Fast Forward Property
and
Decompositions of Graph of Functions

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Fast forward property

Definition **Let** $V = \{0, 1, ..., N-1\}$**. A function** $f : V \to V$ **is fast forward if for each** $m \in \mathbb{N}$ **and each** $x \in V$ **the computational complexity of evaluating the** m**th iterate of** f **at** x**,**

$$f^m(x), \quad \textbf{(e.g. } f^2(x) = f\left(f(x)\right)\textbf{)},$$

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is small (polynomial in $\log N$**).**

Example **The cyclic permutation** $(0, 1, 2, ..., N-1)$ **is fast forward since** $f^m(x) = x + m$ **mod** N**.**

Applications

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Example

Rho methods for factorization (Pollard, 1975).

There compute $f^m(x)$ repeatedly for the function $f(x) = x^2 - 1$ mod N.

The context

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Why?

In the computation of the iteration $f^m(x)$, we need to use $O(N)$ information which is generated in preprocess and stored in memory.

Preliminary

(Recall definitions and results in Tsaban 2003, 2007)

Fast forward permutation

Definition **The fast forward permutation coded by** $(m_0, m_1, ..., m_{\ell-1})$ **is**

$$\pi = \underbrace{(0, ..., s_0 - 1)}_{m_0} \underbrace{(s_0, ..., s_1 - 1)}_{m_1} \cdots \underbrace{(s_{\ell-2}, ..., N - 1)}_{m_{\ell-1}},$$

where $s_i = m_0 + \cdots + m_i$ **for** $0 \leq i \leq \ell - 2$**.**

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Computation of the iteration

$$\pi^m(x) = s_{i(x)} + \left(x - s_{i(x)} + m \bmod \left(s_{i(x)+1} - s_{i(x)}\right)\right),$$

where $s_{i(x)} \leq x < s_{i(x)+1}$ **and the assignments** $x \longrightarrow i(x)$ **and** $i \longrightarrow s_i$ **are preprocessed as lookup tables.**

Arbitrary permutation

For any permutation

$$f = \underbrace{(b_0, b_1, ..., b_{s_0-1})}_{m_0} \underbrace{(b_{s_0}, ..., b_{s_1-1})}_{m_1} \cdots \underbrace{(b_{s_{\ell-2}}, ..., b_{N-1})}_{m_{\ell-1}},$$

define $\sigma(x) = b_x$ **for** $x \in V$**. Then**

$$f = \sigma \circ \pi \circ \sigma^{-1} \quad \textbf{and} \quad f^m = \sigma \circ \pi^m \circ \sigma^{-1}.$$

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$$f = \sigma \circ \pi \circ \sigma^{-1} \quad \textbf{and} \quad f^m = \sigma \circ \pi^m \circ \sigma^{-1}.$$

Thus, f **is fast forward if lookup tables for** σ**,** σ^{-1}**,** $i(x)$ **and** s_i **are stored in memory.**

Fast forward function

Definition **The fast forward function coded by** $(m_0, m_1, ..., m_{\ell-1})$ **and an auxiliary sequence** $(p_0, ..., p_{\ell-1})$**,** $0 \leq p_i < s_i$**, is**

$$\theta(x) = \begin{cases} p_i & \text{if } x = s_i - 1, \\ \pi(x) & \text{otherwise,} \end{cases}$$

where π **is the fast forward permutation codes by** $(m_0, m_1, ..., m_{\ell-1})$**.**

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where π is the fast forward permutation codes by $(m_0, m_1, ..., m_{\ell-1})$.

Note that a cycle of π, $(s_{i-1}, ..., s_i - 1)$, is connected to a previous cycle by the mapping $\theta(s_i - 1) = p_i$ if $p_i < s_{i-1}$. We call this a descent.

