Near Space-Optimal Perfect Hashing Algorithm

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Objective of the Presentation

Present a perfect hashing algorithm that uses the idea of partitioning the input key set into small buckets:

- Key set fits in the internal memory
  - **I**nternal **R**andom **A**ccess memory algorithm (**IRA**)
- Key set larger than the internal memory
  - **E**xternal **C**ache-**A**ware memory algorithm (**ECA**)

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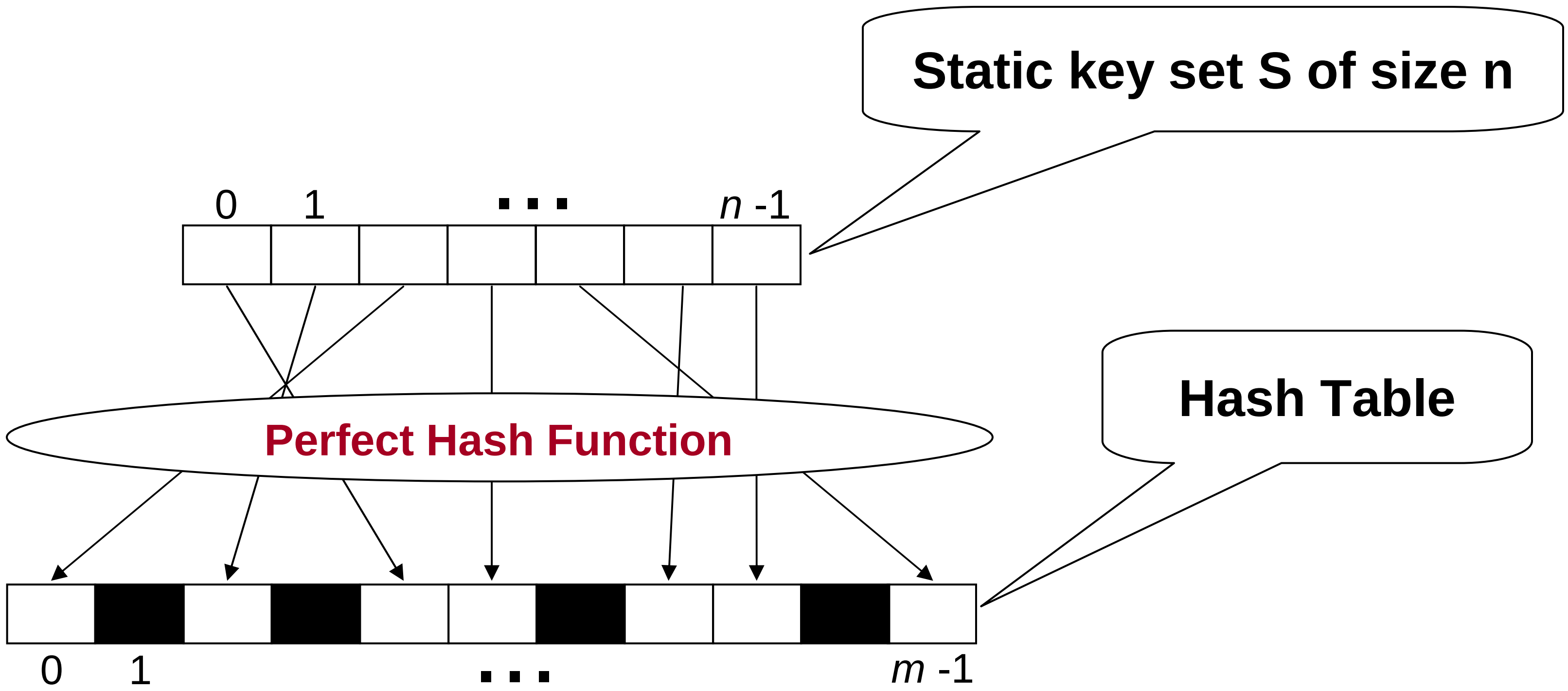
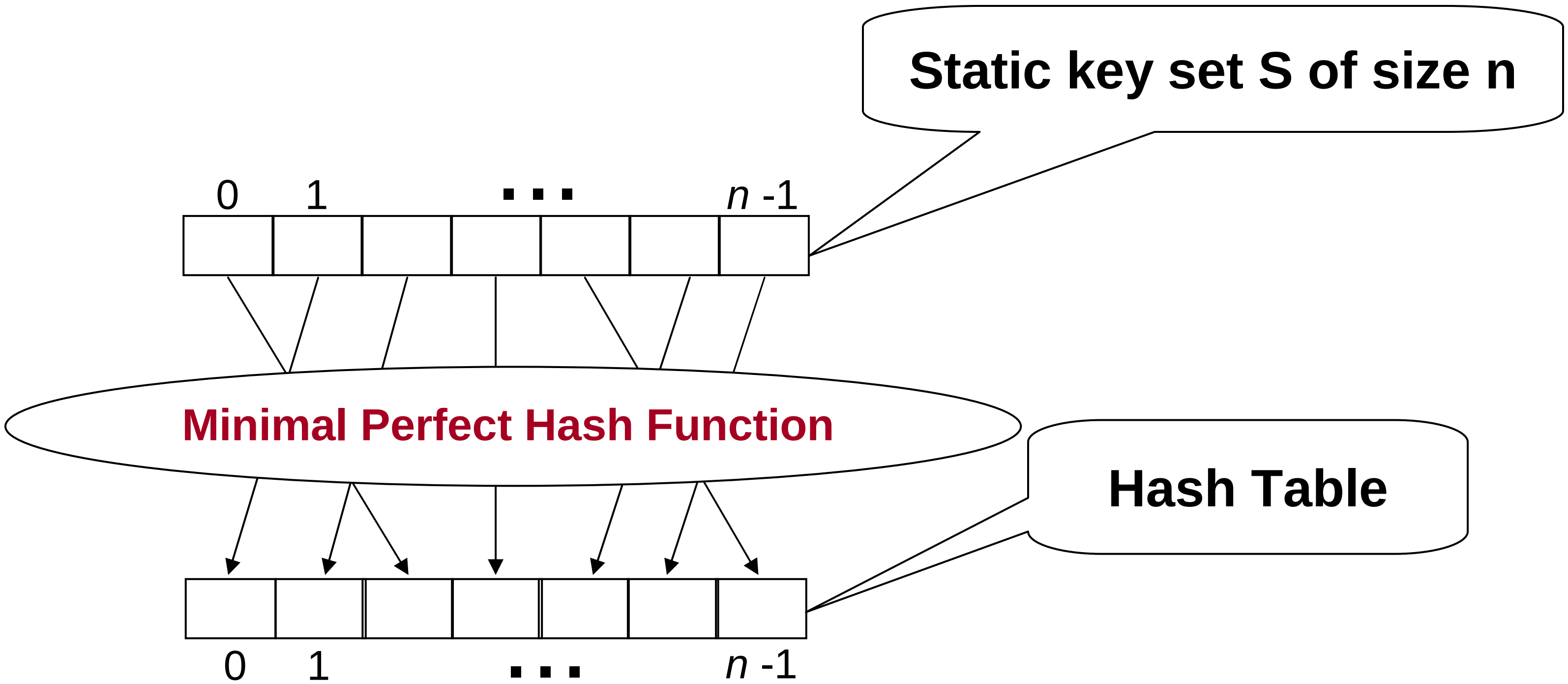
Theoretically well-founded, time efficient, higly scalable and near space-optimal

