SIAM J. COMPUT.
Vol. 14, No. 1, February 1985

006

VORONOI DIAGRAM IN THE LAGUERRE GEOMETRY AND ITS APPLICATIONS*

HIROSHI IMAI†, MASAO IRI† AND KAZUO MUROTA†

Abstract. We extend the concept of Voronoi diagram in the ordinary Euclidean geometry for n points to the one in the Laguerre geometry for n circles in the plane, where the distance between a circle and a point is defined by the length of the tangent line, and show that there is an $O(n \log n)$ algorithm for this extended case. The Voronoi diagram in the Laguerre geometry may be applied to solving effectively a number of geometrical problems such as those of determining whether or not a point belongs to the union of n circles, of finding the connected components of n circles, and of finding the contour of the union of n circles. As in the case with ordinary Voronoi diagrams, the algorithms proposed here for those problems are optimal to within a constant factor. Some extensions of the problem and the algorithm from different viewpoints are also suggested.

Key words. Voronoi diagram, computational geometry, Laguerre geometry, computational complexity, divide-and-conquer, Gershgorin's theorem

Introduction. The Voronoi diagram for a set of n points in the Euclidean plane is one of the most interesting and useful subjects in computational geometry. Shamos and Hoey [15] presented an algorithm which constructs the Voronoi diagram in the Euclidean plane in $O(n \log n)$ time by using the divide-and-conquer technique, and showed many useful applications. Since then, various generalizations of the Voronoi diagram have been considered. Hwang [6] and Lee and Wong [10] considered the Voronoi diagrams for a set of n points under the L_1-metric, and the L_1- and L_∞-metrics, respectively, and gave $O(n \log n)$ algorithms to compute them. Lee and Drysdale [9] studied the Voronoi diagrams for a set of n objects such as line segments or circles, where the distance between a point and an object is defined as the least Euclidean distance from the point to any point of the object, and therefore the edges of these Voronoi diagrams are no longer simple straight line segments but may contain fragments of parabolic or hyperbolic curves. They gave an $O(n(\log n)^2)$ algorithm to construct these diagrams, and Kirkpatrick [7] reduced its complexity to $O(n \log n)$.

Here we extend the concept of usual Voronoi diagram in the Euclidean geometry for n points to the one in the Laguerre geometry for n circles in the plane, where the distance from a point to a circle is defined by the length of the tangent line. Then the edges of these extended diagrams are simple straight line segments which are easy to manipulate. We show that there is an $O(n \log n)$ algorithm for this extended case.

In spite of the unusual distance employed here, the Voronoi diagram in the Laguerre geometry can be applied to solving efficiently a number of geometric problems concerning circles. By using this extended Voronoi diagram, the problem of determining whether or not a point belongs to the union of given n circles can be solved in $O(\log n)$ time and $O(n)$ space with $O(n \log n)$ preprocessing. We can also solve the problem of finding the connected components of given n circles in $O(n \log n)$ time, which can be applied to a problem in numerical analysis, namely, estimating the region where the eigenvalues of a given matrix lie [4]. The problem of finding the contour of the union of n circles can also be solved in $O(n \log n)$ time, which can be applied to image processing and computer graphics. As in the case of the problems connected with the

* Received by the editors December 15, 1981, and in final revised form August 20, 1983.

† Department of Mathematical Engineering and Instrumentation Physics, Faculty of Engineering, University of Tokyo, Tokyo, Japan 113.

ordinary Voronoi diagram, the methods proposed here are optimal to within a constant factor.

Some further generalizations of the problems and the algorithms from different viewpoints are also suggested.

1. Laguerre geometry. Consider the three-dimensional real vector space $\mathbf{R}^3$ where the distance $d(P, Q)$ between two points $P=(x_1, y_1, z_1)$ and $Q=(x_2, y_2, z_2)$ is defined by $d^2(P, Q)=(x_1-x_2)^2+(y_1-y_2)^2-(z_1-z_2)^2$. In the Laguerre geometry [1], a point (x, y, z) in this space $\mathbf{R}^3$ is made to correspond to a directed circle in the Euclidean plane with center (x, y) and radius $|z|$, the circle being endowed with the direction of revolution corresponding to the sign of z. Then the distance between two points in $\mathbf{R}^3$ corresponds to the length of the common tangent of the corresponding two circles. Hereafter we consider the plane with distance so defined. Note here that, so long as the distance $d_L(C_i, P)$ between a circle $C_i=C_i(Q_i; r_i)$ with center $Q_i=(x_i, y_i)$ and radius r_i and a point $P=(x, y)$ is concerned, the direction of the circle has no meaning since the distance $d_L(C_i, P)$ is expressed as

$$d_L^2(C_i, P)=(x-x_i)^2+(y-y_i)^2-r_i^2, \tag{1}$$

$d_L(C_i, P)$ being the length of the tangent segment from P to C_i if P is outside of C_i. Note that, according as a point P lies in the interior of, on the periphery of, or in the exterior of circle C_i, $d_L^2(C_i, P)$ is negative, zero, or positive, respectively. The locus of the points equidistant from two circles C_i and C_j is a straight line, called the *radical axis* of C_i and C_j, which is perpendicular to the line connecting the two centers of C_i and C_j. If two circles intersect, their radical axis is the line connecting the two points of intersection. Typical types of radical axes are illustrated in Fig. 1. If the three centers

