ascending clinching auction. Alternately, agents may supply the mechanism with their valuation, (the budget and set of interest is assumed to be public), and the mechanism will simulate the ascending clinching auction described herein.

Our combinatorial auction is polynomial time and deterministic. This would not be possible if we were to consider the full generality of combinatorial auctions.

Our result is an extension of [7] from the case of multiple identical items to a new combinatorial setting where items are distinct and different agents may be interested in different items. In particular, for the non-combinatorial multi-unit setting of [7, 8], our auction and their auction produce the same allocation.

In light of the impossibility results of Dobzinski et al. [7] we cannot hope to achieve this result when budgets are private knowledge. We further show that the requirement

that budgets are public knowledge is insufficient for Pareto-optimality and incentive compatibility. We show that one cannot avoid the following restrictions:

- If budgets are public but the sets of interest and the valuations are private then no truthful Pareto-optimal auction is possible.
- if budgets are public and private arbitrary valuations are allowed, no truthful and Pareto-optimal auction is possible. This follows by simple reduction to the previous claim on private sets of interest.

In Section 3 we present our mechanism. It is straightforward to show that the mechanism is truthful with respect to valuations. However, it is more involved to prove that the mechanism is Pareto-optimal — we show that the allocation produced by the mechanism is in fact Pareto-optimal in Section 4. In Section 5 we complement our positive result by showing that with public budgets, private valuations, and private sets of interest, there can be no truthful Pareto-optimal auction.

2. PARETO-OPTIMALITY AND TRADE PATHS

The main aim of this section is to introduce the main definitions: trading paths and Pareto-optimality, and to show the equivalence between allocations with no trading paths and Pareto optimal allocations. At the end of the section we will attempt to give some intuition as to why these two notions are related. First, let us define formally the notion of Pareto-optimality.

Definition 2.1 *An allocation (M, P) is Pareto-optimal if for no other allocation (M', P') are all players at least as well off, $M'_i v_i - P'_i \geq M_i v_i - P_i$, including the auctioneer, $\sum_i P'_i \geq \sum_i P_i$, with at least one of these inequalities being strict.*

We now define alternating paths.

Definition 2.2 *Consider a path*

$$\pi = (a_1, t_1, a_2, t_2, \dots, a_{j-1}, t_{j-1}, a_j),$$

in a bipartite graph G . We say that the path π is an alternating path with respect to an allocation M if $(a_i, t_i) \in M$ and $t_i \in S_{i+1}$ for all $1 \leq i < j$. We say that an alternating path is simple if no agent appears more than once along the path. Note that all alternating paths are of even length (even number of edges).

An example of an alternating path from agent i to agent j that includes three items is shown on Figure 2. The solid edge indicates the allocation of an item to an agent whereas a dotted edge indicates that the item belongs to the preference set of the agent.

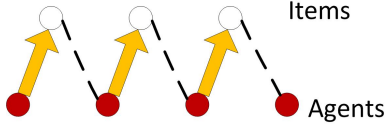


Figure 2: An alternating path

Definition 2.3 A path $\pi = (a_1, t_1, a_2, t_2, \dots, a_{j-1}, t_{j-1}, a_j)$ is called a trading path with respect to the allocation (M, P) if the following hold:

1. π is a simple alternating path with respect to M , (which implies that agent a_i , $i < j$, was allocated item t_i during the course of the auction).
2. The valuation of agent a_j , v_{a_j} is strictly greater than the valuation of agent a_1 , v_{a_1} .
3. The remaining (unused) budget of agent a_j at the conclusion of the auction, $b_{a_j}^*$, is \geq the valuation of agent a_1 , v_{a_1} .

Intuitively, trading paths, as their name suggests, represent possible trades amongst agents. A trading path describes a possible trade, i.e., item t_i is given to agent a_{i+1} for each $i = 1, \dots, j - 1$. After this trade the endpoints of the path are better off and the interior agents are not worse off. (In fact, they can all be made better off by paying a “commission” along the path).

We now move to the following equivalence:

Theorem 2.4 Any allocation (M, P) is Pareto-optimal³ if and only if

1. All items are sold in (M, P) , and
2. There are no trading paths in G with respect to (M, P) .

PROOF. Let \mathcal{Q} be the predicate that (M, P) is Pareto-optimal, \mathcal{R}_1 be the predicate that all items are sold in (M, P) , and \mathcal{R}_2 the predicate that there are no trading paths in G with respect to (M, P) . We seek to show that $\mathcal{Q} \Leftrightarrow \mathcal{R}_1 \cap \mathcal{R}_2$.

$\mathcal{Q} \Rightarrow (\mathcal{R}_1 \cap \mathcal{R}_2)$: to prove this we show that $(\neg \mathcal{R}_1 \cup \neg \mathcal{R}_2) \Rightarrow \neg \mathcal{Q}$.

If both \mathcal{R}_1 and \mathcal{R}_2 are true then this becomes False $\Rightarrow \mathcal{Q}$ which is trivially true.

If the allocation (M, P) does not assign all items ($\neg \mathcal{R}_1$) then it is clearly not Pareto-optimal ($\neg \mathcal{Q}$). We can get a better allocation by assigning all unsold items to any agent i with such items in S_i . This increases the utility of agent i .