Perfect Hash Function



$$S \subseteq U, \text{ where } |U| = u$$

Minimal Perfect Hash Function

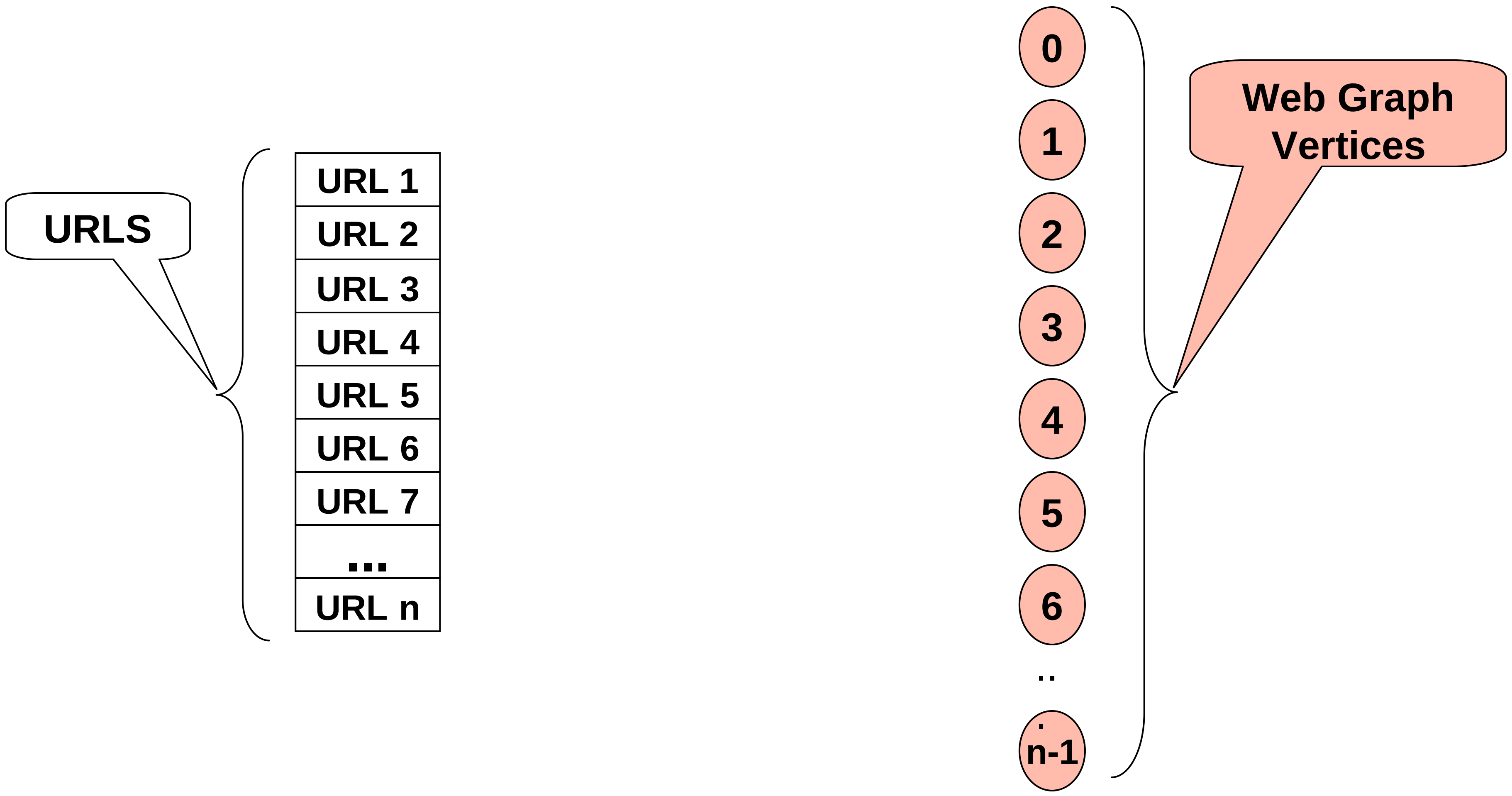


$$S \subseteq U, \text{ where } |U| = u$$

Where to use a PHF or a MPHF?

- Access items based on the value of a key is ubiquitous in Computer Science
- Work with huge static item sets:
  - In data warehousing applications:
    - On-Line Analytical Processing (OLAP) applications
  - In Web search engines:
    - Large vocabularies
    - Map long URLs in smaller integer numbers that are used as IDs

Mapping URLs to Web Graph Vertices



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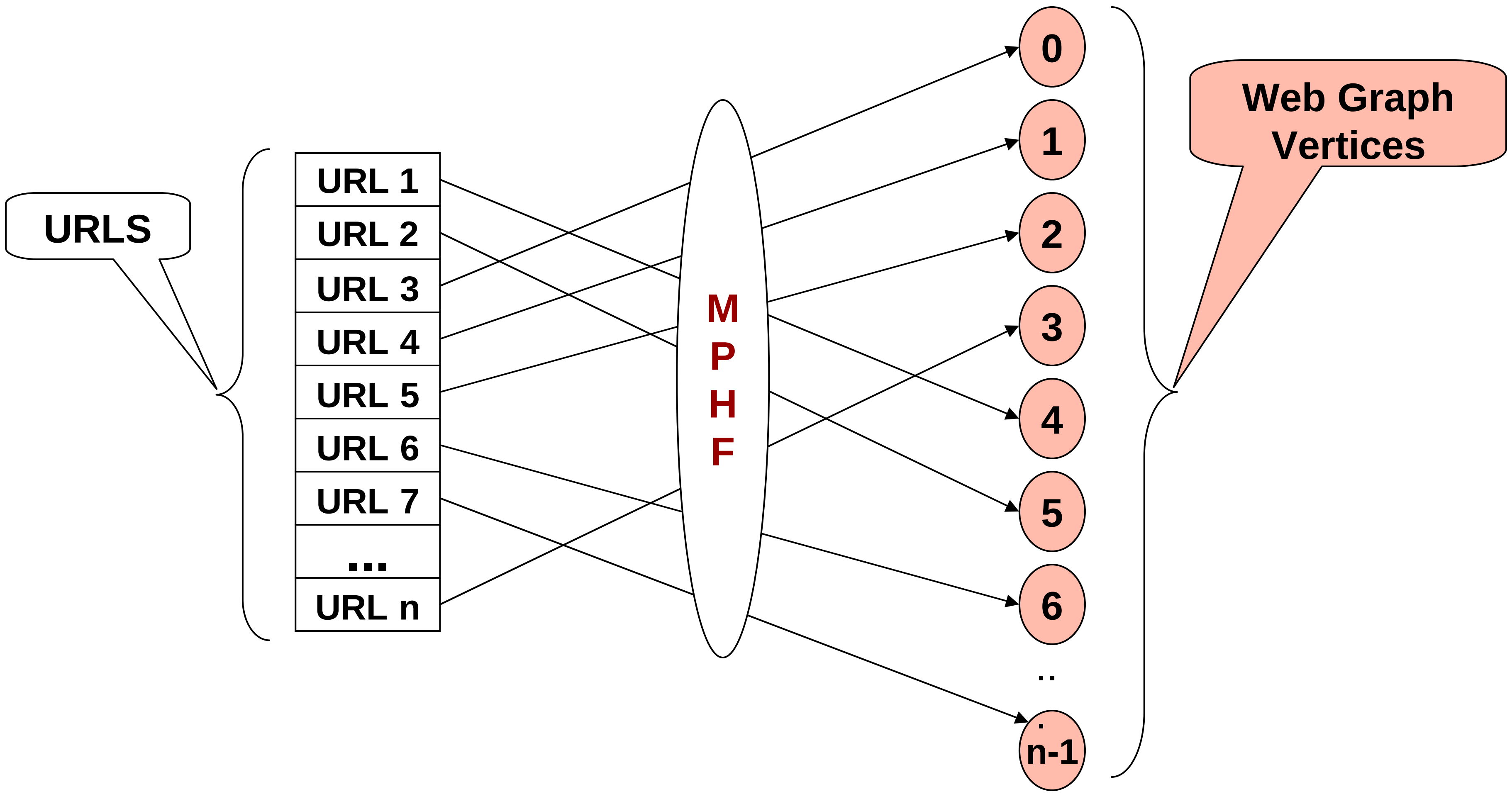


Information Theoretical Lower Bounds for Storage Space

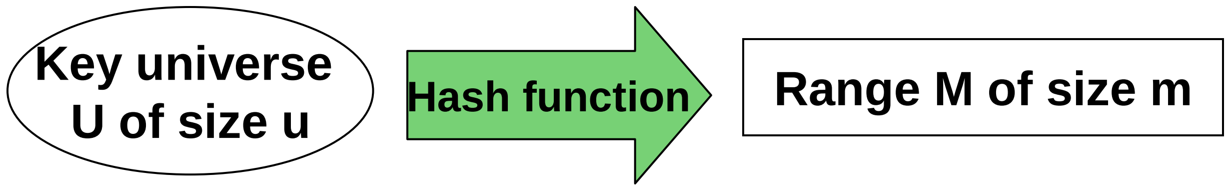
- PHFs (m ≈ n): $\text{Storage Space} \geq \frac{n^2}{m} \log e$

- MPHFs (m = n): $\text{Storage Space} \geq n \log e$

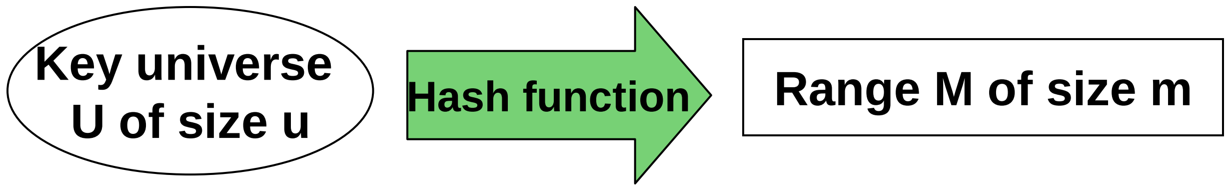
m < 3n

$\log e \approx 1.4427$

Uniform Hashing Versus Universal Hashing



Uniform Hashing Versus Universal Hashing



Uniform hashing

- # of functions from U to M?

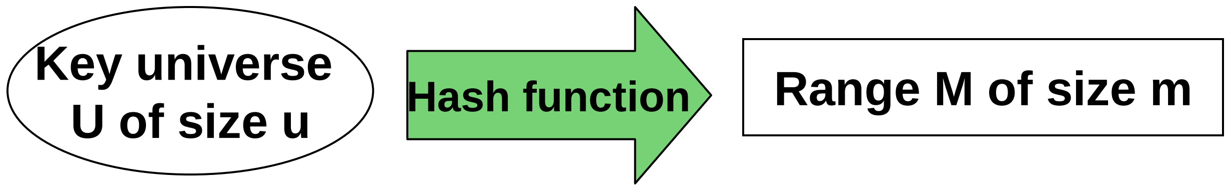
$$m^u$$

- # of bits to encode each fucntion

$$u \log m$$

- Independent functions with values uniformly distributed

Uniform Hashing Versus Universal Hashing



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Universal hashing

- A family of hash functions $\mathcal{H}$ is universal if:
  - for any pair of distinct keys (x_1, x_2) from U and
  - a hash function **h** chosen uniformly from $\mathcal{H}$ then:

$$\Pr(h(x_1) = h(x_2)) \leq \frac{1}{m}$$

Intuition Behind Universal Hashing

- We often lose relatively little compared to using a completely random map (uniform hashing)
- If S of size n is hashed to n^2 buckets, with probability more than ½, no collisions occur
  - Even with complete randomness, we do not expect little $o(n^2)$ buckets to suffice (the birthday paradox)
  - So nothing is lost by using a universal family instead!