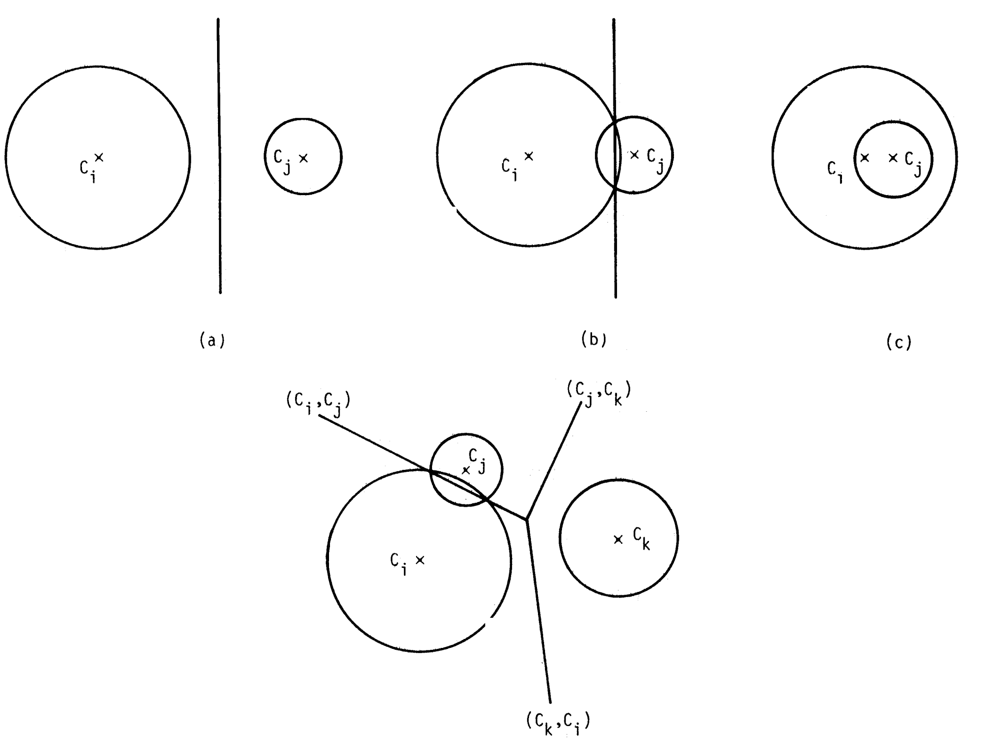

FIG. 1. *Radical axes and radical centers.*

of three circles C_i, C_j and C_k are not on a line, the three radical axes among C_i, C_j and C_k meet at a point, which is called the *radical center* of C_i, C_j and C_k (see Fig. 1(d)).

2. Definition of the Voronoi diagram in the Laguerre geometry. Suppose n circles $C_i = C_i(Q_i; r_i)$ $(Q_i = (x_i, y_i))$ are given in the plane, where the distance between a circle C_i and a point P is defined by $d_L(C_i, P)$ as in § 1. Then the *Voronoi polygon* $V(C_i)$ for circle C_i is defined by

$$V(C_i) = \bigcap_j \{P \in \mathbf{R}^2 \mid d_L^2(C_i, P) \leqq d_L^2(C_j, P)\}. \tag{2}$$

Note that the inequality $d_L^2(C_i, P) \leqq d_L^2(C_j, P)$ determines a half-plane so that $V(C_i)$ is convex. However, note also that $V(C_i)$ may be empty and that C_i may not intersect its polygon $V(C_i)$ when circle C_i is contained in the union of the other circles. The Voronoi polygons for n circles C_i $(i = 1, \cdots, n)$ partition the whole plane, which we shall refer to as the *Voronoi diagram in the Laguerre geometry* (see Fig. 2). A corner of a Voronoi polygon is called a *Voronoi point*, and a boundary edge of the Voronoi polygon is called a *Voronoi edge*. Furthermore, a circle whose corresponding Voronoi polygon is nonempty (empty) is referred to as a *substantial* (*trivial*) *circle*. In Fig. 2, circle C_3 is trivial and all the others are substantial. It is also seen that, in Fig. 2, circle C_2 has no intersection with $V(C_2)$. A circle that intersects the corresponding Voronoi polygon is said to be *proper*, and a circle which is not proper is called *improper*. The following is immediate from the above definitions.

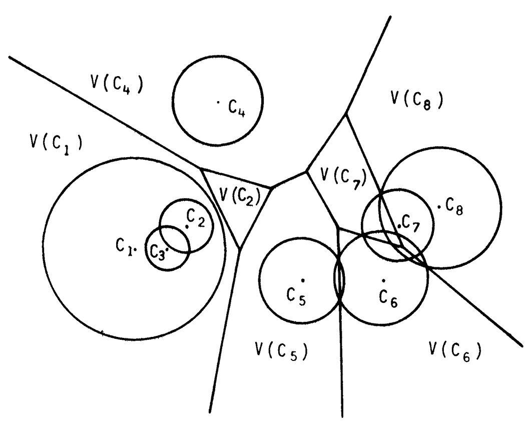

FIG. 2. *Voronoi diagram in the Laguerre geometry.*

LEMMA 1. (i) *A trivial circle is necessarily improper.*

(ii) *An improper circle is contained in the union of the proper circles.*

Obviously, if $r_i = 0$ for all i, the Voronoi diagram in the Laguerre geometry reduces to that in the ordinary Euclidean geometry.

In a Voronoi diagram in the Laguerre geometry, a Voronoi edge is (part of) a radical axis and a Voronoi point is a radical center. Since the diagram is planar, and Euler's formula [5] still holds, we have

LEMMA 2. *There are $O(n)$ Voronoi edges and points in the Voronoi diagram in the Laguerre geometry for n circles.*

In the case of the Voronoi diagram in the ordinary Euclidean geometry for n points P_i $(i = 1, \cdots, n)$, the Voronoi polygon $V(P_j)$ is unbounded iff point P_j is on the boundary of the convex hull of the n points P_i, but, for the Voronoi diagram in the Laguerre geometry for n circles C_i with center Q_i, this statement needs some modification, as in Lemma 3 below. In Fig. 3, the center Q_2 of C_2 lies on the boundary of the convex hull of the centers, but $V(C_2)$ is empty.

