A new approach to Poisson approximation and de-Poissonization

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Outline

Combinatorial scheme

Poisson approximation

- Improvements of Prokhorov's results

Depoissonization

Definition of combinatorial scheme

Let $\{X_n\}_{n \geq n_0}$ be a sequence of random variables. For a wide class of combinatorial problems the probability generating function

$$P_n(w) = \sum_{m=0}^{\infty} \mathbb{P}(X_n = m) w^n$$

satisfies asymptotically

$$P_n(z) = e^{\lambda(z-1)} z^h \left(g(z) + \varepsilon_n(z)\right) \qquad (n \to \infty),$$

where h is a fixed non-negative integer,

- $\lambda = \lambda(n) \to \infty$ with n;
- g is independent of n and is analytic for $|z| \leq \eta$, where $\eta > 1$; $g(1) = 1$ and $g(0) \neq 0$;
- $\varepsilon_n(z)$ satisfies
  $$\varepsilon_n(z) = o(1),$$
  uniformly for $|z| \leq \eta$.

Cauchy formula

$$\mathbb{P}(X_n = m) = \frac{1}{2\pi i} \int_{|z|=r} e^{\lambda(z-1)} \left(g(z) + \varepsilon_n(z)\right) \frac{dz}{z^{n+1}}$$

$$\approx e^{-\lambda} \frac{\lambda^m}{m!} \sum_{j=0}^{k} a_j C_j(\lambda, m) \quad (1)$$

if $g(z) \approx a_0 + a_1(z-1) + a_2(z-1)^2 + \cdots + (z-1)^k$

Charlier polynomials

The Charlier polynomials $C_k(\lambda, m)$ are defined by formula

$$\frac{\lambda^m}{m!} C_k(\lambda, m) = [z^m](z-1)^k e^{\lambda z}, \tag{2}$$

or, equivalently

$$\sum_{m=0}^{\infty} \frac{\lambda^m}{m!} C_k(\lambda, m) z^m = (z-1)^k e^{\lambda z}.$$

Orthogonality relations

Jordan in 1926 proved that Charlier polynomials are orthogonal with respect to Poisson measure $e^{-\lambda}\frac{\lambda^m}{m!}$, that is

$$\sum_{m=0}^{\infty} C_k(\lambda, m)C_l(\lambda, m)e^{-\lambda}\frac{\lambda^m}{m!} = \delta_{k,l}\frac{k!}{\lambda^k},$$

Which means that if a sequence of complex numbers $P_0, P_1, \ldots$ satisfies condition

$$\sum_{j=0}^{\infty} \frac{|P_j|^2}{e^{-\lambda}\frac{\lambda^j}{j!}} < \infty$$

then we can expand

$$P_m = e^{-\lambda}\frac{\lambda^m}{m!}\sum_{j=0}^{\infty} a_j C_j(\lambda, m).$$

Suppose we have a generating function

$$P(z) = \sum_{n=0}^{\infty} P_n z^n$$

then

$$P_m = e^{-\lambda} \frac{\lambda^m}{m!} \sum_{j=0}^{\infty} a_j C_j(\lambda, m).$$

is equivalent to

$$\sum_{n=0}^{\infty} P_n z^n = e^{\lambda(z-1)} \sum_{j=0}^{\infty} a_j (z-1)^j$$

$$P(z) = e^{\lambda(z-1)} f(z).$$

$e^{\lambda(z-1)}$ is a generating function of Poisson distribution. Therefore if

$$P(z) \approx e^{\lambda(z-1)} f(1)$$

we can expect that

$$P_m \approx f(1) e^{-\lambda} \frac{\lambda^m}{m!}.$$

Parseval identity for Charlier polynomials

$$\sum_{m=0}^{\infty} P_m z^m = e^{\lambda(z-1)} f(z) = e^{\lambda(z-1)} \sum_{n=0}^{\infty} a_n (z-1)^n$$

Theorem

Suppose $f(z)$ is analytic in the whole complex plain and $|f(z)| \ll e^{H|z-1|^2}$ as $|z| \to \infty$, then for any $\lambda > 2H$ we have

$$\sum_{n=0}^{\infty} \left| \frac{P_n}{e^{-\lambda} \frac{\lambda^n}{n!}} \right|^2 e^{-\lambda} \frac{\lambda^n}{n!} = \sum_{n=0}^{\infty} \frac{n!}{\lambda^n} |a_n|^2$$