Related Work

- Theoretical Results
  (use uniform hashing)
- Practical Results
  (assume uniform hashing for free)
- Heuristics

Theoretical Results

Work	Gen. Time	Eval. Time	Size (bits)
Mehlhorn (1984)	Expon.	Expon.	O(n)
Schmidt and Siegel (1990)	Not analyzed	O(1)	O(n)
Hagerup and Tholey (2001)	O(n+log log u)	O(1)	O(n)

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Theoretic ECA (CIKM 2007)	**O(n)**	**O(1)**	**O(n)**

Practical Results – Assume Uniform Hashing For Free

Work	Gen. Time	Eval. Time	Size (bits)
Czech, Havas & Majewski (1992)	**O(n)**	**O(1)**	**O(n log n)**
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IRA (WADS 2007)	**O(n)**	**O(1)**	**O(n)**
Heuristic ECA (CIKM 2007)	**O(n)**	**O(1)**	**O(n)**

Empirical Results

Work	Application	Gen. Time	Eval. Time	Size (bits)
Fox, Chen & Heath (1992)	**Index data in CD-ROM**	**Exp.**	**O(1)**	**O(n)**
Lefebvre & Hoppe (2006)	**Sparse spatial data**	**O(n)**	**O(1)**	**O(n)**
Chang, Lin & Chou (2005, 2006)	**Data mining**	**O(n)**	**O(1)**	**Not analyzed**

Internal Random Access and External Cache-Aware Memory Algorithms

- Near space optimal
- Evaluation in constant time
- Function generation in linear time
- Simple to describe and implement
- Known algorithms with near-optimal space either:
  - Require exponential time for construction and evaluation, or
  - Use near-optimal space only asymptotically, for large *n*
- Acyclic random hypergraphs
  - Used before by Majewski et all (1996): O(n log n) bits
- We proceed differently: O(n) bits

  (we changed space complexity, close to theoretical lower bound)

Random Hypergraphs (r-graphs)

- 3-graph:

0 1

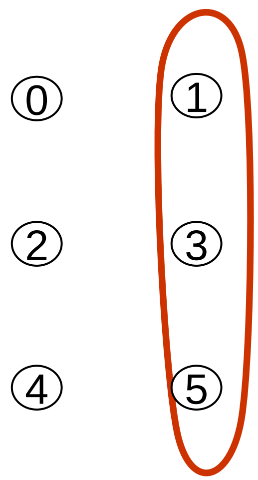
2 3

4 5

- 3-graph is induced by three uniform hash functions

Random Hypergraphs (r-graphs)

- 3-graph:

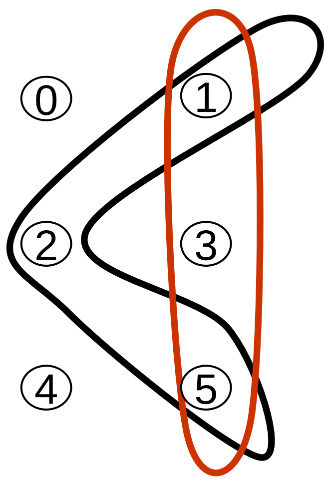


$h_0(jan) = 1 \quad h_1(jan) = 3 \quad h_2(jan) = 5$

- 3-graph is induced by three uniform hash functions

Random Hypergraphs (r-graphs)

- 3-graph:



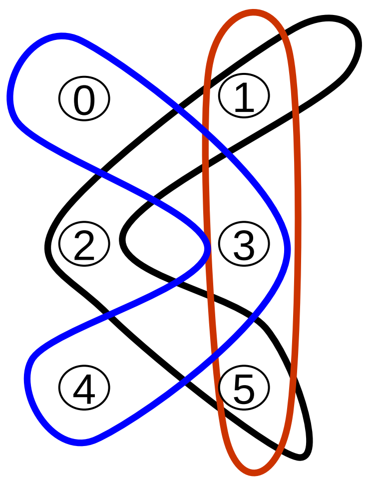
$h_0(jan) = 1 \quad h_1(jan) = 3 \quad h_2(jan) = 5$

$h_0(feb) = 1 \quad h_1(feb) = 2 \quad h_2(feb) = 5$

- 3-graph is induced by three uniform hash functions

Random Hypergraphs (r-graphs)

- 3-graph:



$h_0(jan) = 1 \quad h_1(jan) = 3 \quad h_2(jan) = 5$

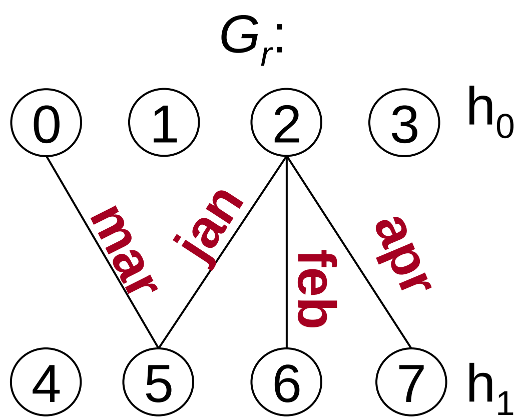
$h_0(feb) = 1 \quad h_1(feb) = 2 \quad h_2(feb) = 5$

$h_0(mar) = 0 \quad h_1(mar) = 3 \quad h_2(mar) = 4$

- 3-graph is induced by three uniform hash functions
- Our best result uses 3-graphs

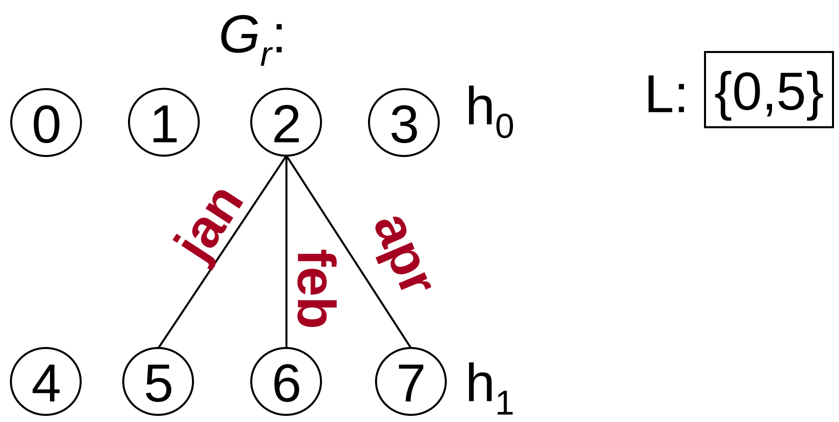
The Internal Random Access memory algorithm ...

Acyclic 2-graph

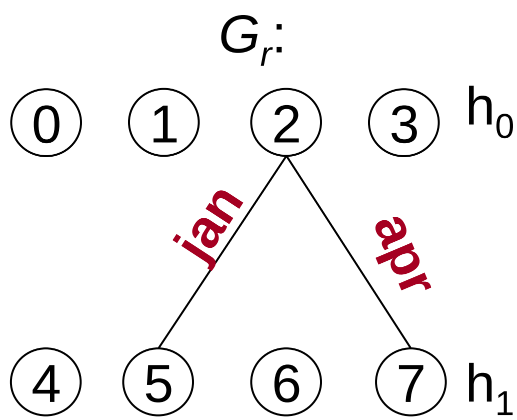


L:Ø

Acyclic 2-graph

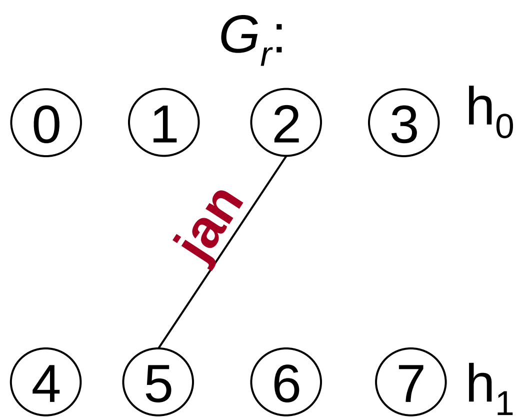


Acyclic 2-graph



	0	1
L:	{0,5}	{2,6}

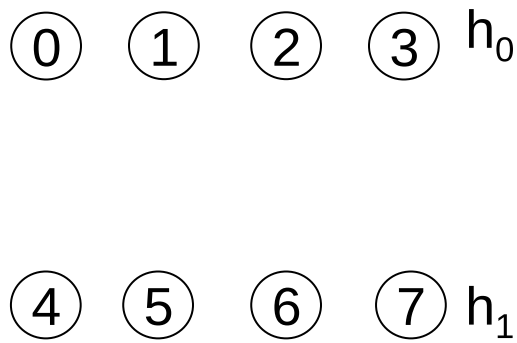
Acyclic 2-graph



	0	1	2
L:	{0,5}	{2,6}	{2,7}

Acyclic 2-graph

G_r:



	0	1	2	3
L:	{0,5}	{2,6}	{2,7}	{2,5}

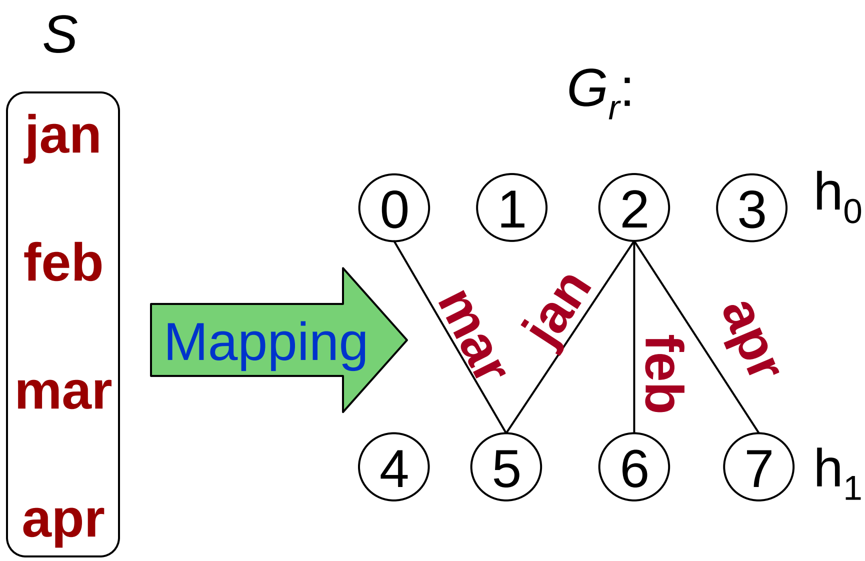
G_r is acyclic

Internal Random Access Memory Algorithm (r=2)

S

jan
feb
mar
apr

Internal Random Access Memory Algorithm (r=2)



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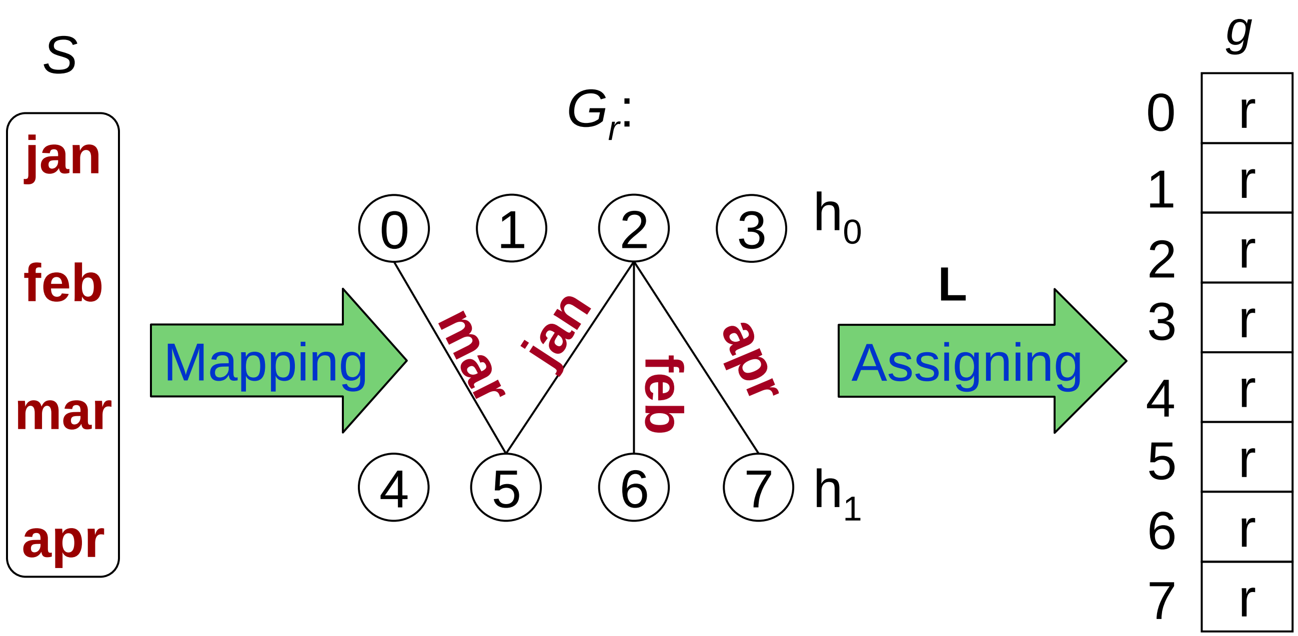

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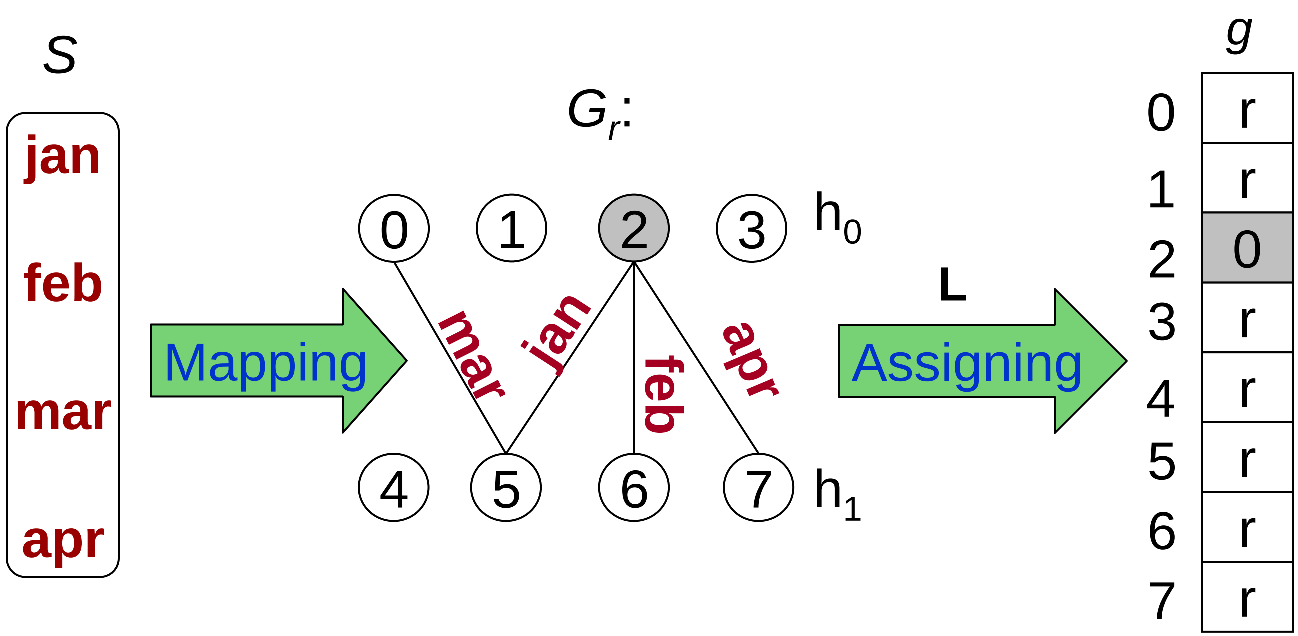

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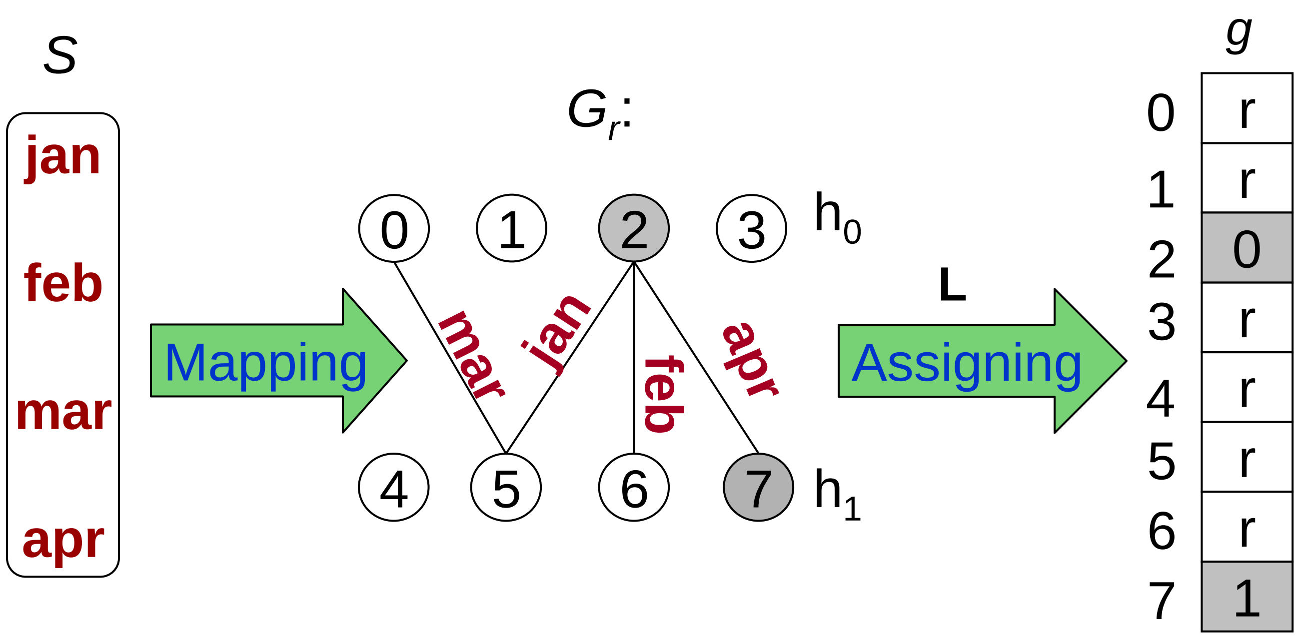