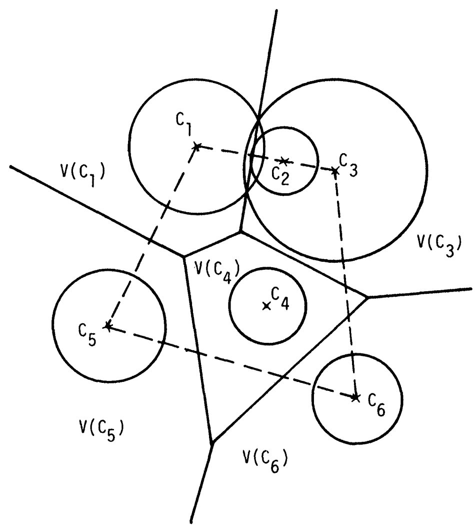

FIG. 3. *Relations between the convex hull and Voronoi polygons.*

LEMMA 3. *In the Voronoi diagram in the Laguerre geometry, the Voronoi polygon $V(C_i)$ is nonempty and unbounded if the center Q_i of the circle C_i is at a corner of the convex hull of the centers $Q_i, \cdots, Q_n$. Furthermore, if the center Q_j of a circle C_j is on the boundary of this convex hull but not at a corner, its Voronoi polygon $V(C_j)$ is either unbounded or empty. If the center Q_k of a circle C_k is not on the boundary of this convex hull, its Voronoi polygon $V(C_k)$ is either bounded or empty.*

Proof. Consider the Voronoi diagram in the Laguerre geometry for n circles $C_i(Q_i; r_i)$ $(Q_i=(x_i, y_i);\ i=1, \cdots, n)$, where we can assume $y_i \neq y_j$ $(i \neq j)$ without loss of generality. First recall (cf. (1), (2)) that a point $P=(x, y)$ belongs to $V(C_1)$ iff

$$d_L^2(C_1, P) \leqq d_L^2(C_i, P), \qquad i=1, \cdots, n,$$

i.e.,

$$(x_i - x_1)x + (y_i - y_1)y \leqq R_i, \qquad i=1, \cdots, n, \tag{3}$$

where

$$R_i = (x_i^2 + y_i^2 - r_i^2 - x_1^2 - y_1^2 + r_1^2)/2.$$

Next, note that the center Q_1 of C_1 lies on the boundary (including the corners) of the convex hull of $\{Q_i \mid i=1, \cdots, n\}$ iff

$$\exists(\alpha, \beta)(\neq(0,0)):\ \ \alpha(x_i - x_1) + \beta(y_i - y_1) \leqq 0, \qquad i=1, \cdots, n, \tag{4}$$

since all the centers Q_i $(i=2, \cdots, n)$ lie on one side with respect to a line passing through (x_1, y_1).

Suppose that $V(C_1) \neq \emptyset$ and $(x_0, y_0) \in V(C_1)$. Then, $V(C_1)$ $(\neq \emptyset)$ is unbounded iff a half line starting from (x_0, y_0) is contained in $V(C_1)$, i.e.,

$$\exists(\alpha, \beta)(\neq(0,0)), \forall M(>0):\ \ (x, y) = (x_0 + M\alpha,\ y_0 + M\beta) \text{ satisfies (3)},$$

which is easily seen to be equivalent to (4) above, so that $V(C_1)$ ($\neq\emptyset$) is unbounded iff the center Q_1 of C_1 lies on the boundary of the convex hull.

When the center Q_1 lies at a corner of the convex hull, there exist two distinct pairs of (α, β), say, (α_1, β_1) and (α_2, β_2) such that (4) holds and that the vectors $(x_i - x_1, y_i - y_1)$ $(i = 2, \cdots, n)$ can be represented as linear combinations of (α_1, β_1) and (α_2, β_2) with nonpositive coefficients one of which is strictly negative. The assertion that $V(C_1) \neq \emptyset$ easily follows from the fact that (3) holds for $(x, y) = (M\alpha, M\beta)$ with a sufficiently large $M(>0)$, where $(\alpha, \beta) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2)$. □

3. Construction of the Voronoi diagram in the Laguerre geometry. We shall show that the Voronoi diagram in the Laguerre geometry can be constructed in $O(n \log n)$ time. The algorithm is based on the divide-and-conquer technique, which is very much like the one proposed initially by Shamos and Hoey [15] in constructing the Voronoi diagram in the ordinary Euclidean geometry for n points, but which is different in some essential points. We shall briefly review Shamos and Hoey's algorithm first, and then explain the difference.

Shamos and Hoey's algorithm works as follows. For a given set $S = \{P_1, P_2, \cdots, P_n\}$ of n distinct points, we sort them lexicographically by their (x, y)-coordinates with the x-coordinate as the first key. Then, renumbering the indices of the points in that order, we divide S into two subsets $L = \{P_1, P_2, \cdots, P_{[n/2]}\}$ and $R = \{P_{[n/2]+1}, \cdots, P_n\}$. We recursively construct the Voronoi diagrams $V(L)$ and $V(R)$ for points in L and R, respectively, and merge $V(L)$ and $V(R)$. If we can merge $V(L)$ and $V(R)$ in $O(n)$ time, the Voronoi diagram $V(S)$ can be computed in $O(n \log n)$ time.

By virtue of the manner of partitioning S into L and R, there exists a unique unicursal polygonal line, called the *dividing* (*polygonal*) *line*, such that every point to the left [right] of it is closer to some point in L [R] than to any point in R [L]. Once this dividing line is found, we can obtain the diagram $V(S)$ in $O(n)$ time simply by discarding that part of Voronoi edges in $V(L)$ and $V(R)$ which lies, respectively, to the right and to the left of the dividing line.

Hence, the main problem in merging $V(L)$ and $V(R)$ is to find the dividing polygonal line in $O(n)$ time, which is actually possible by virtue of the following properties (Lemmas 4 and 5) of the dividing line.