Evaluating $\theta^m(x)$

Case 1. if m is small ($s_{i(x)} \leq x + m < s_{i(x)+1}$), then

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Case 2. if $x + m \geq s_{i(x)+1}$ and $p_{i(x)+1} \geq s_{i(x)}$. Then

$$\theta^m(x) = p_{i(x)+1} + \Big((x + m - s_{i(x)+1}) \textbf{ mod } (s_{i(x)+1} - p_{i(x)+1}) \Big) .$$

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Case 3. if $x + m \ge s_{i(x)+1}$ and $p_{i(x)+1} < s_{i(x)}$. Then $\theta^m(x)$ is computed recursively by

$$\theta^m(x) = \theta^{x+m-s_{i(x)+1}}(p_{i(x)+1})$$

(here $\theta^0 = I$). (case of descent)

The complexity

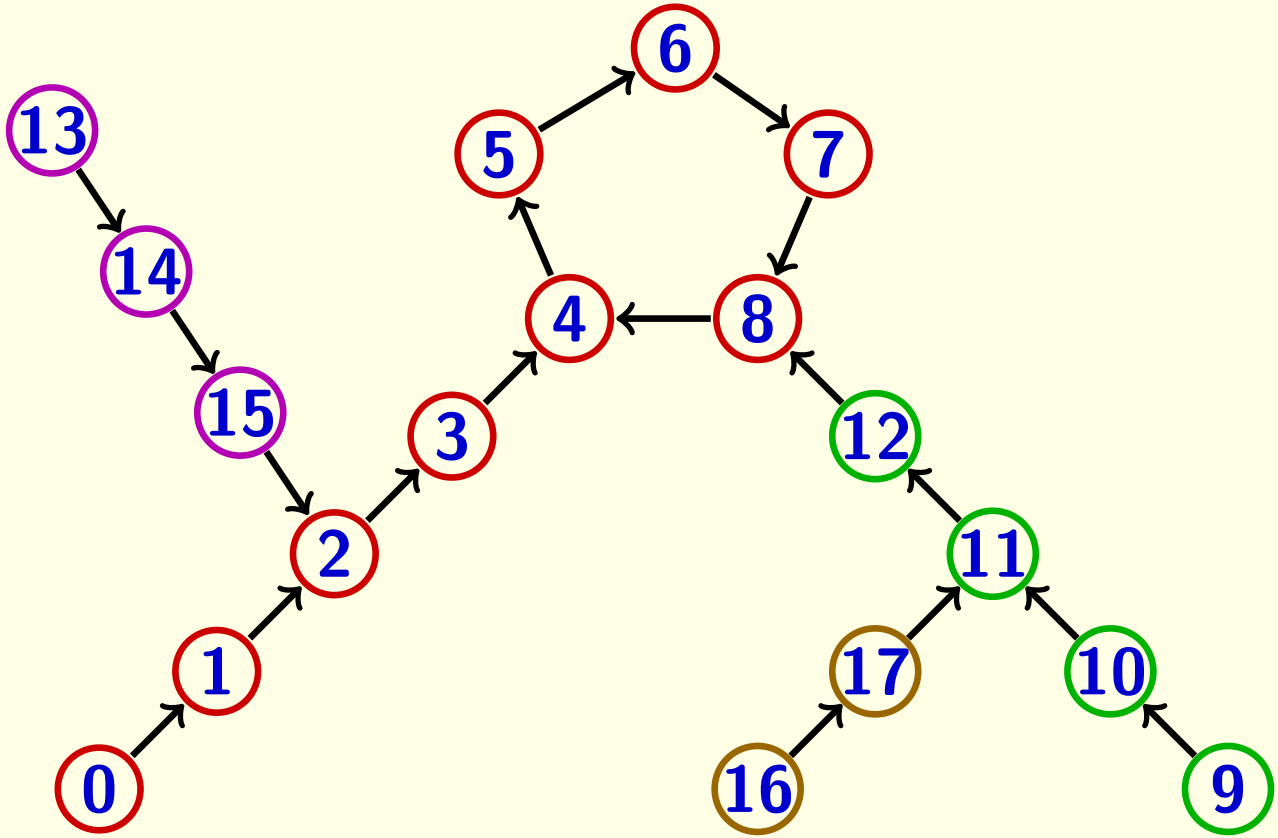
Theorem **(Tsaban, 2007)**

The complexity of evaluating $\theta^m(x)$ is measured by the number of descents (recursions) on the tour $x, \theta(x), \theta^2(x), \theta^3(x), ...$

Notation of the number of descents: $D_\theta(x)$.