If $\neg \mathcal{R}_2$ then there exists a trading path π in G with respect to (M, P) , let $\pi = (a_1, t_1, a_2, t_2, \dots, a_{j-1}, t_{j-1}, a_j)$, as $v_{a_j} > v_{a_1}$ and $b_{a_j}^* \geq v_{a_1}$ then we can decrease the payment of agent a_1 by v_{a_1} , increase the payment of agent a_j

³We remark that an analogous (but simpler) claim made in the proceedings version of the multi unit auction with budget paper [7] was incorrect but was corrected in [8].

by the same v_{a_1} , and move item t_i from agent a_i to agent a_{i+1} for all $i = 1, \dots, j - 1$. In this case, the utility of agents a_1, a_2, \dots, a_{j-1} is unchanged, the utility of agent a_j increases by $v_{a_j} - v_{a_1} > 0$, and the utility of the auctioneer is unchanged. The sum of payments by the agents is likewise unchanged. This contradicts the assumption that (M, P) is Pareto-optimal.

We now prove that $(\mathcal{R}_1 \cap \mathcal{R}_2) \Rightarrow \mathcal{Q}$. Again we show the contrapositive: $\neg \mathcal{Q} \Rightarrow (\neg \mathcal{R}_1 \cup \neg \mathcal{R}_2)$. Assume $\neg \mathcal{Q}$, i.e., assume that (M, P) is not Pareto-optimal. Further assume \mathcal{R}_1 , that M assigns all items. We will show $\neg \mathcal{R}_2$, i.e. that there is a trading path with respect to (M, P) . Since (M, P) is not Pareto-optimal, there must be some other allocation (M', P') that is not worse for all players (including the auctioneer) and strictly better for at least one player. We can assume that (M', P') assigns all items as well, as otherwise we can take an even better allocation that would assign all items.

By Lemma 2.5 (see below) we know that M and M' are related by a set of simple paths and cycles. On a path, the first agent gives up one item, whereas the last agent receives one item more, after items are exchanged along the path. Cycles represent giving up one item in return for another by passing items around along it. Cycles don't change the number of items assigned to the bidders along the cycles so we will ignore them. Let x_1, \dots, x_z and y_1, \dots, y_z denote the start and end agents along these z alternating paths. Note that the same agent may appear multiple times amongst x_i 's or multiple times amongst y_i 's, but cannot appear both as an x_i and as a y_i (we can concatenate two such paths into one). Such an alternating path represents a shuffle of items between agents where agent x_j loses an item, and agent y_j gains an item when moving from M to M' . In general, these two items may be entirely different.

Assume there are no trading paths with respect to (M, P) . Then it must be the case that for every one of these z alternating paths either

- α . $v_{y_j} \leq v_{x_j}$ holds. Define $I = \{j | v_{y_j} \leq v_{x_j}\}$.
- β . $b_{y_j}^* < v_{x_j}$ holds (where $b_{y_j}^*$ is the budget left over for agent y_j at the end of the mechanism). Define $J = \{j | b_{y_j}^* < v_{x_j}\}$.

Now, no bidder is worse off in (M', P') (in comparison to (M, P)), and the auctioneer is not worse off, and, by assumption, either

- A. Some bidder is strictly better off. Or,
- B. The auctioneer is strictly better off.

First, we rule out case B above: Consider the process of changing (M, P) into (M', P') as a two stage process: at first, the agents x_1, \dots, x_z give up items. During this first stage, the payments made by these agents must decrease (in sum) by at least $Z^- = \sum_{i=1}^z v_{x_i}$. The 2nd stage is that agents y_1, \dots, y_z receive their extra items. In the 2nd stage, the maximum extra payment that can be received from agents y_1, \dots, y_z is no more than

$$Z^+ = \sum_{j \in I} v_{y_j} + \sum_{j \in J} b_{y_j}^* \leq \sum_{j \in I} v_{x_j} + \sum_{j \in J} v_{x_j} = Z^-, \quad (1)$$

by definition of sets I and J above. Thus, the total increase in revenue to the auctioneer is $Z^+ - Z^- \leq 0$. This rules

out case B. Moreover, as the auctioneer cannot be worse off, $Z^+ = Z^-$ and from Equation (1) we conclude that

$$\sum_{j \in I} v_{y_j} + \sum_{j \in J} b_{y_j}^* = \sum_{j \in I} v_{x_j} + \sum_{j \in J} v_{x_j}. \quad (2)$$

By definition, we have for $j \in I$, that $v_{y_j} \leq v_{x_j}$; and for $j \in J$, we have that $b_{y_j}^* < v_{x_j}$. Thus, if $J \neq \emptyset$ then the lefthand side of Equation (2) is strictly less than the righthand side, a contradiction.