LEMMA 4. *The dividing line is composed of two rays extending to infinity and some finite line segments. Each element* (*a ray or a segment*) *is contained in the intersection of* $V(P_i)$ *in* $V(L)$ *and* $V(P_j)$ *in* $V(R)$ *for some pair of* $P_i \in L$ *and* $P_j \in R$ *and is the perpendicular bisector of* P_i *and* P_j. □

LEMMA 5. *Each of the two rays is the perpendicular bisector of a pair of consecutive points on the boundary of* CH(S), *the convex hull of points of* S, *such that one is in* L *and the other in* R. □

Lemma 4 implies that, given a ray, we can find the dividing line in $O(n)$ time by tracing it from the ray to the other by means of a special scanning scheme, i.e., by the clockwise and counterclockwise scanning scheme [9]. Lemma 5, on the other hand, enables us to find a ray in $O(n)$ time from CH(S), which, in turn, can be found in $O(n)$ time from CH(L) and CH(R) [14], [15].

Most of the above ideas for the Euclidean Voronoi diagram, with suitable modifications, can be carried over to obtain an efficient algorithm for constructing the Voronoi diagram in the Laguerre geometry for n circles $C_i(Q_i; r_i)$ as follows.

The first problem is how to partition the set of given circles into two subsets. We partition the set S of n circles C_i into two sets L and R with respect to the coordinates of the centers Q_i of C_i. That is, we sort centers Q_i $(i = 1, \cdots, n)$ lexicographically by their (x, y)-coordinates with the x-coordinate as the first key and divide them into two subsets. Then, the locus of points equidistant (in the Laguerre geometry) from L and R, which we call the dividing line (see Fig. 4), enjoys the same property as in the Euclidean case, as stated below.

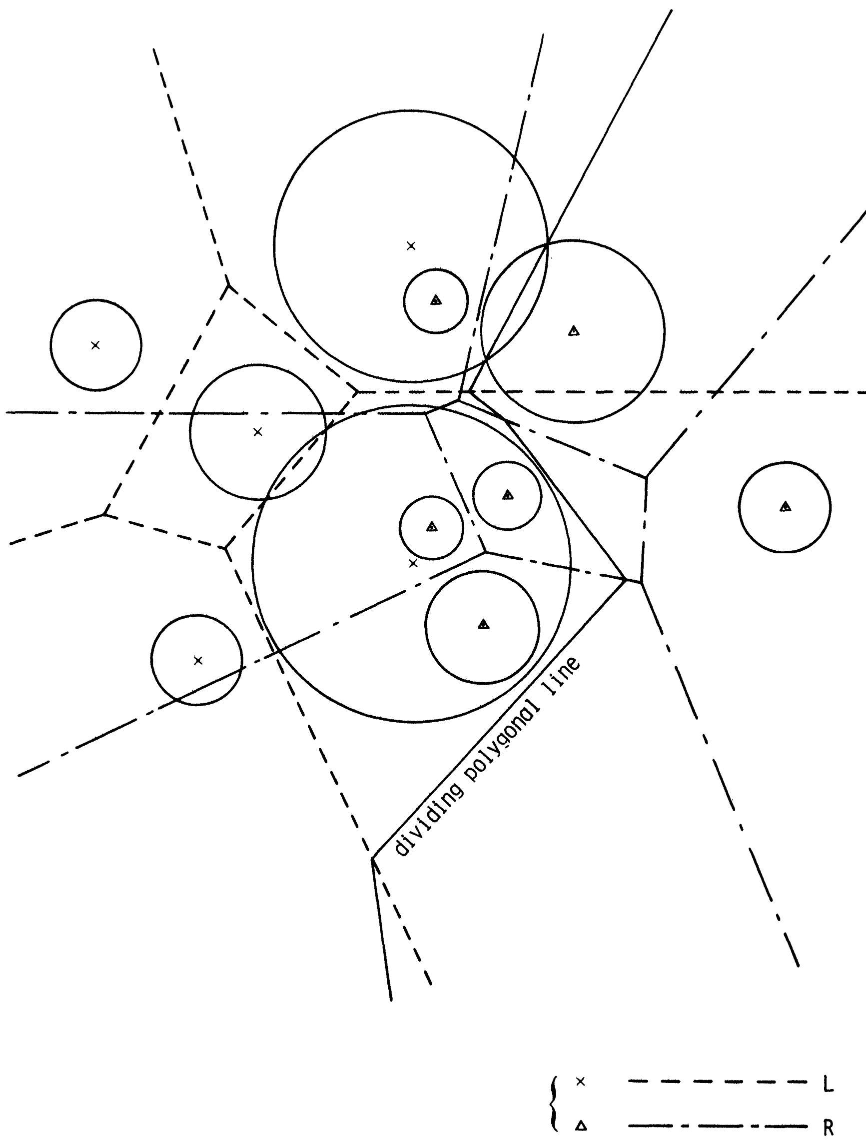

FIG. 4. *Merging the Voronoi diagrams in the Laguerre geometry.*

LEMMA 6. *The dividing polygonal line is unicursal, consisting of two rays and several finite line segments. Every point to the left [right] of this polygonal line is closer (in the sense of the Laguerre geometry) to some circle in L [R] than to any circle in R [L].*

Proof. By rotating clockwise the axes, if necessary, by a sufficiently small angle, we can assume that $x_i \neq x_j$ $(i \neq j)$. Then there exists no Voronoi edge parallel to the x-axis.

It suffices to prove that, for any t, there exists one and only one intersection point $P=(s, t)$ of the dividing line with the line $y=t$, i.e., the dividing line is monotone and hence unicursal. By the assumption that $x_i \neq x_j$ $(i \neq j)$, there exists at least one such point $P=(s, t)$, since the point $(-\infty, t)$ is nearer to L than to R whereas the point $(+\infty, \mathrm{t})$ is nearer to R than to L.