Example for number of descents

$D_\theta(x) = 0$ **for** $x = 0, ..., 8$**;**



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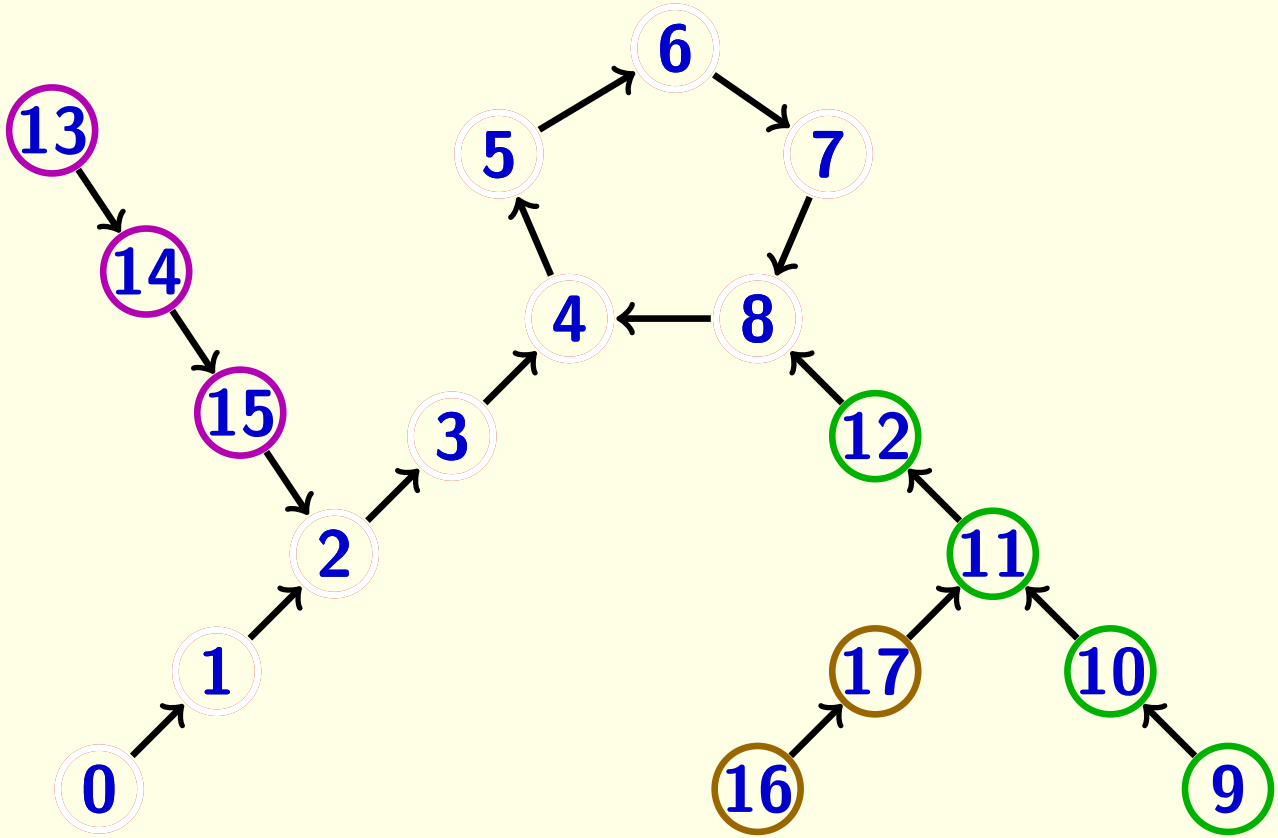