Therefore, case A must hold and it must be that $J = \emptyset$, we will conclude the proof of the theorem by showing that these two are inconsistent. So, we have that

$$\begin{aligned} M'_a v_a - P'_a &= M_a v_a - P_a && \text{for agents } a \\ &&& \text{whose utility is unchanged} \\ M'_a v_a - P'_a &> M_a v_a - P_a && \text{for some agent } \hat{a} \\ \sum_a P'_a &= \sum_a P_a. \end{aligned}$$

We can now derive that

$$\begin{aligned} \sum_a M'_a v_a &> \sum_a M_a v_a - \left(\sum_a P'_a - \sum_a P_a \right) \\ &= \sum_a M_a v_a. \\ \Rightarrow \sum_a (M'_a - M_a) v_a &> 0. \end{aligned} \quad (3)$$

Now, whenever $a = x_j$ we decrease $M'_a - M_a$ by one, whenever $a = y_j$ we increase $M'_a - M_a$ by one. Thus, rewriting Equation (3) we get that

$$\begin{aligned} \sum_a (|\{j|a = y_j\}| - |\{j|a = x_j\}|) v_a &> 0 \\ \Rightarrow \sum_{j=1}^z v_{y_j} - \sum_{j=1}^z v_{x_j} &> 0 \\ \Rightarrow \sum_{j=1}^z v_{y_j} &> \sum_{j=1}^z v_{x_j}. \end{aligned} \quad (4)$$

But, Equation (4) is inconsistent with Equation (2) as $J = \emptyset$ implies that $I = \{1, \dots, z\}$. \square

The following technical lemma was required in the proof of Theorem 2.4 above; its proof is deferred to the full version of this paper.

Lemma 2.5 *Let M and M' be two allocations that allocate all items, then, the symmetric difference between these two allocations, $M \oplus M'$, can be decomposed into a set of simple alternating paths (with respect to M) and alternating cycles (also with respect to M) that are edge disjoint. Moreover, there are no two simple alternating paths such that one ends and the other begins at the same agent.*

The following comments will hopefully help in clarifying the notion of Pareto-optimality we use.

1. Pareto-optimality as given in Definition 2.1, allows any alternative allocation and pricing, even allowing positive transfers to the bidders. If Pareto-optimality was defined only with respect to any other allocation with no positive transfers, i.e., “such that $P'_i \geq 0$ for all i ”, then “Pareto-optimal” assignments could in fact contain trading paths.

2. Pareto-optimality as given in Definition 2.1 is a more desirable social goal than “Pareto-optimal” with no positive transfers. If we only insisted on “Pareto-optimal” assignments that forbid positive transfers then we could get assignments that are inefficient in the sense that, after the auction, bidders could trade amongst themselves and improve their utilities.

3. However, it may also be desirable that no agent actually get paid from the mechanism. Thus, it may be desirable that the actual allocation has no positive transfers ($P_i \geq 0$ for all i), yet at the same time is Pareto-optimal in the strong sense of Definition 2.1. By Theorem 2.4 after such allocation is presented, no agents will desire to trade amongst themselves. The existence of such allocation is the claim of Theorem 4.1 that applies to the auction we present in the next section.

3. DYNAMIC CLINCHING

In this section we describe our auction in detail. It can be viewed as an incentive compatible ascending auction (where incentive compatible means ex-post Nash) in the spirit of the mechanism proposed by [7] for the simpler case of multiunit auction. The ascending auction raises the price of unsold items until all items are clinched, i.e., allocated to the bidders. We describe the mechanism as a direct revelation mechanism and assume that the private value \tilde{v}_a is equal to the bid v_a .

We make extensive use of maximal B -matchings (on bipartite graphs) in the course of our auction. A B -matching [14] in a graph G is a subgraph of G where vertices have degree constraints, i.e. $B_v \geq 0$ is an upper bound on the degree of v in the B -matching. Given the vector B , and a graph $G = (V, E)$ one can compute a maximal weight B -matching in polynomial time. Often in the paper we denote such B -matchings simply by *matchings*.

The main details of the auction are described by Algorithms 1, 2 and 3.

Throughout the algorithm there is always some current price p (initially zero); a set of unsold items U ; current number of unsold items, m (initially equal to total number of items); and current remaining budgets $b = (b_1, b_2, \dots, b_n)$, where b_a is the remaining budget for agent $1 \leq a \leq n$. We also denote by d_i the current demand of agent i . The auction repeatedly computes a B -matching of unsold items, i.e., a maximal matching that assigns to agent i at most d_i items from $U \cap S_i$.

A key tool used in our auction is that of S -avoid matchings. These are maximal matchings that try to avoid, if at all possible, assigning any items to bidders in some set S . Finding such a matching requires a min-cost max-flow computation, where there is high cost to direct flow through a vertex of S . The mechanism will sell items only when the S -avoid matching will still assign items to agents of S . All items assigned to agents of S will actually be sold to agents of S .

The algorithm also keeps a set of active agents A — those with current demand greater than zero. Not all active agents are in the same position with respect to the auction. The auction will distinguish a set of *value limited agents* V —

those with valuation equal to the current price:

$$A = \{1 \leq a \leq n | d_a > 0\}, \quad (5)$$

$$V = \{1 \leq a \leq n | d_a > 0, v_a = p\}. \quad (6)$$

For every agent $1 \leq i \leq n$, the algorithm makes use of values D_i , D_i^+ , and d_i , these values are functions of the current values of p , m , b . D_i is equal to the number of items that agent i is interested in purchasing at current p , m , and b (Equation 7). Later on, we omit these arguments in the description of the algorithm. In Equation 8 we define D_i^+ , which is equal to the number of items that agent a_i would be interested in purchasing if the price were increased by an infinitesimally small amount, thus $D_i^+ \leq D_i$. In addition, the algorithm maintains a vector of variables $d = (d_1, d_2, \dots, d_n)$. The current demand d_i of agent i will either be equal to D_i or to D_i^+ depending on the state of the execution.