For such a point $P=(s, t)$ let $C_i(Q_i; r_i)$ be the circle in L that is nearest to the point P and $C_j(Q_j; r_j)$ the circle in R that is nearest to P. Since $x_i < x_j$, we see by elementary calculation that, for some $\varepsilon > 0$,

$$(s+\varepsilon, t) \in V(C_j) \text{ and } (s-\varepsilon, t) \in V(C_i). \tag{5}$$

Suppose that there were more than one intersection point, say, $P_1=(s_1, t)$, $P_2=(s_2, t), \cdots, P_k=(s_k, t)$ $(s_1<s_2<\cdots<s_k;\ k \geqq 2)$. It follows from (5) that the points (s, t) with $s=s_1+\varepsilon$ $(<s_2)$ are nearer to R than to L, whereas the points with $s=s_2-\varepsilon$ $(>s_1)$ are nearer to L than to R. Therefore, there exists one and only one intersection point $P=(s, t)$ of the dividing line with the line $y=t$. The Lemma then follows by the continuity arguments. □

It should be noted that the property of the above Lemma 6 does not hold for the Voronoi diagram for line segments, i.e., that there may appear an "L-island" in the R-region and vice versa, which makes the problem quite complicated [9].

The second problem is to trace the dividing line from a given ray to the other ray in linear time. Since a statement similar to Lemma 4 holds for the Voronoi diagram in the Laguerre geometry, we can simply utilize the ordinary clockwise and counterclockwise scanning scheme by taking advantage of the fact that the Voronoi edges are straight lines.

The last problem is to find a ray in $O(n)$ time. The ray is found just as in the ordinary Voronoi diagram from the convex hull of the centers (cf. Lemma 5), provided that the new hull edge is not degenerate (i.e., not collinear). In the degenerate case, however, the property of Lemma 5, as it stands, does not necessarily hold, and something more is needed. For example, consider the case shown in Fig. 5(i), where one of the new hull edges is degenerate. Let l be the line of the new degenerate hull edge of the convex hull of the centers. Even if Q_4 and Q_5 are the closest pair of centers on l such that $C_4 \in L$ and $C_5 \in R$, the radical axis of C_4 and C_5 does not appear in the Voronoi diagram (Fig. 5(ii)). In place of Lemma 5, we have the following Lemma 7 in the Laguerre geometry.

LEMMA 7. *Consider the line l of the new hull edge (in the degenerate case, edges) of the convex hull of the centers $Q_1, \cdots, Q_n$. Let L_l and R_l be sets of circles in L and R, respectively, with their centers on l. Let $C_{i*} \in L_l \subseteq L$ and $C_{j*} \in R_l \subseteq R$ be two circles which have the corresponding Voronoi edge e^* in the Voronoi diagram $V(L_l \cup R_l)$ in the Laguerre geometry for the subset $L_l \cup R_l$ of circles. Then, e^*, which is the radical axis of C_{i*} and C_{j*}, is a ray of the dividing line in merging $V(L)$ and $V(R)$.*

Proof. From the Lemma 3, it is obvious that the two circles corresponding to a ray of $V(L \cup R)$ have their centers on the boundary of the convex hull of $Q_1, \cdots, Q_n$. Therefore, the edge e^* is the only candidate for the ray of the dividing line. □

In order to find the ray of the dividing line in $O(n)$ time, we find the Voronoi edge e^* in the diagram $V(L_l \cup R_l)$ for circles in $L_l \cup R_l$ in linear time in the following way. Note that the Voronoi edges of $V(L_l \cup R_l)$ are all parallel.

First, we construct the diagrams $V(L_l)$ and $V(R_l)$ for circles in L_l and in R_l, respectively, from the diagrams $V(L)$ and $V(R)$, which can be done in linear time as follows. Considering a part of the diagram $V(L)$ far from the line l, we see that two circles in L_l share a Voronoi edge in $V(L_l)$ iff they share a Voronoi edge in $V(L)$.