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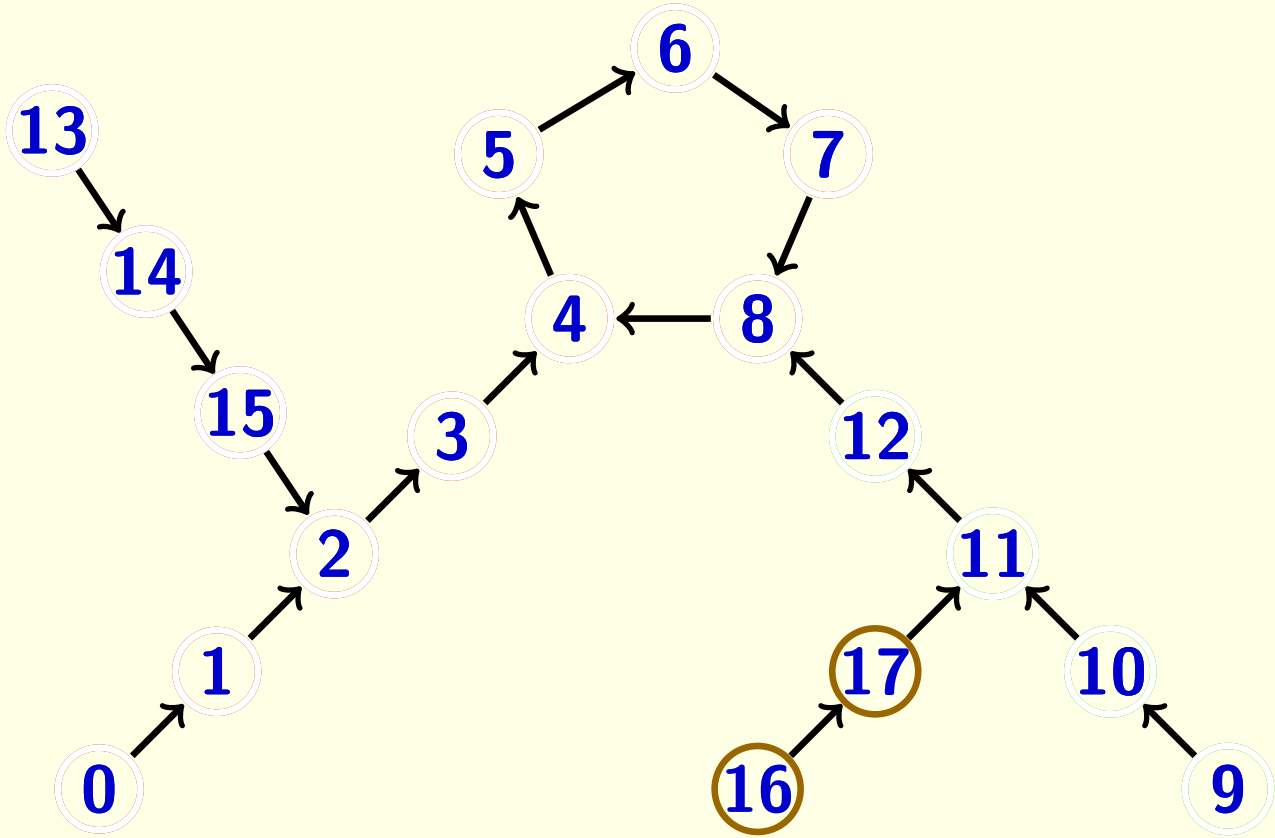
$D_\theta(x) = 2$ **for** $x = 16, 17$**.**



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If there is a fast forward function θ such that

$$f = \sigma \circ \theta \circ \sigma^{-1},$$

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$$f^m = \sigma \circ \theta^m \circ \sigma^{-1}$$

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How to find such a θ?

An approach proposed by Tsaban

Orbit decomposition

Definition **The orbit of an element x in $U \subset V$ is the simple tour $\{x, f(x), f^2(x), ..., f^k(x)\}$ such that $f^{k+1}(x) = f^i(x)$ for some $0 \leq i \leq k$ or $f^{k+1}(x) \notin U$.**

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Definition **An orbit decomposition of f is $C_0, C_1, ..., C_{\ell-1}$ defined as follows: C_0 is an orbit in V and C_i is an orbit in $V - C_0 \cup \cdots \cup C_{i-1}$ for $i > 0$.**

Given an arbitrary function f, and

$$\underbrace{(b_0, b_1, ..., b_{s_0-1})}_{C_0} \quad \underbrace{(b_{s_0}, ..., b_{s_1-1})}_{C_1} \quad \ldots \quad \underbrace{(b_{s_{\ell-2}}, ..., b_{N-1})}_{C_{\ell-1}}$$

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is an orbit decomposition of f. Then

$$f = \sigma \circ \theta \circ \sigma^{-1},$$

where $\sigma(x) = b_x$ and θ is the fast forward function coded by $(|C_0|, |C_1|, ..., |C_{\ell-1}|)$ and the auxiliary sequence $(p_0, ..., p_{\ell-1})$ such that $f(b_{s_i-1}) = b_{p_i}$.