Formally:

$$D_i[p, m, b] = \begin{cases} \min\{m, |S_i|, \lfloor b_i/p \rfloor\} & \text{if } p \leq v_i \\ 0 & \text{if } p > v_i \end{cases} \quad (7)$$

$$D_i^+[p, m, b] = \lim_{\epsilon \rightarrow 0^+} D_i[p + \epsilon, m, b]; \quad (8)$$

Observe that the algorithm repeatedly tries to sell items at the current level of demand for all the players. When no items can be sold then the demand of some active bidder a is reduced from D_a to D_a^+ , for . This will reflect in the decrease by 1 of the demand of bidder a . When neither action can be done, the price increases. A step through execution of this mechanism, at the critical points, for the instance described in Figure 1, appears in Figure 3.

In general, the auction prefers to sell items only at the last possible moment (alternately phrased, the highest possible price) at which this item can still be sold while still preserving incentive compatibility. We will prove that the auction will in fact sell all items (Lemma 4.2).

Once a price has been updated, the auction checks to see if it must sell items to value limited bidders. Such bidders will receive no real benefit from the item (their valuation is equal to their payment), but this is important so as to increase the utility of the auctioneer. Our definition of Pareto-optimality includes all bidders and the auctioneer. To check if this is necessary, the auction computes a V -avoid matching, trying to avoid the bidders in V . If this cannot be done, then items are sold to these V bidders. After items are sold to value limited bidders, these bidders effectively disappear and we are only left to consider selling to active bidders. The main loop of the mechanism checks whether any items must be sold to any of the currently active bidders. This is where incentive compatibility comes into play. The auction sells an item to some bidder, a , at the lowest price where the remaining bidders total demand is such that an item can be assigned to a without creating a shortage. Again, this makes use of the $\{a\}$ -avoid matching, if in the $\{a\}$ -avoid matching some item is matched to a then a must be sold that item. If no item can be sold than the demand d_i of a bidder with $d_i > D^+(i)$ is reduced by setting $d_i = D^+(i)$. When neither action can be done, the price increases.

4. PARETO-OPTIMALITY

The goal of this section is to show that the final allocation produced by Algorithm 1 sells all items and contains no trading paths, thus proving the following theorem:

Algorithm 1 Combinatorial Auction with Budgets

```

1: procedure COMBINATORIAL AUCTION WITH BUDGETS( $v, b, \{S_i\}$ )
   Implicitly defined  $D_a, D_a^+, A$ , and  $V$  — see Equations (7), (8), (5) and (6).
    $B(\neg\{a\})$  - number of items assigned to agents in  $A \setminus \{a\}$ 
   in  $\{a\}$ -avoid matching
2:    $p \leftarrow 0$ 
3:   while ( $A \neq \emptyset$ ) do
4:     Sell( $V$ )
5:      $A \leftarrow A - V$ 
6:     repeat
7:       if  $\exists i | B(\neg\{i\}) < m$  then Sell( $i$ )
8:       else
9:         For arbitrarily agent  $i$  with  $d_i > D_i^+(p)$ :
10:           $d_i \leftarrow D_i^+(p)$ 
11:        end if
12:      until  $\forall i: (d_i = D^+(i))$  and  $B(\neg\{i\}) \geq m$ 
13:      Increase  $p$  until for some  $i$ ,  $D_i(p) \neq D_i^+(p)$ 
14:    end while
15: end procedure

```

Algorithm 2 Computing an avoid matching, can be done via min cost max flow

```

1: procedure  $S$ -AVOID MATCHING
   Construct interest graph  $G$ :
   • Active agents,  $A$ , on left, capacity constraint of agent  $a \in A = d_a$ 
   • Unsold items,  $U$ , on right, capacity constraint 1.
   • Edge  $(a, t)$  from agent  $a \in A$  to unsold item  $t \in U$  iff  $t \in S_a$ .
   Return maximal  $B$ -matching with minimal number of items assigned to agents in  $S$ , amongst all maximal  $B$ -matchings.
2: end procedure

```

Algorithm 3 Selling to a set S

```

1: procedure SELL( $S$ )
2:   repeat
3:     Compute  $Y = S$ -AVOID MATCHING
4:     For arbitrary  $(a, t)$  in  $Y$ ,  $a \in S$ , sell item  $t$  to agent  $a$ .
5:   until  $B(\neg S) \geq m$ 
6: end procedure