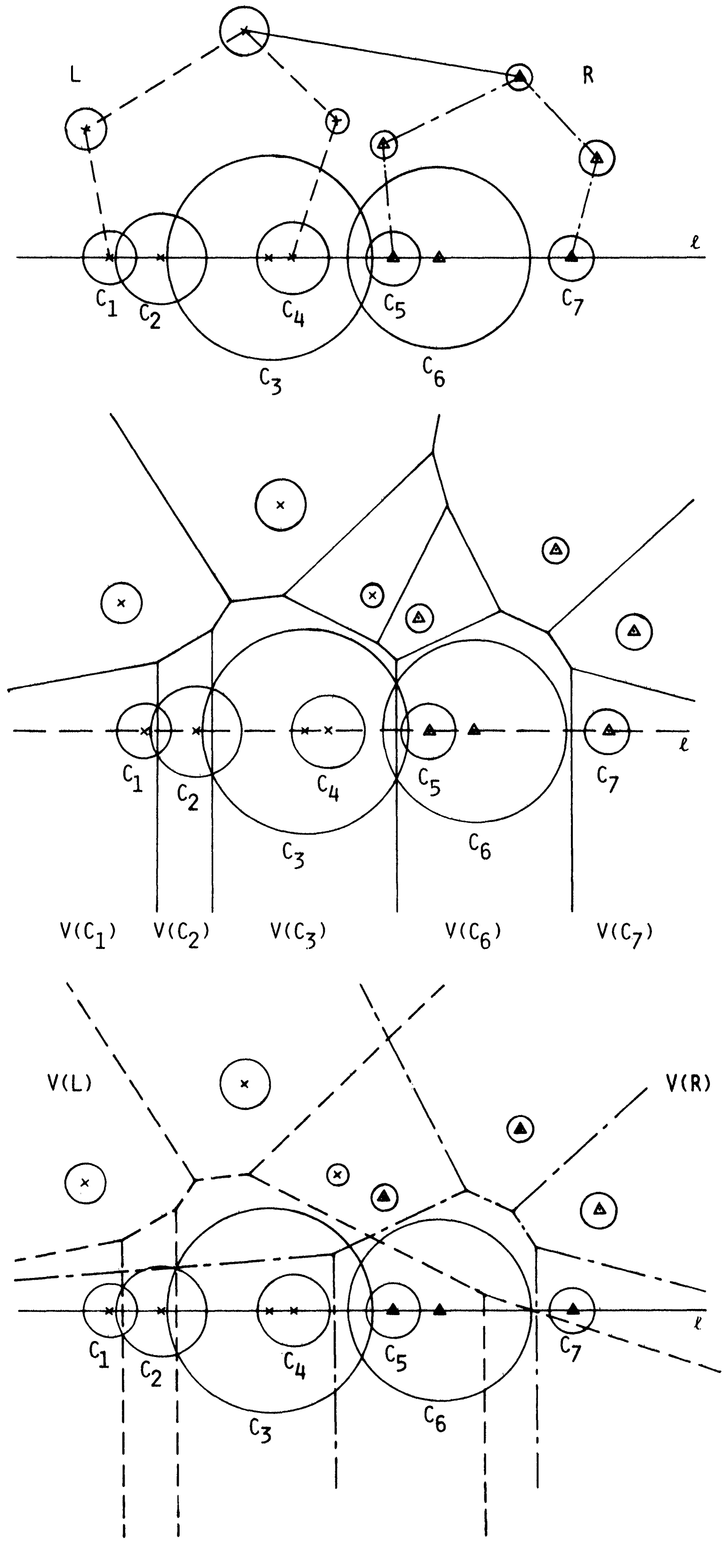

FIG. 5. *Finding a ray in a degenerate case.* (i) *Degenerate new hull edge* l ($L_l=\{C_1, C_2, C_3, C_4\}$, $R_l=\{C_5, C_6, C_7\}$). (ii) $V(L\cup R)$. (iii) $V(L)$ *and* $V(R)$. (iv) $V(L_l)$ *and* $V(R_l)$.

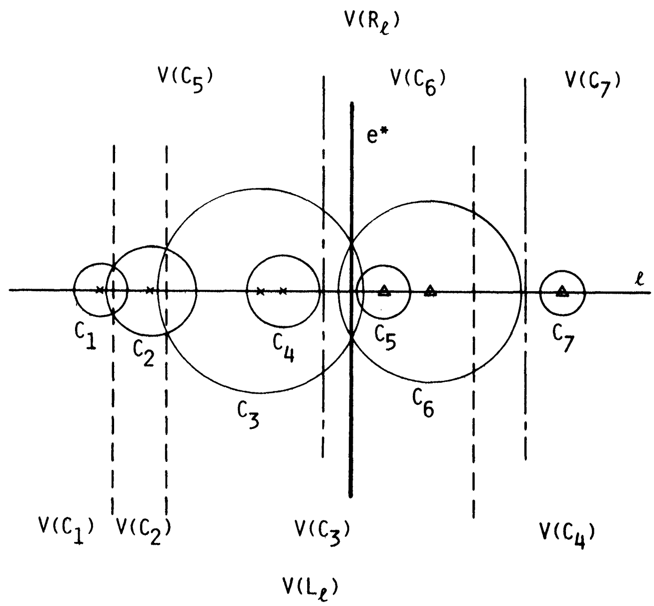

FIG. 5. (*cont.*)

Hence, the diagram $V(L_l)$ can be constructed simply by picking out the Voronoi edges (rays) of pairs of circles in L_l in the diagram $V(L)$. (In the example of Fig. 5, the $V(L_l)$ shown in Fig. 5(iv) by broken lines can be obtained by extending that part (consisting of parallel lines) of $V(L)$ which is far down to the bottom in Fig. 5(iii).) A similar construction is valid for the diagram $V(R_l)$.

Next, we can find e^* from $V(L_l)$ and $V(R_l)$ in linear time as follows. Since all the Voronoi edges in both diagrams $V(L_l)$ and $V(R_l)$ are perpendicular to l, we can merge the diagrams $V(L_l)$ and $V(R_l)$ to obtain $V(L_l \cup R_l)$ in linear time in a way similar to that in which we merge two sorted lists into a single sorted list. In the merged diagram of $V(L_l)$, and $V(R_l)$, each region between two neighbouring edges is the intersection of two Voronoi regions, one in $V(L_l)$ and the other in $V(R_l)$. For each region of the merged diagram, with which is associated a pair $(C_i \in L_l, C_j \in R_l)$ of circles, we examine whether or not there exists a point equidistant (in the Laguerre geometry) from C_i and C_j within the region; if there exists one, the radical axis of C_i and C_j is the ray e^*. (In the example of Fig. 5(iv), the ray e^*, lying in the intersection of $V(C_3)$ in $V(L_l)$ and $V(C_6)$ in $V(R_l)$, is equidistant from C_3 and C_6.) Since the number of those regions in that diagram is $O(n)$, we can find e^*, which is the ray of the dividing line, in $O(n)$ time. $V(L_l \cup R_l)$ is ready to obtain from $V(L_l)$, $V(R_l)$ and e^*.

Thus, it has been shown that the Voronoi diagram in the Laguerre geometry for n circles can be constructed in $O(n \log n)$ time.

4. Applications.

Problem 1. Given n circles in the plane, determine whether a given point P is contained in their union or not.

Once we have constructed the Voronoi diagram in the Laguerre geometry for the given n circles $C_i (i = 1, \cdots, n)$, we have only to find the Voronoi polygon $V(C_j)$ containing P and check if P lies in C_j. If P is not in C_j, then for any circle C_i, $d_L^2(C_i, P) \geqq d_L^2(C_j, P) > 0$, and therefore P is not in any circle. Since we can construct the Voronoi diagram in the Laguerre geometry in $O(n \log n)$ time, and locate a point in a polygonal subdivision of the plane in $O(\log n)$ time and $O(n)$ storage, using $O(n \log n)$ preprocessing [8], [12], we can solve this problem completely in $O(\log n)$ time and $O(n)$ storage with $O(n \log n)$ preprocessing.