Application of the Parseval identity

$$P(z) = e^{\lambda(z-1)} g(z)$$

Theorem

Suppose $g(z)$ is analytic in the whole complex plane and

$$|g(z)| \leqslant A e^{H|z-1|^2}, \tag{3}$$

for all $z \in \mathbb{C}$ with some positive constants A and H. Then uniformly for all $N, n \geqslant 0$ and $\lambda \geqslant (2+\epsilon)H$ with $\epsilon > 0$ we have

$$\left| P_n - e^{-\lambda} \frac{\lambda^n}{n!} \left(\sum_{j=0}^{N} a_j C_j(\lambda, n) \right) \right| \leqslant A \frac{\left((2+\epsilon)H\right)^{(N+1)/2}}{\lambda^{(N+2)/2}}$$

Theorem

Under the conditions of the previous theorem

$$\sum_{n=0}^{\infty}\left|P_n - e^{-\lambda}\frac{\lambda^n}{n!}\sum_{j=0}^{N} a_j C_j(\lambda, n)\right| \leqslant A\frac{\big((2+\epsilon)H\big)^{(N+1)/2}}{\lambda^{(N+1)/2}}$$

for all $n, N \geqslant 0$.

Parseval identity for Charlier polynomials. Integral form.

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$$\sum_{n=0}^{\infty} \left| \frac{P_n}{e^{-\lambda} \frac{\lambda^n}{n!}} \right|^2 e^{-\lambda} \frac{\lambda^n}{n!} = \int_0^{\infty} I(\sqrt{r/\lambda}) e^{-r} \, dr,$$

where

$$I(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(1 + re^{it})|^2 \, dt.$$

Consequences of the Parseval identity

Suppose

$$P(z) = \sum_{n=0}^{\infty} P_n z^n.$$

$$I(P,\lambda;r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |P(1+re^{it})e^{-\lambda re^{it}}|^2\,dt.$$

Theorem

$$\sum_{n=0}^{\infty} |P_n| \leqslant \left(\int_0^{\infty} I(P,\lambda;\sqrt{r/\lambda})e^{-r}\,dr\right)^{1/2} \qquad (4)$$

and

$$|P_n| \leqslant \frac{1}{\sqrt{\lambda}}\left(\int_0^{\infty} I(P,\lambda;\sqrt{r/\lambda})re^{-r}\,dr\right)^{1/2}\sqrt{Z(n)}, \qquad (5)$$

for all $n \geqslant 0$ *and*

$$Z(n) \leqslant e^{-\frac{(m-\lambda)^2}{2(m+\lambda)}}$$

Further inequalities

Theorem

If we additionally assume that $P(1) = 0$, then

$$\sum_{n=0}^{\infty} |P_0 + P_1 + \cdots + P_n| \leqslant \sqrt{\lambda} \left(\int_0^{\infty} I(P, \lambda; \sqrt{r/\lambda}) r^{-1} e^{-r} \, dr \right)^{1/2}, \tag{6}$$

and

$$|P_0 + P_1 + \cdots + P_n| \leqslant \left(\int_0^{\infty} I(P, \lambda; \sqrt{r/\lambda}) e^{-r} \, dr \right)^{1/2} \sqrt{Z(n)} \tag{7}$$

for all $n \geqslant 0$.

Generalized binomial distribution

Suppose

$$S_n = I_1 + I_2 + \cdots + I_n, \tag{8}$$

where the X_j's are independent Bernoulli random variables with

$$\mathbb{P}(I_j = 1) = 1 - \mathbb{P}(I_j = 0) = p_j.$$

Then

$$\sum_{0 \le m \le n} \mathbb{P}(S_n = m) z^m = \prod_{1 \le j \le n} (1 + p_j(z-1)) = e^{\lambda(z-1)} g(z).$$

We will use notation

$$\lambda = p_1 + p_2 + \cdots + p_n.$$

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Example of application to Poisson approximation

$$\theta := \frac{p_1^2 + p_2^2 + \cdots + p_n^2}{p_1 + p_2 + \cdots + p_n}, \quad \text{and} \quad \lambda := p_1 + p_2 + \cdots + p_n$$

Theorem

Suppose $\theta < 1$ then the following inequalities hold

$$\sum_{m=0}^{\infty} \left| \frac{\mathbb{P}(S_n = m)}{e^{-\lambda}\frac{\lambda^m}{m!}} - 1 \right|^2 e^{-\lambda}\frac{\lambda^m}{m!} \leqslant \frac{e}{2}\frac{\theta^2}{(1-\theta)^3},$$

$$\frac{1}{2}\sum_{m=0}^{\infty} \left| \mathbb{P}(S_n = m) - e^{-\lambda}\frac{\lambda^m}{m!} \right| \leqslant \frac{\sqrt{e}}{2^{3/2}}\frac{\theta}{(1-\theta)^{3/2}}$$

Since $\sqrt{e}/2^{3/2} = 0.582\ldots$ the bound of the above theorem could be sharper than that of Barbour-Hall inequality if $\theta \leqslant 0.3$ and λ is large enough.