```

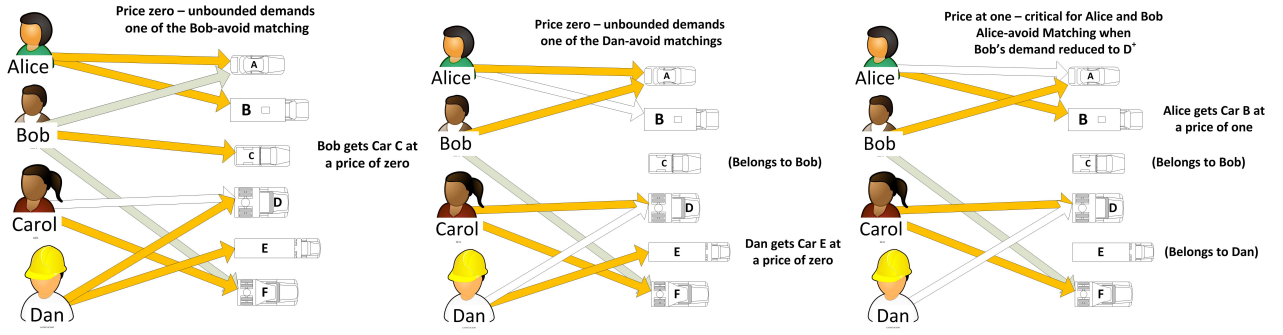


Figure 3: Significant points during mechanism execution, input from Figure 1. Example continues in Figure 4.

Theorem 4.1 *The allocation (M^*, P^*) produced by Algorithm 1 is Pareto-optimal. Moreover, the mechanism makes no positive transfers.*

We first show that the auction sells all items.

Lemma 4.2 *If every item appears in $\cup_{i=1}^n S_i$ then the auction will sell all items.*

PROOF. We prove that throughout the auction, there is always a matching that can sell all remaining items at the current price without exceeding the budget of any agent. As prices only increase, eventually all items must be sold. The lines below refer to Algorithm 1 unless stated otherwise.

Initially, all items can be sold at price zero. The d_a capacity constraints are all equal to $|S_a|$.

Furthermore, we argue that it is always true that all unsold items can be sold to active agents at the current price without violating the capacity constraints. We prove this invariant by case analysis of the following events:

- Increase in price followed by setting $d_a = D_a$ for each active bidder $a \in A$: The repeat loop in lines 6 – 12 ends with all the d_a 's set to D_a^+ and $B(\neg\{a\}) \geq m$ for all agents a .

Any increment in price in line 13 will set D_a equal to the previous D_a^+ , which means that the new d_a 's are equal to the old ones. Thus, any matching valid at the old price is valid at the new price.

- The $\text{Sell}(V)$ operation (line 4 of Algorithm 1, Algorithm 3) sells items to agents in V only if all other unsold items can be matched to agents not in V .
- Setting $A \leftarrow A - V$ (line 5 of Algorithm 1) is OK because nothing is sold to $a \in V$ at any higher price.
- The $\text{Sell}(a)$ operation (line 7 of Algorithm 1, Algorithm 3) sells items to agent a only if all other unsold items can be matched to other agents.
- Setting $d_i \leftarrow D_i^+(p)$ (line 10) is done only if $B(\neg\{a\}) \geq m$, i.e., all unsold items can be matched to the other agents (not including a).

Thus, the mechanism will sell all items. \square

As a warm up for the proof for the single-valued combinatorial auction case we show a short proof of the theorem in the multiunit case, i.e., identical preference sets for all players.

Theorem 4.3 *The allocation (M^*, P^*) produced by Algorithm 1 is Pareto-optimal for Multi-unit auction with budgets. Moreover, the mechanism makes no positive transfers.*

PROOF. The proof is by contradiction. Assume a final allocation with any two bidders i, j , such that $v_j > v_i$, $b_j \geq v_i$, and bidder i allocated with at least one item. Denote by p the price paid for the last item allocated to bidder i from a call to $\text{Sell}(V)$ ($i \in V$) or $\text{Sell}(i|d(A/i) < m)$ ($i \notin V$). Denote by M_k the number of items allocated to agent k at any time after the last item was allocated to i .

We distinguish two cases:

1. Agent $i \in V$ when it receives the item. Observe $j \notin V$. We derive a contradiction to the allocation of the item to i from the following:

$$\begin{aligned} m &= \# \text{ items to be sold to } V + \sum_{k \in A/\{V \cup j\}} M_k + M_j \\ &< \# \text{ items to be sold to } V + \sum_{k \in A/\{V \cup j\}} D_k + D_j. \end{aligned}$$

The inequality stems from $D_j \geq M_j + 1$, since $b_j \geq v_i = p$ at the end of the auction and all M_j items have been sold to bidder j at price p or higher, and of course $\forall k, D_k \geq M_k$.

2. Agent $i \notin V$ when it receives the item. We derive a similar contradiction from the following:

$$\begin{aligned} m &= \# \text{ items to be sold to } i + \sum_{k \in A/\{i \cup j\}} M_k + M_j \\ &< \# \text{ items to be sold to } i + \sum_{k \in A/\{i \cup j\}} d_k + d_j. \end{aligned}$$

The inequality follows from $d_j \geq M_j + 1$, since from $i \notin V$ and from the fact that all M_j items have been allocated to bidder j at price p or higher we derive $b_j \geq v_i > p$, and of course $\forall k, d_k \geq M_k$.

\square

We next give the proof of the Theorem 4.1 for agents with different preference sets. We need to show that there are no trading paths in the final allocation (M^*, P^*) produced by Algorithm 1.