Problem 2. Partition the set of n circles into the connected components. That is, find the connected components of the intersection graph of the n circles, i.e. the graph whose vertices are the circles and which has an edge between two vertices iff the circles corresponding to them intersect in the plane.

This problem arises in numerical analysis when we estimate the eigenvalues of a matrix by means of Gershgorin's theorem [4]. Though the intersection graph can have $O(n^2)$ edges, we can solve this problem in $O(n \log n)$ time as follows with the help of the Voronoi diagram in the Laguerre geometry.

Since an improper circle is contained in the union of the proper circles (Lemma 1) and does not affect the connectedness of the other circles, we first consider only proper circles. For the connectedness of proper circles, we have:

LEMMA 8. *For any pair of proper circles C and C' in the same connected component, there exists a sequence $C = C_1, C_2, \cdots, C_k = C'$ of proper circles such that every pair of consecutive circles intersect each other so that they have the corresponding Voronoi edge.*

Proof. Consider the connected component S_I which consists of proper circles and contains C and C'. Since the union of circles in S_I is a connected region and is partitioned into $C_i \cap V(C_i)$ $(C_i \in S_I)$ [i.e., $\bigcup\{C_i \mid C_i \in S_I\} = \bigcup\{C_i \cap V(C_i) \mid C_i \in S_I\}$], we can take a path within this connected region from a point in $C \cap V(C)$ to a point in $C' \cap V(C')$. Considering a sequence $C = C_1, C_2, \cdots, C_k = C'$ of circles in the order in which this path passes through $C_i \cap V(C_i)$ $(C_i \in S_I)$, we can see that every pair of consecutive circles in this sequence intersect each other so that they have the corresponding Voronoi edge. $\square$

We construct a subgraph G of the intersection graph of the n circles which is guaranteed by Lemma 8 to carry the same information as the intersection graph so far as the connected components of the proper circles are concerned. For each pair of proper circles (C_i, C_j) having a common Voronoi edge, we put an edge connecting C_i and C_j in G if the two circles C_i and C_j have a nonempty intersection in the plane. The graph G can be constructed in $O(n)$ time since there exist only $O(n)$ Voronoi edges. Furthermore, the connected components of G can easily be found in $O(n)$ time.

In order to find which components the improper circles belong to, we first make a list of all the improper circles, among which the trivial circles are found in the course of the construction of the diagram and the substantial but improper circles are found by scanning all the Voronoi edges. Next, for each improper circle C_i, we find a proper circle that intersects C_i by locating the center Q_i of C_i in the diagram; if $Q_i \in V(C_j)$, then C_j is a proper circle that contains Q_i, i.e., intersects C_i. The set of centers of improper circles can be located in the diagram in $O(n \log n)$ time by means of the simple algorithm which makes use of a balanced tree [11]. Thus, the total time to find the partition of n circles into the connected components is $O(n \log n)$.

This algorithm is optimal to within a constant factor. In fact, we have

LEMMA 9. *Any algorithm which finds the partition of n circles into the connected components makes at least $\Omega(n \log n)$ comparisons under the linear decision tree model.*[1]

[1] A referee has kindly informed the authors that this lemma holds true not only under the linear decision tree model but also under the more precise algebraic computation tree model, based on the recent result by Ben-Or (see M. Ben-Or, *Lower bounds for algebraic computation trees*, Proc. 15th ACM Symposium on Theory of Computing, Boston, 1983, pp. 80–86).

Proof. This follows immediately from the fact that the element-uniqueness problem, i.e., to determine whether given n real numbers are distinct, reduces in linear time to the connected-component problem, where the lower bound of $\Omega(n \log n)$ is known for the element-uniqueness problem under the above model of computation [3]. □

Problem 3. Find the contour of the union of n given circles in the plane.

This kind of problem is sometimes encountered in image processing and computer graphics. First, we construct the Voronoi diagram in the Laguerre geometry for n circles and then collect that part of the periphery of each circle C_i which lies in the Voronoi polygon $V(C_i)$ for $i=1, \cdots, n$. The validity of this algorithm is obvious. Concerning the number of circular arcs on the contour, we have the following.

LEMMA 10. *The number of circular arcs on the contour is $O(n)$.*

Proof. To distinct pairs of consecutive arcs of the contour, there correspond distinct Voronoi edges (i.e., radical axes), the number of which is $O(n)$. □

This algorithm is optimal for the contour problem. In fact, we have

LEMMA 11. *The complexity of finding the contour of the n circles in the plane is $\Omega(n \log n)$ under the decision tree model.*

Proof. We show that sorting n real numbers $x_1, x_2, \cdots, x_n$ reduces to this problem in $O(n)$ time. First, find $x_* = \min(x_i)$ and $x^* = \max(x_i)$, and let $R = x^* - x_* \geqq 0$. Then, consider n circles with centers $(x_i, 0)$ and radii R (see Fig. 6). The contour of the union of these circles consists of circular arcs, and the order of arcs, according to which the contour can be traced unicursally, gives us the sorted list of n numbers. □

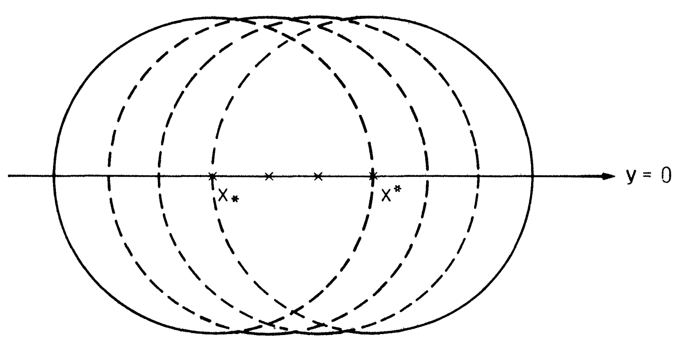

FIG. 6. *Reduction of sorting to finding the contour of the union of circles.*

5. Discussion. Consider the Voronoi diagram in the Laguerre geometry for n circles $C_i(Q_i; r_i)$ $(Q_i = (x_i, y_i);\ i = 1, \cdots, n)$. This diagram will remain invariant if $r_i^2 (i=1, \cdots, n)$ are replaced simultaneously by $r_i^2 - R$ with some constant R; in other words, this diagram can be regarded as the Voronoi diagram for n points $Q_i = (x_i, y_i)$ in the plane where, with some constant R, a distance $d(Q_i, P)$ between Q_i and a point $P = (x, y)$ is defined by