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Kolmogorov distance

$$\theta := \frac{p_1^2 + p_2^2 + \cdots + p_n^2}{p_1 + p_2 + \cdots + p_n}, \quad \text{and} \quad \lambda := p_1 + p_2 + \cdots + p_n$$

Theorem

Whenever $\theta < 1$ we have

$$\left| \mathbb{P}(S_n \leqslant j) - \sum_{m \leqslant j} e^{-\lambda} \frac{\lambda^m}{m!} \right| \leqslant \frac{\sqrt{e}}{2^{1/2}} \frac{\theta}{(1-\theta)^{3/2}} \sqrt{Z(j)},$$

where

$$Z(n) = \min \left\{ \sum_{j \leqslant n} e^{-\lambda} \frac{\lambda^m}{m!}, \sum_{j > n} e^{-\lambda} \frac{\lambda^m}{m!} \right\} \leqslant e^{-\frac{(m-\lambda)^2}{2(m+\lambda)}}$$

Compound poisson distribution

$$\lambda_3 := p_1^3 + p_2^3 + \cdots + p_n^3$$

Theorem

Suppose $\theta < 1/3$ *then*

$$\sum_{m=0}^{\infty} \left| \mathbb{P}(S_n = m) - [z^m] \left[e^{\lambda(z-1) - \frac{\lambda_2}{2}(z-1)^2} \right] \right| \leqslant \frac{\lambda_3}{\lambda^{3/2}} \sqrt{\frac{2e}{3}} \frac{1}{(1-3\theta)^2},$$

$$\left| \mathbb{P}(S_n = m) - [z^m] \left[e^{\lambda(z-1) - \frac{\lambda_2}{2}(z-1)^2} \right] \right| \leqslant \frac{\lambda_3}{\lambda^2} \sqrt{\frac{8e}{3}} \frac{\sqrt{Z(m)}}{(1-3\theta)^{5/2}}.$$

Generalized binomial distribution in combinatorics

Can be used if the discrete random variable X_n is Bernoulli decomposable

$$X_n = I_1 + I_2 + \cdots + I_n$$

This happens if a probability generating function $F_n(z)$ of a discrete random variable X_n is a polynomial whose root are real and negative

Example

- Hypergeometric distribution.
- Number of cycles in a random permutation

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Advantages and disadvantages of this approach

Advantages

- Quick proofs.
- Very accurate explicit constants.
- Non-uniform estimates for distribution functions.

Disadvantage

- The generating function $P(z)$ should be defined on all complex pane and satisfy condition

$$P(1+z) \ll e^{\lambda|z|^2}$$

for some λ.

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Improvements of Prokhorov's results

Depoissonization

Prokhorov's theorem

Suppose $\mathcal{B}(n,p)$– Bernoulli distribution. If $npq \to \infty$ then

$$\mathcal{B}(n,p) \to \mathcal{N}(\sqrt{pqn}, pn)$$

If np is not very large then

$$\mathcal{B}(n,p) \to \mathcal{P}(pn)$$

Prokhorov in 1953 proved

$$\frac{1}{2}\sum_{j\geq 0}\left|\binom{n}{j}p^j(1-p)^{n-j} - e^{-np}\frac{(np)^j}{j!}\right|$$
$$= \frac{p}{\sqrt{2\pi e}}\left(1 + O\left(\min(1, p + (np)^{-1/2})\right)\right)$$

Further refinements of Prokhorov's result

Later Le Cam in 1960 proved that if probabilities p_j satisfy condition $\max_{1 \leqslant j \leqslant n} p_j \leqslant 1/4$ we have

$$d_{TV}(S_n, \mathcal{P}(\lambda)) = \frac{1}{2} \sum_{j \geq 0} \left| P(S_n = j) - e^{-\lambda} \frac{\lambda^j}{j!} \right| \leqslant 8 \frac{\lambda_2}{\lambda}.$$