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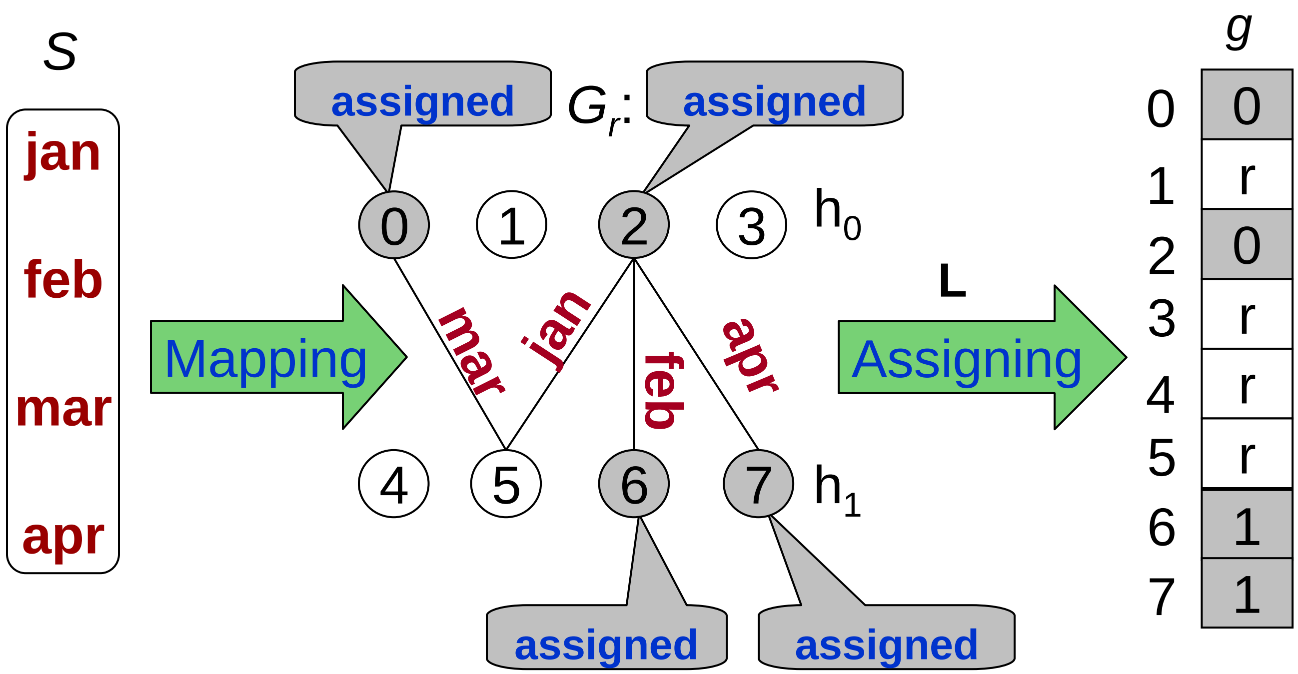

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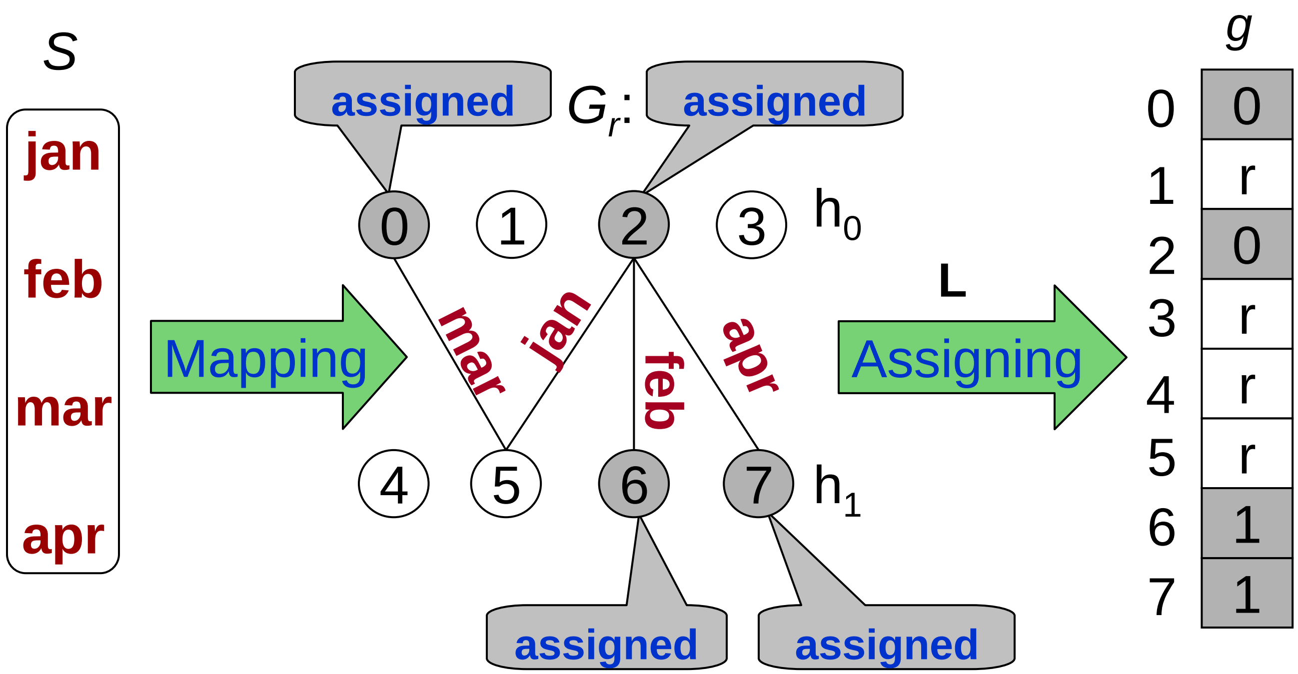
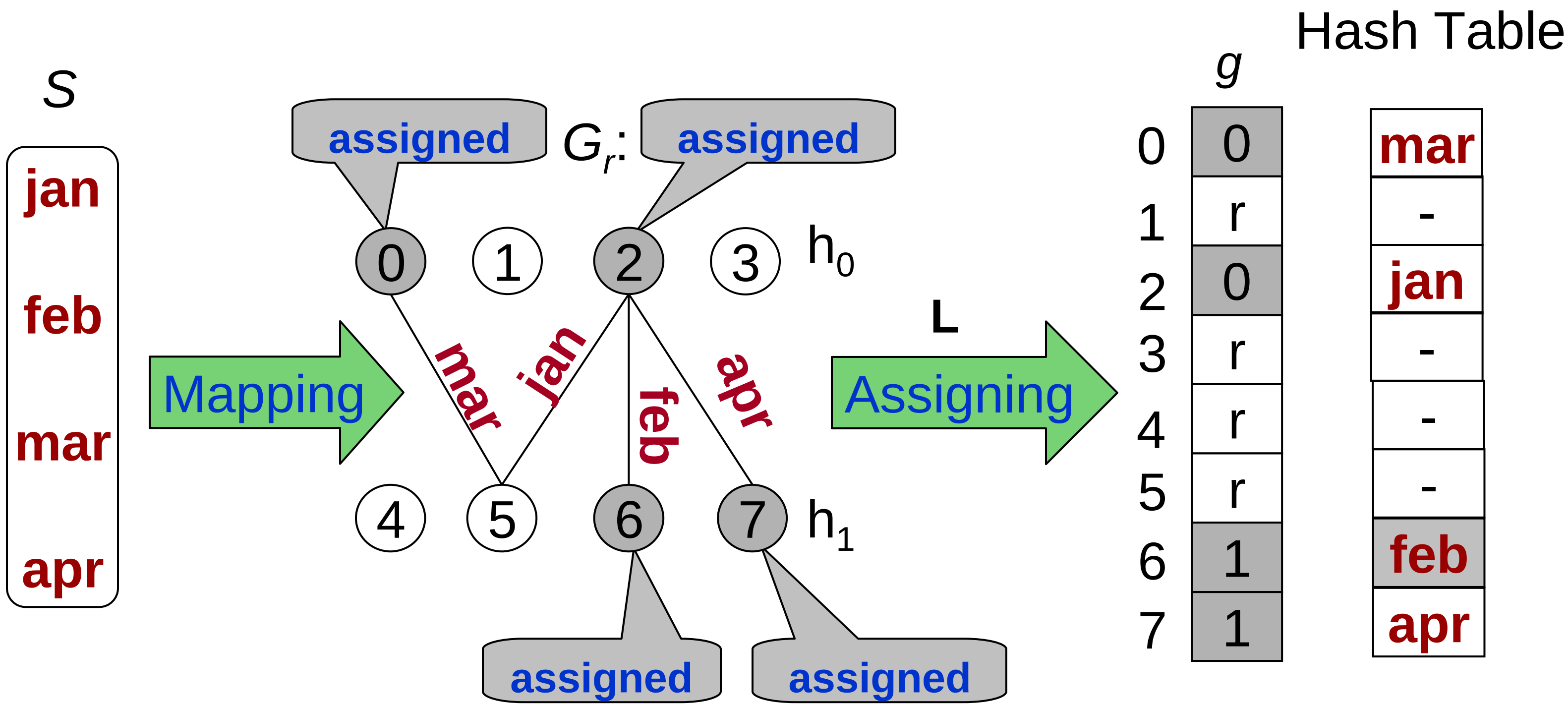


$$i = (g[h_0(\text{feb})] + g[h_1(\text{feb})]) \bmod \mathbf{r} = (g[2] + g[6]) \bmod \mathbf{2} = 1$$