$$d^2(Q_i, P) = (x - x_i)^2 + (y - y_i)^2 - r_i^2 + R.$$

On the other hand, the two-dimensional section (with $z = 0$) of the Voronoi diagram in the three-dimensional Euclidean space for n points $P_i = (x_i, y_i, z_i)$ $(i = 1, \cdots, n)$ is a kind of Voronoi diagram for n points $Q_i = (x_i, y_i)$ $(i = 1, \cdots, n)$, which we will call the *section diagram* (or, the generalized Dirichlet tessellation [13]), with the distance $d(Q_i, P)$ between Q_i and a point $P = (x, y)$ defined by

$$d^2(Q_i, P) = (x - x_i)^2 + (y - y_i)^2 + z_i^2.$$

Hence, by setting $r_i^2 = R - z_i^2$ with sufficiently large constant R, the algorithm we presented here can be applied to the construction in $O(n \log n)$ time of the section with the plane $z = 0$ of the Voronoi diagram for n points in the three-dimensional Euclidean space.

More generally, we can consider the section of the Voronoi diagram in the k-dimensional space with the distance $d_G(P_i, P_j)$ between two points, $P_i = x_i$ and $P_j = x_j \in \mathbf{R}^k$, defined by

$$d_G^2(P_i, P_j) = (x_i - x_j)' G(x_i - x_j),$$

where G is a $k \times k$ symmetric matrix [13], [16]. We can apply the algorithm presented here to such section diagrams even if G is not positive definite (for example, $G = \mathrm{diag}[1, -1, -1]$). Here, it should be noted that the Voronoi diagram in the Laguerre geometry itself is the section with the plane $z = 0$ of the Voronoi diagram for n points $P_i = (x_i, y_i, z_i)$ in three-dimensional space where the square of distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is defined by $(x_1 - x_2)^2 + (y_1 - y_2)^2 - (z_1 - z_2)^2$. Nevertheless, it would be worth while to consider the Voronoi diagram in the Laguerre geometry in connection with the circles since, then, the Voronoi edges and the Voronoi points have the geometrical and physical meanings of radical axes and radical centers, respectively.

Concluding remarks. We have shown that the Voronoi diagram in the Laguerre geometry can be constructed in $O(n \log n)$ time, and is useful for geometric problems concerning circles. Brown [2] considered a technique of inversion which is also useful for geometrical problems for circles. In fact, it can be applied to the problems treated in the present paper. However, our approach is intrinsic in the plane and would be of interest in itself. We have also discussed the relation between the Voronoi diagram in the Laguerre geometry and the two-dimensional section of the Voronoi diagram in the three-dimensional Euclidean space.

Acknowledgment. The authors thank the referees for many helpful comments, without which the paper might have been less readable.

REFERENCES

[1] W. BLASCHKE, *Vorlesungen über Differentialgeometrie* III, Springer, Berlin, 1929.
[2] K. Q. BROWN, *Geometric transforms for fast geometric algorithms*, Ph.D. thesis, Computer Science Department, Carnegie-Mellon Univ., Pittsburgh, PA, 1979.
[3] D. P. DOBKIN AND R. J. LIPTON, *On the complexity of computations under varying sets of primitives*, J. Comput. System Sci., 18 (1979), pp. 86–91.
[4] D. K. FADDEEV AND V. N. FADDEEVA, *Computational Methods of Linear Algebra*, Fizmatgiz, Moscow, 1960; translated by R. C. Williams, Freeman, San Francisco, 1963.
[5] F. HARARY, *Graph Theory*, Addison-Wesley, Reading, MA, 1969.
[6] F. K. HWANG, *An $O(n \log n)$ algorithm for rectilinear minimal spanning trees*, J. Assoc. Comput. Mach., 26 (1979), pp. 177–182.
[7] D. G. KIRKPATRICK, *Efficient computation of continuous skeletons*, Proc. 20th IEEE Symposium on Foundations of Computer Science, San Juan, 1979, pp. 18–27.
[8] ———, *Optimal search in planar subdivision*, this Journal, 12 (1983), pp. 28–35.
[9] D. T. LEE AND R. L. DRYSDALE, III, *Generalization of Voronoi diagrams in the plane*, this Journal, 10 (1981), pp. 73–87.
[10] D. T. LEE AND C. K. WONG, *Voronoi diagrams in $L_1(L_\infty)$ metrics with 2-dimensional storage applications*, this Journal, 9 (1980), pp. 200–211.
[11] D. T. LEE AND C. C. YANG, *Location of multiple points in a planar subdivision*, Inform. Proc. Letters, 9 (1979), pp. 190–193.

[12] R. J. LIPTON AND R. E. TARJAN, *Applications of planar separator theorem*, Proc. 18th IEEE Symposium on Foundations of Computer Science, Providence, RI, 1977, pp. 162–170.

[13] R. E. MILES, *The random division of space*, Suppl. Adv. Appl. Prob. (1972), pp. 243–266.

[14] F. P. PREPARATA AND S. J. HONG, *Convex hulls of finite sets of points in two or three dimensions*, Comm. ACM, 20 (1977), pp. 87–93.

[15] M. I. SHAMOS AND D. HOEY, *Closest-point problems*, Proc. 16th IEEE Symposium on Foundations of Computer Science, Berkeley, CA, 1975, pp. 151–162.

[16] R. SIBSON, *A vector identity for the Dirichlet tessellation*, Math. Proc. Camb. Phil. Soc., 87 (1980), pp. 151–155.