Kerstan in 1964 later sharpened the constant in Le Cam's inequalities proving that whenever $\max_{1 \leqslant j \leqslant n} p_j \leqslant 1/4$ we have

$$d_{TV}(S_n, \mathcal{P}(\lambda)) \leqslant 1.05 \frac{\lambda_2}{\lambda}$$

Barbour-Hall inequality

Finally Barbour and Hall 1984 applying Stein-Chen's method established their famous inequality

$$\frac{1}{2}\sum_{j\geq 0}\left|P(S_n = j) - e^{-\lambda}\frac{\lambda^j}{j!}\right| \leqslant (1 - e^{-\lambda})\theta,$$

where as before

$$\theta = \frac{\lambda_2}{\lambda}$$

Let us denote

$$d_{TV}^{(\alpha)}(\mathcal{L}(S_n), Po(\lambda_1)) = \frac{1}{2}\sum_{m=0}^{\infty}\left|P(S_n = m) - e^{-\lambda}\frac{\lambda^m}{m!}\right|^{\alpha}.$$

Theorem

Suppose $\theta := \frac{\lambda_2}{\lambda_1} = o(1)$ *and* $\lambda_1 \to \infty$ *then*

$$d_{TV}^{(\alpha)}(\mathcal{L}(S_n), Po(\lambda_1)) = \frac{\theta^{\alpha}\lambda_1^{\frac{1-\alpha}{2}}}{2^{\alpha+1}(2\pi)^{\alpha/2}}\left(J^{(\alpha)}(\theta) + O\left(\frac{1}{\lambda_1^{(\alpha+1)/2}} + \frac{1}{\lambda_1}\right)\right),$$

where $J^{(\alpha)}(\theta)$ *is the is an explicitly defined function.*

Depoissonization

$$G(z) = e^{-z} \sum_{m=0}^{\infty} \frac{g_m}{m!} z^m$$

If $G(z)$ is analytic in circle $|z - n| < n + \epsilon$ where $\epsilon > 0$ then

$$g_n = \sum_{j=0}^{\infty} \frac{G^{(j)}(n)}{j!} n^j C_j(n, n)$$

How close is $G(n)$ to g_n?

Inequality estimating closeness of de-Poissonization

$$G(z) = e^{-z} \sum_{m=0}^{\infty} \frac{g_m}{m!} z^m$$

Theorem

$$\left| g_n - \sum_{j=0}^{k} \frac{G^{(j)}(n)}{j!} n^j C_j(n,n) \right| \leqslant c(n) \left(\sum_{j=k+1}^{\infty} \frac{|G^{(j)}(n)|^2 (j+1)}{j!} n^j \right)^{1/2}$$

Example

Suppose g_n is the mean value of number of steps in exhaustive search algorithm that is needed to find a maximum independent set in a random graph

$$G'(z) = G(pz) + e^{-z} \quad \text{with} \quad p < 1$$

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Integral form of depoissonization inequality

$$G(z) = e^{-z} \sum_{m=0}^{\infty} \frac{g_m}{m!} z^m$$

Theorem

$$|g_n - G(n)| \leqslant c(n) \left(\int_0^{\infty} r e^{-r} \int_{-\pi}^{\pi} |G(n + e^{it}\sqrt{rn}) - G(n)|^2 \, dt \, dr \right)^{1/2}$$

here

$$c(n) := \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{4\pi n}} \to \frac{1}{\sqrt{2}}, \quad as \quad n \to \infty$$

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Comparison with the results of Jacket and Spankowsky

This form of the depoissonization inequality is consistent with a general theorem of Jacket and Spankowsky of 1998.

Theorem (basic depoissonization lemma)

If for $|\arg z| \leqslant \theta > 0$

$$|G(z)| \ll |z|^{\beta}$$

and for $|\arg z| > \theta$

$$|G(z)e^{z}| \ll \exp(\alpha|z|)$$

then

$$g_n = G(n) + O(n^{\beta-1/2})$$

Generalization of the de-Poissonization inequality

$$G(z) = e^{-z} \sum_{m=0}^{\infty} \frac{g_m}{m!} z^m$$

Theorem

$$\left| g_n - \sum_{j=0}^{k} \frac{G^{(j)}(n)}{j!} n^j C_j(n,n) \right|$$

$$\leqslant c(n) \left(\int_0^\infty r e^{-r} \int_{-\pi}^{\pi} \left| G(n + e^{it}\sqrt{rn}) - \sum_{j=0}^{k} \frac{G^{(j)}(n)}{j!} (e^{it}\sqrt{rn})^j \right|^2 dt\, dr \right)$$

Generalizations

Suppose

$$F(z) = \sum_{x=0}^{n} f_x z^x = (p + zq)^n g(z)$$

where $p + q = 1$ and $0 < p < 1$.
Similar approach can be used applying Parseval identity for Kravchuk polynomials.
This can be useful for

- analyzing the distribution of the digit sum function
- approximation of generalized binomial distribution by simple binomial distribution

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For Further Reading I

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