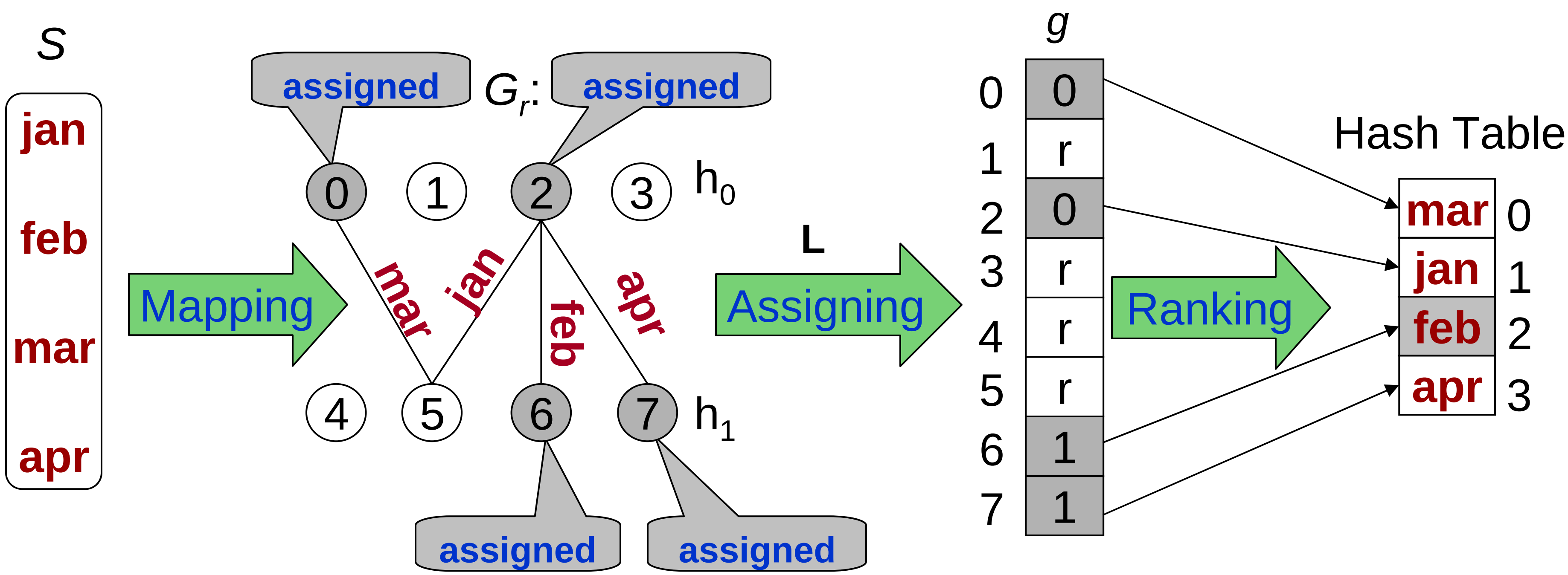
Internal Random Access Memory Algorithm: PHF



i = (g[h_0(feb)] + g[h_1(feb)]) mod **r** = (g[2] + g[6]) mod **2** = 1

phf(feb) = $h_{i=1}$ (feb) = 6

Internal Random Access Memory Algorithm: MPHF

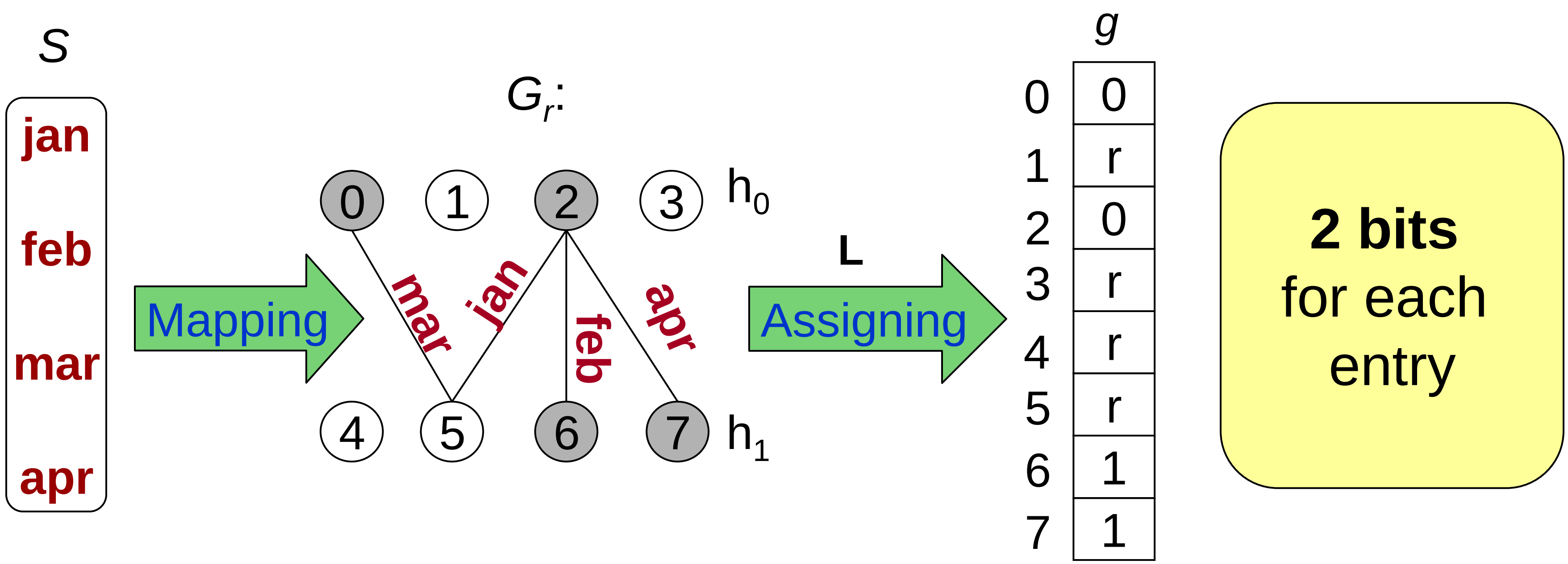


$i = (g[h_0(\text{feb})] + g[h_1(\text{feb})]) \bmod \mathbf{r} = (g[2] + g[6]) \bmod \mathbf{2} = 1$

$\text{phf}(\text{feb}) = h_{i=1}(\text{feb}) = 6$

$\text{mphf}(\text{feb}) = \text{rank}(\text{phf}(\text{feb})) = \text{rank}(6) = 2$

Space to Represent the Function



Space to Represent the Functions (r = 3)

- PHF g: [0,m-1] $\rightarrow$ {0,1,2}
  - m = cn bits, c = 1.23 $\rightarrow$ **2.46** n bits
  - (log 3) cn bits, c = 1.23 $\rightarrow$ **1.95 n** bits (arith. coding)
  - Optimal: **0.89n** bits
- MPHF g: [0,m-1] $\rightarrow$ {0,1,2,3} (ranking info required)
  - 2m + εm = (2+ ε)cn bits
  - For c = 1.23 and ε = 0.125 $\rightarrow$ **2.62 n** bits
  - Optimal: **1.44n** bits.