Consider the set of all trading paths Π in the final allocation M^* .

Definition 4.4 We define the following for every $\pi \in \Pi$:

- Let Y^π be the S -avoid matching used the first time some item t is sold to some agent a where (a, t) is an edge along π . Y^π is either a V -avoid matching (line 4 of Algorithm 1) or an a -avoid matching for some agent-item edge (a, t) along π (line 7 of Algorithm 1).
- If Y^π is a V -avoid matching, let V^π be this set of value limited agents.
- If Y^π is an a -avoid matching, let a^π be this agent.
- Let $F^\pi \subset M^*$ be the set of edges (a, t) such that item t was sold to agent a at or subsequent to the first time that some item t' was sold to some agent a' for some edge $(a', t') \in \pi$ ((a', t') is itself in F^π).
- Let m^π be the number of unsold items just before the first time some edge along π was sold. I.e., m^π is equal to the number of items matched in F^π .
- Let p^π be the price at which item[s] were sold from Y^π .
- Let b_a^π be the remaining budget for agent a before any items are sold in $\text{Sell}(V^\pi)$ or $\text{Sell}(a^\pi)$.

We partition Π into two classes of trading paths:

1. Π_V is the set of trading paths such that $\pi \in \Pi_V$ iff Y^π is some V^π -avoid matching used in $\text{Sell}(V^\pi)$ (line 4 of Algorithm 1).
2. Π_{-V} is the set set of trading paths such that $\pi \in \Pi_{-V}$ iff Y^π is some a^π -avoid matching used in $\text{Sell}(a^\pi)$ (line 7 of Algorithm 1).

Lemma 4.5 $\Pi_V = \emptyset$.

PROOF. We need the following Claim:

Claim 4.6 Let $\pi = (a_1, t_2, \dots, a_{j-1}, t_{j-1}, a_j) \in \Pi_V$ be a trading path, and let (a_i, t_i) be the last edge belonging to Y^π along π . Then the suffix of π starting at a_i , (a_i, t_i, \dots, a_j) , is itself a trading path.

PROOF. This trivially follows as the valuation of a_i is equal to current price when $\text{Sell}(V^\pi)$ was done (p^π), and the valuation of a_1 is $\geq p^\pi$ as edge (a_1, t_1) was unsold prior to this $\text{Sell}(V^\pi)$ and does belong to the final F^π . \square

From the Claim above we may assume, without loss of generality, that if $\Pi_V \neq \emptyset$ then $\exists \pi \in \Pi_V$ such that the first edge along π was also the first edge sold amongst all edges of π , furthermore, all subsequent edges do not belong to Y^π .

As agents $a \in V^\pi$ will not be sold any further items after this $\text{Sell}(V^\pi)$, the items assigned to a_1 in Y^π are the same items assigned to a_1 in F^π .

We seek a contradiction to the assumption that Y^π was a V^π -avoid matching. Note that the matching F^π is a V^π -avoid matching by itself, because exactly the items assigned to V -type agents in Y^π are sold. We now show how to

construct from F^π another matching that assigns less items to V -type agents.

We show that the number of items assigned to agent a_1 in F^π (which is the same as in Y^π) can be reduced by one by giving agent a_{k+1} item t_k for $k = 1, \dots, j-1$. This is also a full matching but it remains to show that this does not exceed the capacity constraints for agent a_j , d_{a_j} .

As $d_{a_j} = D_{a_j}$ for all $a \in A$ when $\text{Sell}(V^\pi)$ is done, agent a_j has remaining budget $\geq v_1$ at the conclusion of the auction, and all items assigned to agent a_j in F^π are at price $\geq p^\pi = v_1$. This implies that at the time of $\text{Sell}(V^\pi)$ we have $D_{a_j} >$ the number of items assigned to a_j in F^π . Thus, we can increase the number of items allocated to a_j by one without exceeding the demand constraint $d_{a_j} = D_{a_j}$.

Now, note that a_j is not V -type agent, so the new matching constructed assigns less items to V type agents then the matching F^π . Hence, F^π is not an V^π -avoid matching, and in turn neither Y^π is V^π -avoid matching.

We've shown that $\Pi_V = \emptyset$. It remains to show that $\Pi_{-V} = \emptyset$.

Assume $\Pi_{-V} \neq \emptyset$. Order $\pi \in \Pi_{-V}$ by the first time at which some edge along π was sold. We know that this occurs within some $\text{Sell}(a^\pi)$ for some a^π and that $a^\pi \notin V$. Let us define $\pi = (a_1, t_1, a_2, t_2, \dots, a_{j-1}, t_{j-1}, a_j)$ be the last path in this order, and let $e = (a^\pi, t^\pi) = (a_i, t_i)$.

Recall that Y^π is the a^π -avoid matching used when item t^π was sold to agent a^π . Also, $F^\pi \subset M^*$ is the set of edges added to M^* in the course of the auction from this point on (including the current $\text{Sell}(a_i)$).