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$9 \to 8 \to 7 \to 6 \to 5 \to 4 \to 3 \to 2 \to 1 \to 0 \circlearrowleft$

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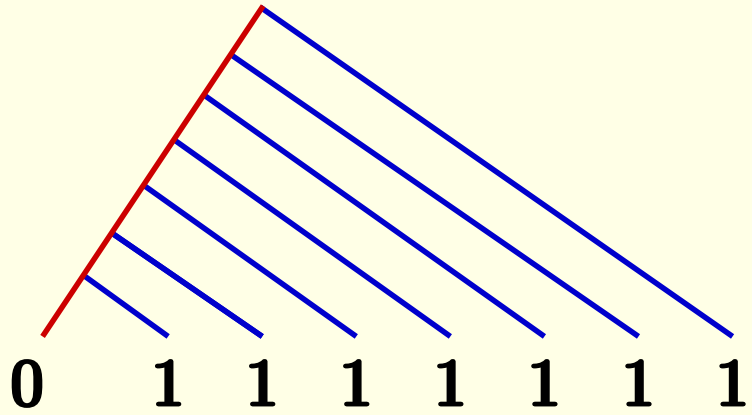
... or even the worst

$(0)(1)(2)(3)(4)(5)(6)(7)(8)(9)$**,** $\quad D_\theta(9) = 9$**.**

Example **Two different orbit decompositions:**

(here the numbers indicate the numbers of descents)



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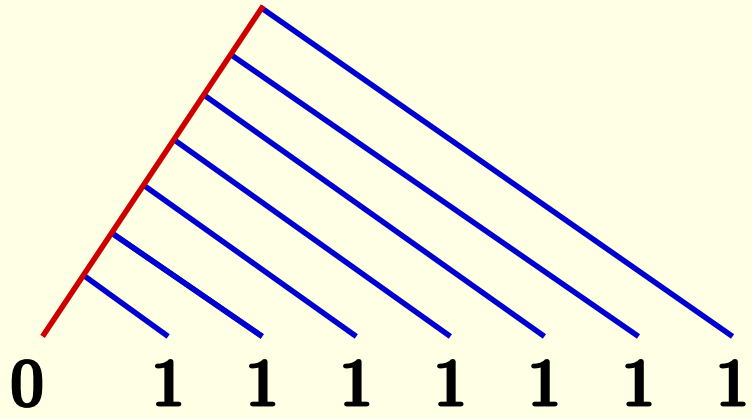

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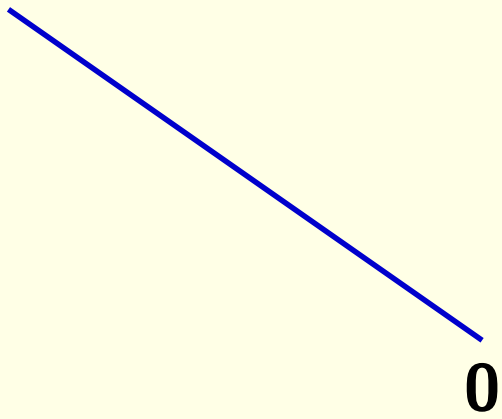

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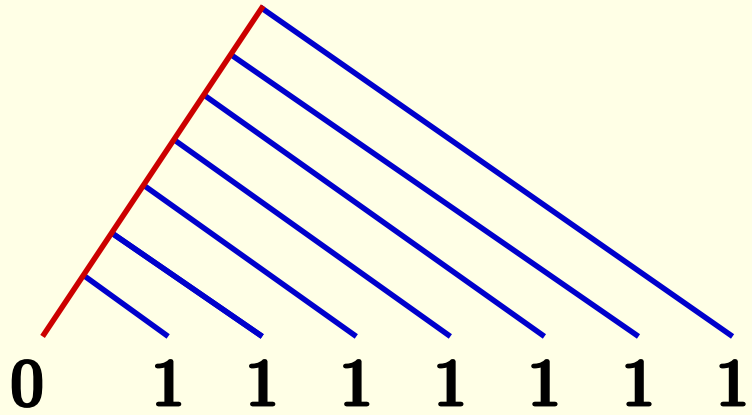