Use of Acyclic Random Hypergraphs

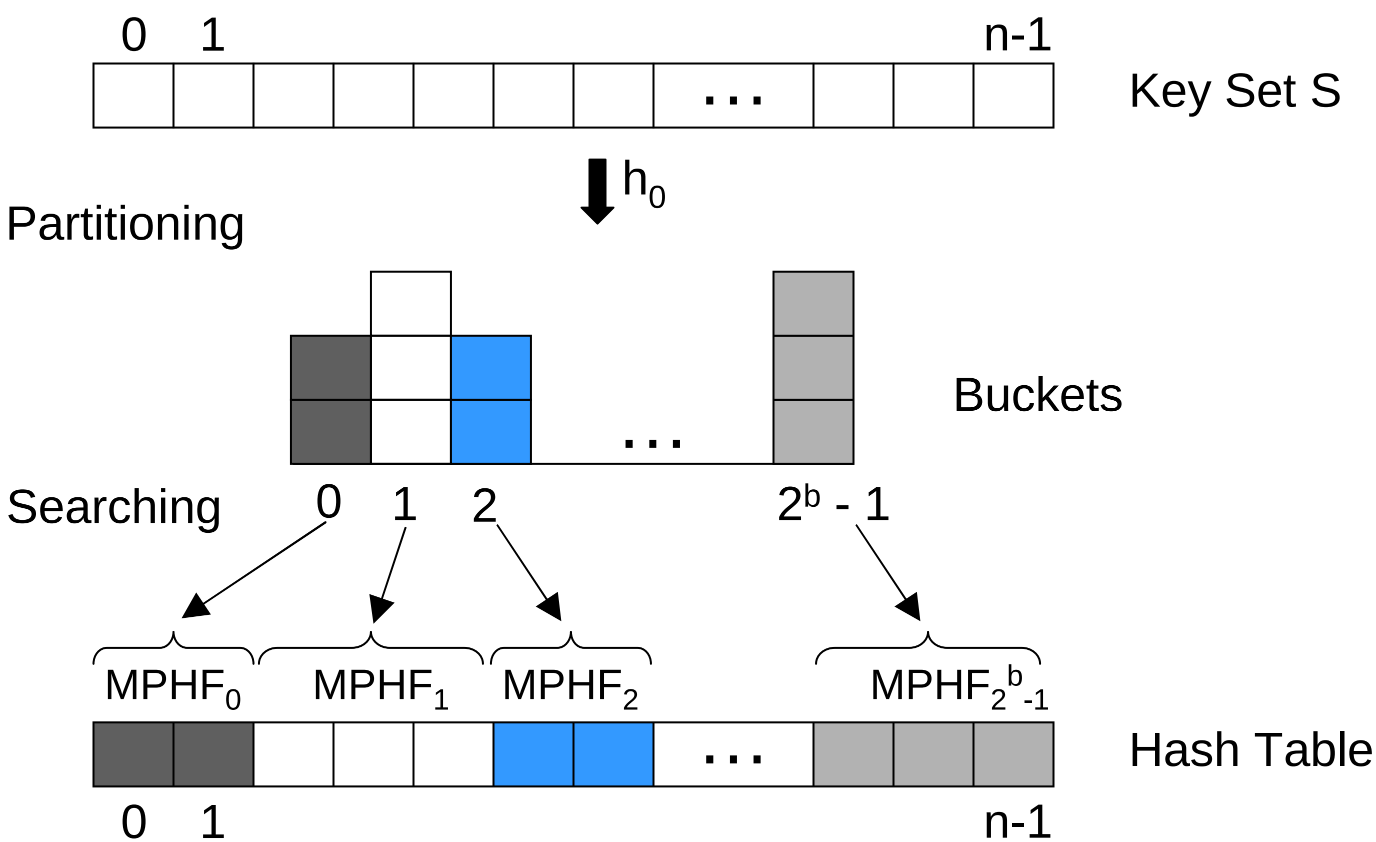
- Sufficient condition to work
- Repeatedly selects h_0, h_1..., h_{r-1}
- For r = 2, m = cn and c ≥ 2.09: Pr_a = 0.29
- For r = 3, m = cn and c≥1.23: Pr_a tends to 1
- Number of iterations is $1/Pr_a$
  - r = 2: 3.5 iterations
  - r = 3: 1.0 iteration

The External Cache-Aware memory algorithm ...

External Cache-Aware Memory Algorithm (ECA)

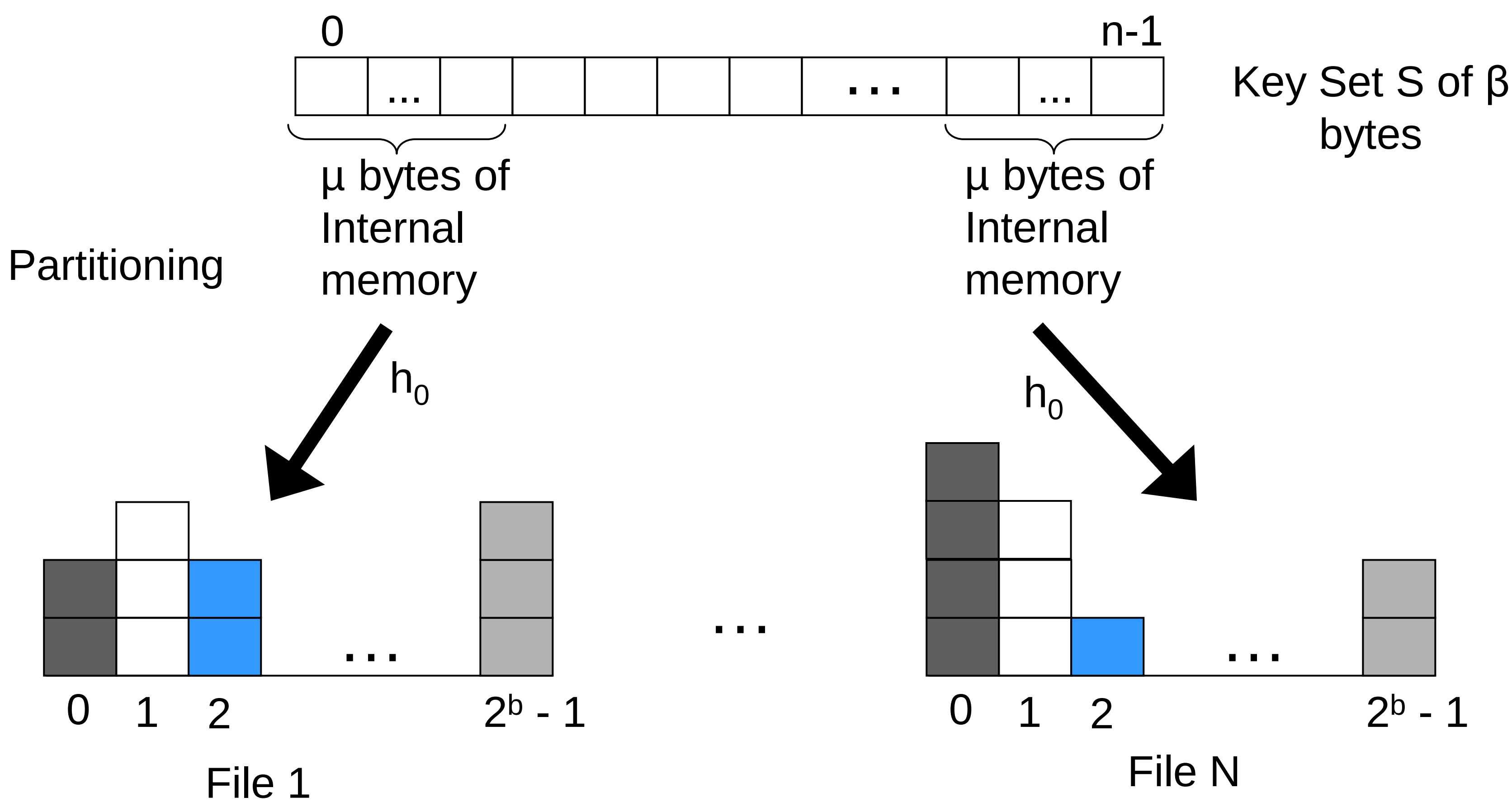
- First MPHF algorithm for very large key sets (in the order of billions of keys)
- This is possible because
  - Deals with external memory efficiently
  - Generates compact functions (near space-optimal)
  - Uses a little amount of internal memory to scale
  - Works in linear time
- Two implementations:
  - Theoretical well-founded ECA (uses uniform hashing)
  - Heuristic ECA (uses universal hashing)

External Cache-Aware Memory Algorithm (ECA)