Lemma 4.7 Let π , $a^\pi = a_i$, $t^\pi = t_i$, be as above, we argue that when Y^π was computed as an a^π -avoid matching there was another full matching X with the following properties:

1. The suffix of π from a_i to a_j :

$$\pi[a_i, \dots, a_j] = (a_i, t_i, a_{i+1}, t_{i+1}, \dots, a_{j-1}, t_{j-1}, a_j),$$
is an alternating path with respect to X . (I.e., edges (a_k, t_k) , $i \leq k \leq j-1$, belong to X).
2. The number of items assigned to a_i in X is equal to the number of items assigned to a_i in Y^π .
3. The number of items assigned to a_j in X is equal to the number of items assigned to a_j in F^π .

The proof will be given in the full version of this paper.

Corollary 4.8 $\Pi_{-V} = \emptyset$.

PROOF. Assume $\pi \in \Pi_{-V} \neq \emptyset$ and let $a^\pi = a_i$, $t^\pi = t_i$, we now seek to derive a contradiction as follows:

- When Y^π was computed there was also an alternate full matching Y' with fewer items assigned to agent a_i , contradicting the assumption that Y^π is an a_i avoid matching. Or,
- We show that the remaining budget of agent a_j at the end of the auction, $b_{a_j}^*$, has $b_{a_j}^* < v_1$, contradicting the assumption that π is a trading path.

Let X be a matching as in Lemma 4.7 and F^π be as defined in Definition 4.4. Also, let $X(a)$, $F^\pi(a)$, be the

number of items assigned to agent a in full matchings X , F^π , respectively.

We consider the following cases regarding d_{a_j} when Y^π , the a_i -avoid matching, was computed:

1. $d_{a_j} > X(a_j)$: then, like in Lemma 4.5, we can decrease the number of items sold to a_i by assigning item t_k to agent a_{k+1} for $k = i, \dots, j-1$, without exceeding the d_{a_j} demand constraint.
2. $d_{a_j} = X(a_j)$, by subcase analysis we show that $b_{a_j}^\pi \leq (X(a_j) + 1)p^\pi$:

- (a) $D_{a_j} = D_{a_j}^+$: Observe that $X(a_j) < m$, the current number of unsold items. This follows because $X(a_i) = Y^\pi(a_i) \geq 1$ by assumption that t_i was assigned to a_i in Y^π . This means that $d_{a_j} = X(a_j) < m$ so

$$\begin{aligned} X(a_j) &= d_{a_j} = \lfloor b_{a_j}^\pi / p^\pi \rfloor > b_{a_j}^\pi / p^\pi - 1 \\ \Rightarrow b_{a_j}^\pi &< (X(a_j) + 1)p^\pi. \end{aligned}$$

- (b) $D_{a_j} \neq D_{a_j}^+$: Observe that $a_j \notin V$ as $v_{a_j} > v_{a_i}$ and $a_i \notin V$. As $a_j \notin V$, the only reason that $D_{a_j} \neq D_{a_j}^+$ is because the remaining budget of agent a_j , $b_{a_j}^\pi$, is an integer multiple of the current price p^π . Then, $D_{a_j}^+ = D_{a_j} - 1$ and $D_{a_j} = \lfloor b_{a_j}^\pi / p^\pi \rfloor = b_{a_j}^\pi / p^\pi$, it follows that

$$\begin{aligned} X(a_j) &= d_{a_j} \geq D_{a_j}^+ = D_{a_j} - 1 = b_{a_j}^\pi / p^\pi - 1 \\ \Rightarrow b_{a_j}^\pi &\leq (X(a_j) + 1)p^\pi. \end{aligned}$$

Note that the current price $p^\pi < v_{a_i}$ because we assume that a_i was sold t_i as a result of $\text{Sell}(a_i)$ and not $\text{Sell}(V)$. It is also true that $v_{a_i} \leq v_{a_1}$ as (a_i, t_i) was the first edge that was sold along π . By condition 3 of Lemma 4.7 we can deduce that

$$b_{a_j}^\pi \leq (X(a_j) + 1)p^\pi = (F^\pi(a_j) + 1)p^\pi.$$

Agent a_j is sold exactly $F^\pi(a_j)$ items at a price not lower than p^π , to at the end of the auction the remaining budget for agent a_j , $b_{a_j}^*$, is $\leq p^\pi$. This contradicts the assumption that π is a trading path since

$$b_{a_j}^* \leq p^\pi < v_{a_i} \leq v_{a_1}.$$

□

5. MAPPING THE FRONTIER

In this paper we gave a mechanism that is incentive compatible with respect to valuation, and produces a Pareto-optimal allocation, but with various restrictions and assumptions:

- public budgets;
- public sets of interest;
- moreover, agents are restricted to have a step function valuation for items, if the item is in S_i then its valuation is v_i , otherwise zero.

This poses the question: can we remove these assumptions/restrictions? Just how far can we go?

As for private budgets, it was shown by [7] that even for the multi unit case, one cannot achieve incentive compatibility with respect to valuation along with bidder rationality, auctioneer rationality, and obtain a Pareto-optimal allocation.

We argue that even if one assumes public budgets, the other restrictions are also necessary. This is summarized in the following theorems:

Theorem 5.1 *There is no truthful, bidder rational, auctioneer rational and Pareto-optimal auction with public budgets, b_a , private valuations, v_a , and private sets of interest, S_a .*

PROOF. We assume the step function valuations (as done throughout this paper). We also say that an agent wins an item if the item is assigned to the agent.