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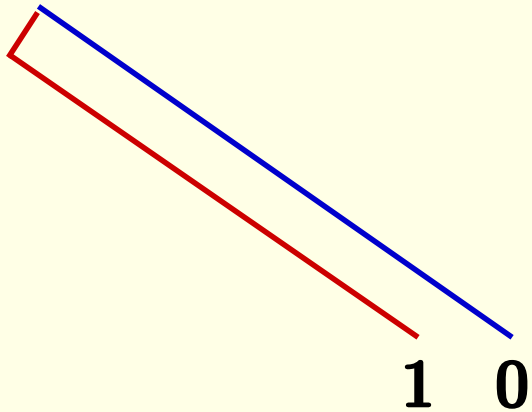

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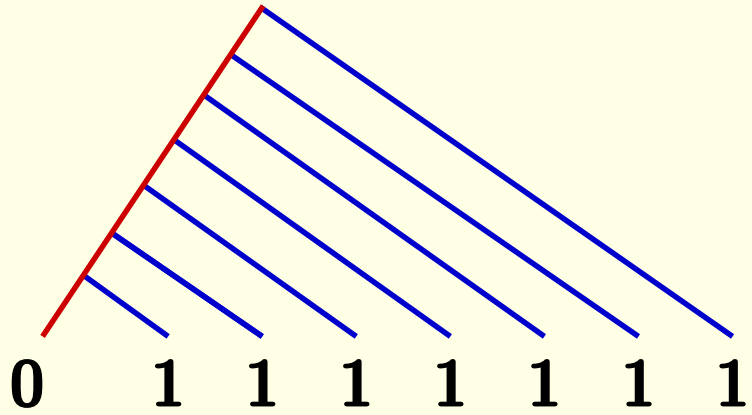

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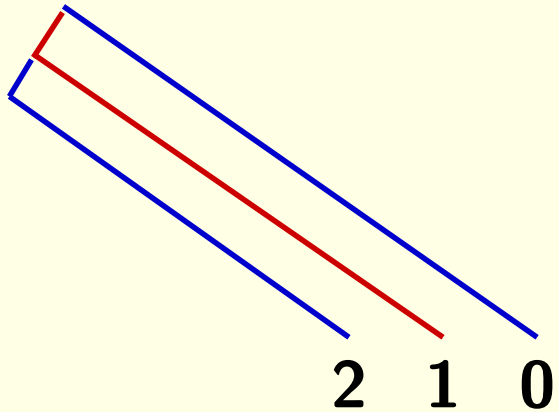

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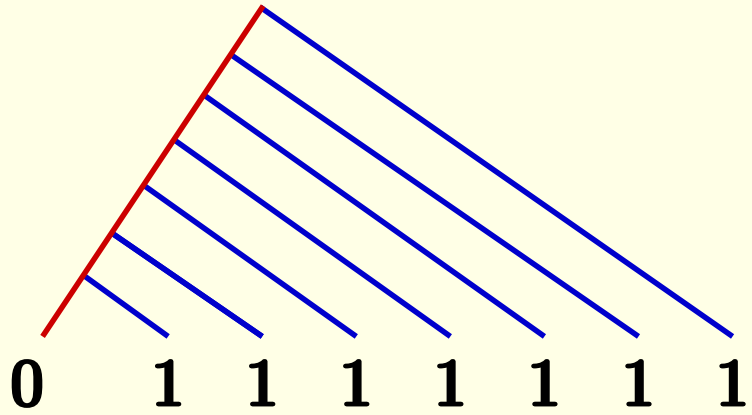

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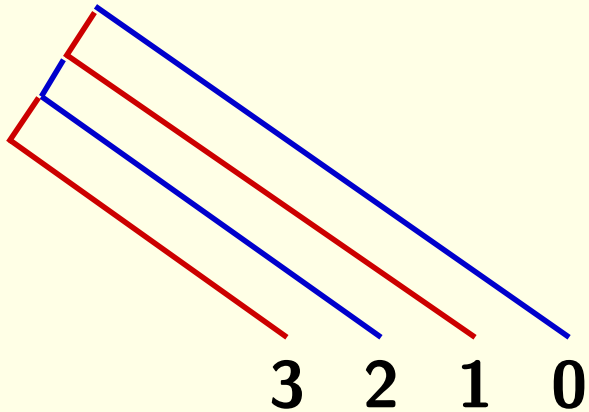