$MPHF(x) = MPHF_i(x) + offset[i];$

Key Set Does Not Fit In Internal Memory

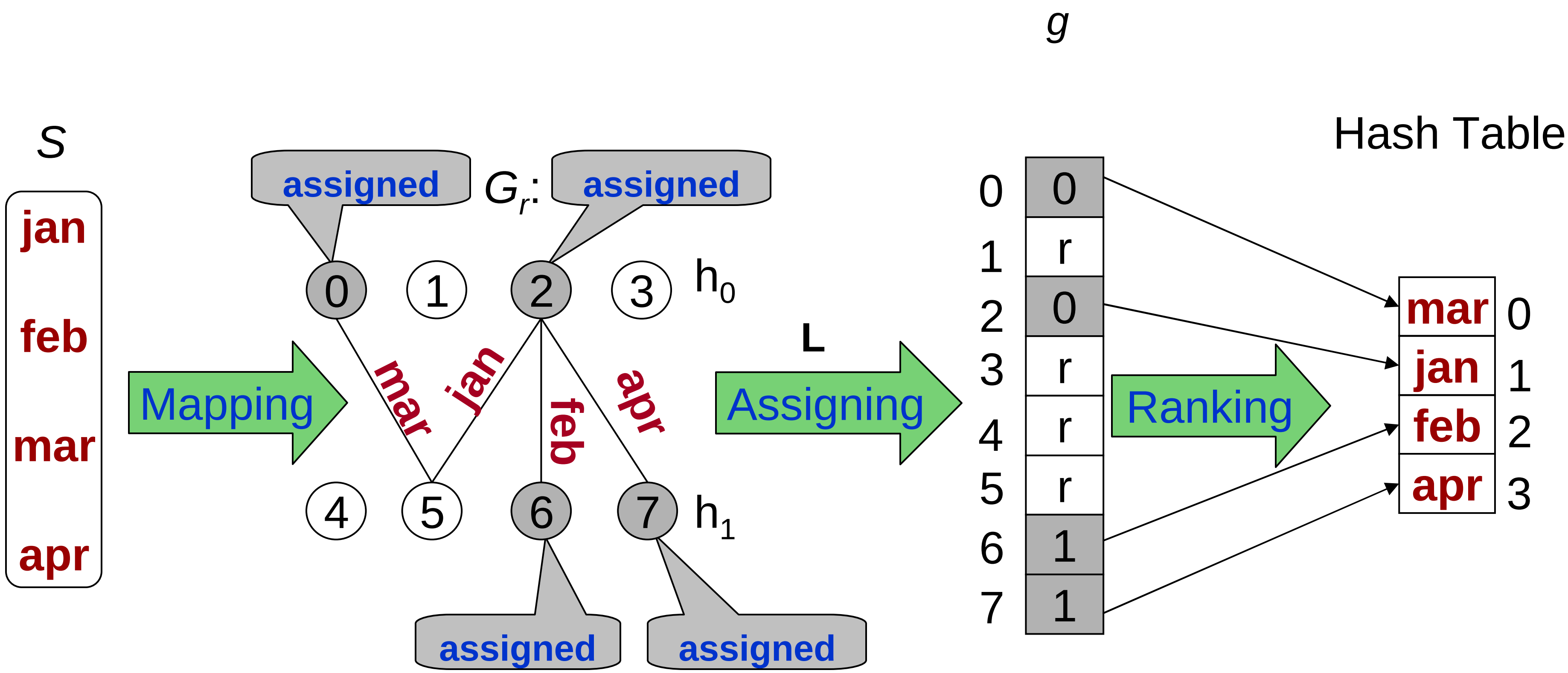


$N = \beta/\mu$ b = Number of bits of each bucket address Each bucket ≤ 256

Important Design Decisions

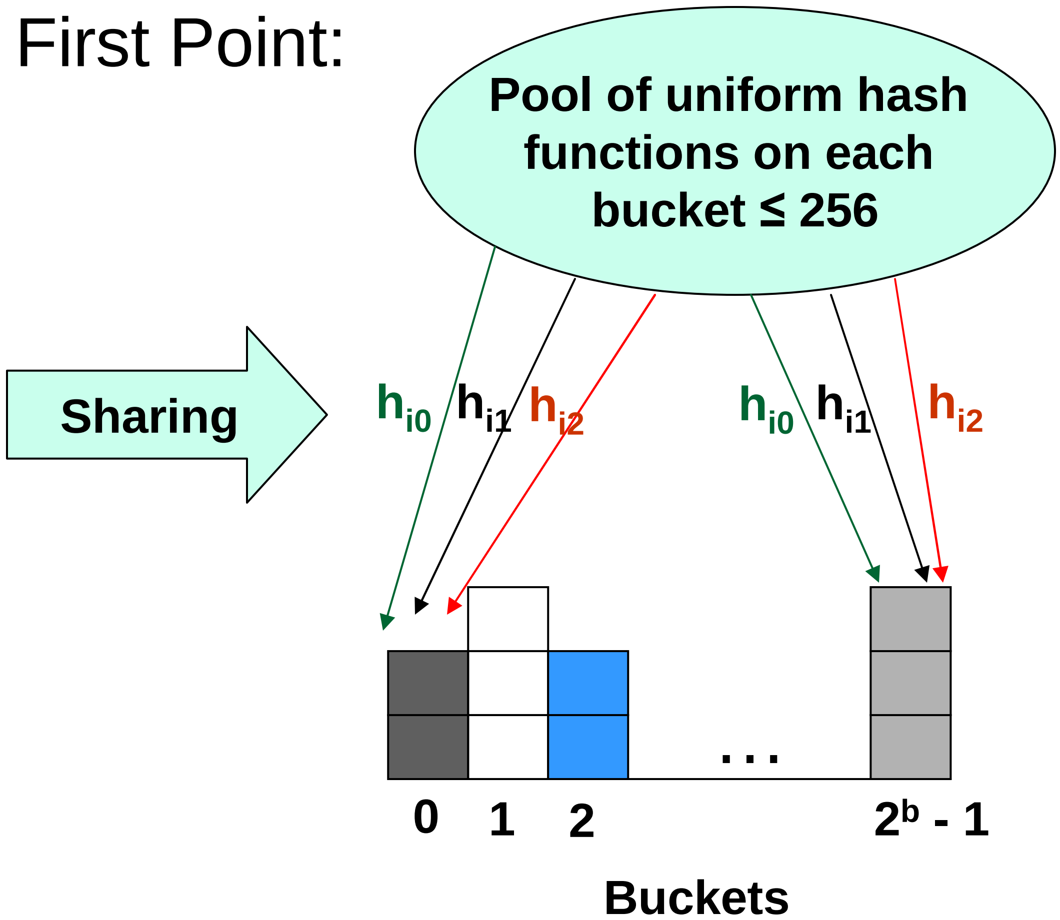
- We map long URLs to a fingerprint of fixed size using a hash function
- Use our IRA linear time and near space-optimal algorithm to generate the MPHF of each bucket
- How do we obtain a linear time complexity?
  - Using internal radix sorting to form the buckets
  - Using a heap of N entries to drive a N-way merge that reads the buckets from disk in one pass

Use the Internal Random Access Memory Algorithm for Each Bucket



Why the ECA Algorithm is Well-Founded?

First Point:



Why the ECA Algorithm is Well-Founded?

Second Point:

We have shown how to create that pool based on the linear hash functions proposed by Alon et al (JACM 1999)

$$f(x,s,\Delta) = \left(\sum_{j=1}^{k} t_j [y_j(x) \oplus \Delta] + s \sum_{j=k+1}^{2k} t_j [y_{j-k}(x) \oplus \Delta] \right) \bmod p$$

$$h_{i0}(x) = f(x,s,0) \bmod |B_i|$$

$$h_{i1}(x) = f(x,s,1) \bmod |B_i| + |B_i|$$

$$h_{i2}(x) = f(x,s,2) \bmod |B_i| + 2|B_i|$$

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Pool

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Pool   Random numbers

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Computed by a linear hash function

$$f(x,s,\Delta) = \left(\sum_{j=1}^{k} t_j [y_j(x) \oplus \Delta] + s \sum_{j=k+1}^{2k} t_j [y_{j-k}(x) \oplus \Delta] \right) \bmod p$$

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Pool Random numbers

Why the ECA Algorithm is Well-Founded?

Second Point:

Computing fingerprints of 128 bits with the linear hash functions

$$h'(x) = 10010101110011011010000111000110$$

$$1101110101001000$$

$$0011000111000110$$

$$0000000111010110$$

$$0011111111000110$$

$$1111111111000110$$

$$0000000000000110$$

$$h_0(x) = h'(x)[96{,}127] >> (32 - b)$$

$$y_6(x) = h'(x)[80{,}95]$$

.

.

.

$$y_1(x) = h'(x)[0{,}15]$$

Why the ECA Algorithm is Well-Founded?

Third Point:

How to keep maximum bucket size smaller than *l* = 256?

$$b \leq \log(n) - \log(l / \log l) + O(1)$$

$$l \geq \log n \log \log n$$

The Heuristic ECA Algorithm

- Uses a universal pseudo random hash function proposed by Jenkins (1997):
  - Faster to compute
  - Requires just one random integer number as seed

Experimental Results

- Metrics:
  - Generation time
  - Storage space
  - Evaluation time
- Collection:
  - 1.024 billions of URLs collected from the web
  - 64 bytes long on average
- Experiments
  - Commodity PC with a cache of 4 Mbytes
  - 1.86 GHz, 1 GB, Linux, 64 bits architeture

Generation Time of MPHFs (in Minutes)

n (millions)	32	128	512	1024
Theoretic ECA	1.3 ± 0.002	6.2 ± 0.02	27.6 ± 0.09	57.4 ± 0.06
Heuristic ECA	0.95 ± 0.02	5.1 ± 0.01	22.0 ± 0.13	46.2 ± 0.06

Related Algorithms

- Botelho, Kohayakawa, Ziviani (2005) - BKZ
- Fox, Chen and Heath (1992) – FCH
- Czech, Havas and Majewski (1992) – CHM
- Pagh (1999) - PAGH

All algorithms coded in the same framework

Generation Time

Algorithms	Generation Time (sec)
IRA (r = 3)	6.7 ± 0
Theoretic ECA	9.0 ± 0.3
Heuristic ECA	6.4 ± 0.3
BKZ	12.8 ± 1.6
CHM	17.0 ± 3.2
FCH	2,400.1 ± 711.6
PAGH	42.8 ± 2.4

3,541,615 URLs

Generation Time and Storage Space

Algorithms	Generation Time (sec)	Space (bits/key)
IRA (r = 3)	6.7 ± 0	2.6
Theoretic ECA	9.0 ± 0.3	3.3
Heuristic ECA	6.4 ± 0.3	3.1
BKZ	12.8 ± 1.6	21.8
CHM	17.0 ± 3.2	45.5
FCH	2,400.1 ± 711.6	4.2
PAGH	42.8 ± 2.4	44.2

3,541,615 URLs

Generation Time, Storage Space and Evaluation Time

Algorithms	Generation Time (sec)	Space (bits/key)	Evaluation time (sec)
IRA (r = 3)	6.7 ± 0	2.6	2.1
Theoretic ECA	9.0 ± 0.3	3.3	4.9
Heuristic ECA	6.4 ± 0.3	3.1	2.7
BKZ	12.8 ± 1.6	21.8	2.3
CHM	17.0 ± 3.2	45.5	2.3
FCH	2,400.1 ± 711.6	4.2	1.7
PAGH	42.8 ± 2.4	44.2	2.3

3,541,615 URLs

Key length = 64 bytes

C Minimal Perfect Hashing Library

- Why to build a library?
  - Lack of similar libraries in the free software community
  - Test the applicability of our algorithm out there
- Feedbacks:
  - 1,883 downloads (until Apr 15th, 2008)
  - Incorporated by Debian
- Library address: http://cmph.sourceforge.net

Conclusions

- Three implementations were developed:
  - Theoretic ECA (external memory)
  - Heuristic ECA (external memory)
  - IRA (internal memory, used in ECA algorithm)
- Near space-optimal functions in linear time
- Function evaluation in time O(1)
- First theoretically well-founded algorithm that is practical and will work for every key set from U with high probability

Parallel Version of the ECA Algorithm

PCs	2	4	8	10	14
Speedup	1.8	3.5	7.0	8.7	12.2

1 billion URLs using 14 PCs in 5 minutes