For the proof of the theorem consider two agents, 1 and 2, and two items t_1, t_2 . Let $S_1 = \{t_1\}$ and $S_2 = \{t_1, t_2\}$.

Recall the uniqueness result of [7]:

Theorem 5.2 (Theorem 5.1 of [7]) *Let A be a truthful, bidder-rational, auctioneer rational, and Pareto-optimal unit auction (identical items) with 2 players with known (public) budgets b_1, b_2 that are generic⁴ then if $v_1 \neq v_2$ the allocation produced by A is identical to that produced by the Dynamic clinching auction of [7] (and, in particular, with our auction when applied to these inputs).*

For all the details of the proof please see [8], as the original publication [7] includes only a sketch.

The next two lemmas allow us to argue about the payments of the auction. The proofs will be given in the full version of this paper.

Lemma 5.3 *Consider any Pareto-optimal, incentive compatible, bidder rational and auctioneer rational combinatorial auction that produces an allocation (M, P) : if agent 2 wins both items then the payment P_1 by agent 1 is zero.*

Lemma 5.4 *Consider any Pareto-optimal, incentive compatible, bidder rational and auctioneer rational combinatorial auction that produces an allocation (M, P) : if agent 2 does not win item t_1 then $P_2 = 0$.*

The next lemma will tell us which matching is chosen. The proof will be given in the full version of this paper.

Lemma 5.5 *If $b_1 < b_2$, $v_1 < v_2$, and $v_1 < v_2$, then any incentive compatible, Pareto-optimal, bidder rational and auctioneer rational combinatorial auction has to assign both items to agent 2.*

We are now ready to prove the main result of this section.

Consider the case of two agents, 1 and 2, and two items t_1, t_2 . Let $S_1 = \{t_1, t_2\}$ and $S_2 = \{t_1, t_2\}$. Additionally, Fix $v_1 = 10$, $v_2 = 11$, $b_1 = 4$ and $b_2 = 5$. In this case, by Theorem 5.2, the allocation must coincide with the result of the dynamic clinching auction of [7].

I.e., both agents get one of the two items, $p_1 = 3$, and $p_2 = 2$. Without loss of generality assume that item t_1

⁴Not all pairs of values are generic, but for our purposes assume that this holds for every such pair.

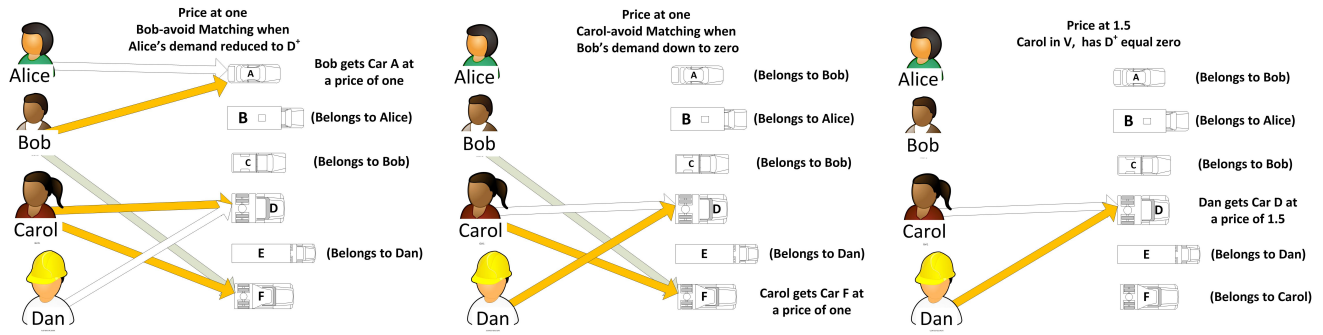


Figure 4: Significant points during mechanism execution, input from Figure 1. Sales at cost > 0 .

is assigned to agent 1 with probability at least $\frac{1}{2}$ (if the mechanism is randomized).

Now, assume that the true set of interest for agent 1 was in fact $S_1 = \{t_1\}$. We argue that agent 1 now has incentive to lie about S_1 :

- if agent 1 reports her true set of interest – then by Lemma 5.5 both items end up assigned to agent 2, and by Lemma 5.3 $P_1 = 0$, so her utility is zero as well;
- if agent 1 lies and reports $\{t_1, t_2\}$ as her set of interest – then with probability $\leq \frac{1}{2}$ her utility is equal to $0 - 3$, and with probability at least $\frac{1}{2}$ her utility is equal to $10 - 3 = 7$, so on average his utility is at least $-3 \cdot \frac{1}{2} + 7 \cdot \frac{1}{2} = 2$.

This concludes the proof as agent 1 has incentive to lie in any incentive compatible, Pareto-optimal, bidder rational and auctioneer rational combinatorial auction. \square

Corollary 5.6 *There is no truthful, bidder rational, auctioneer rational and Pareto-optimal auction with public budgets, b_a , and private item-dependent valuations v_{at} .*

PROOF. This follows immediately from Theorem 5.1. Consider the case where the private valuations v_{at} are zero for any $t \notin S_a$, and v_a for $t \in S_a$. \square

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