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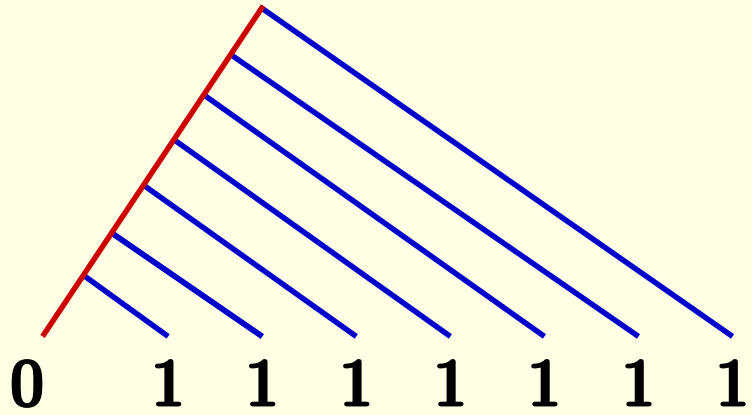

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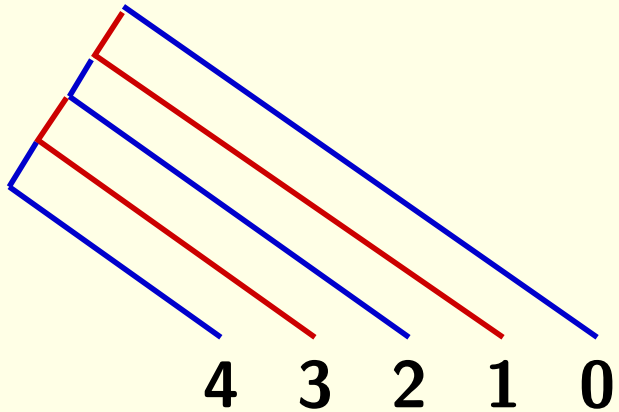

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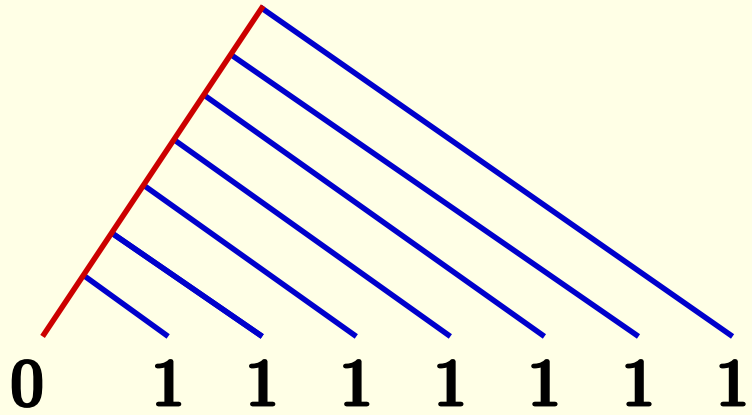

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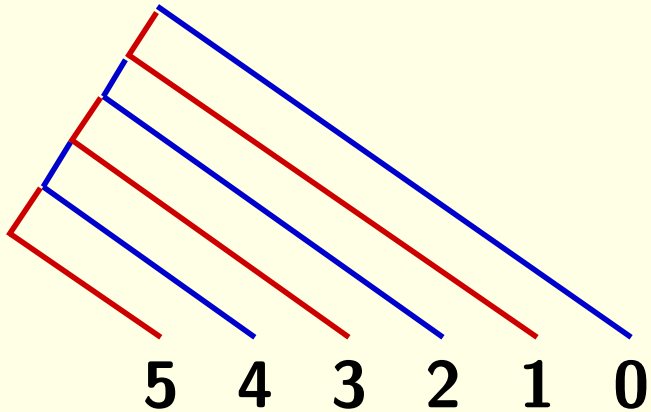

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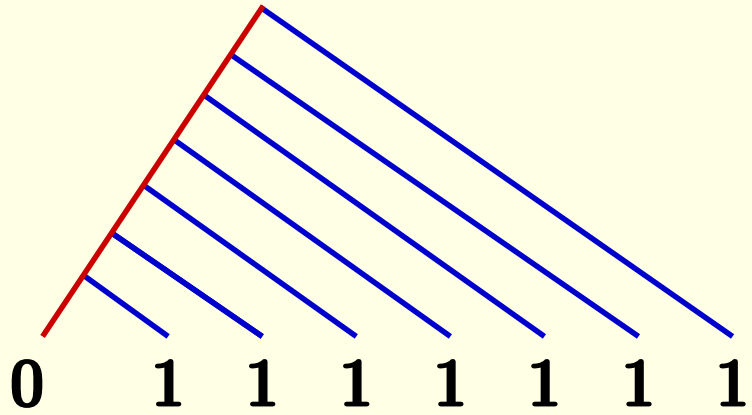

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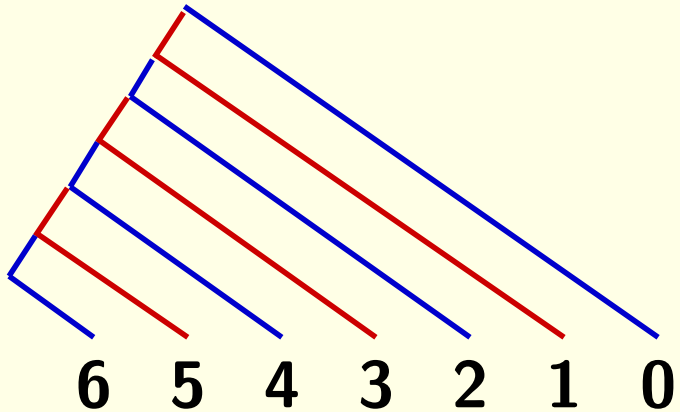

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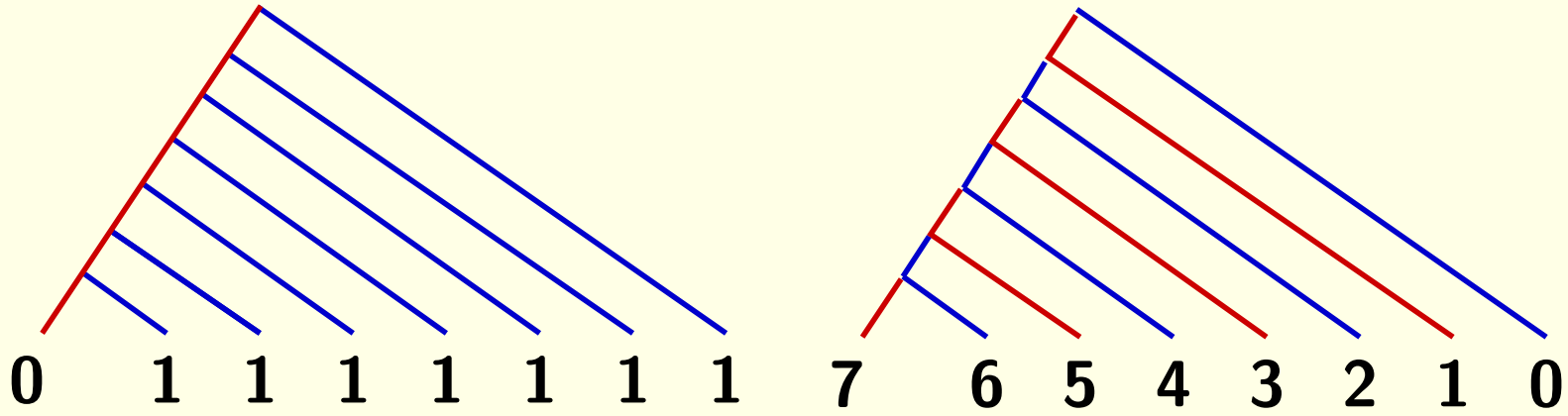

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Greedy orbit decomposition

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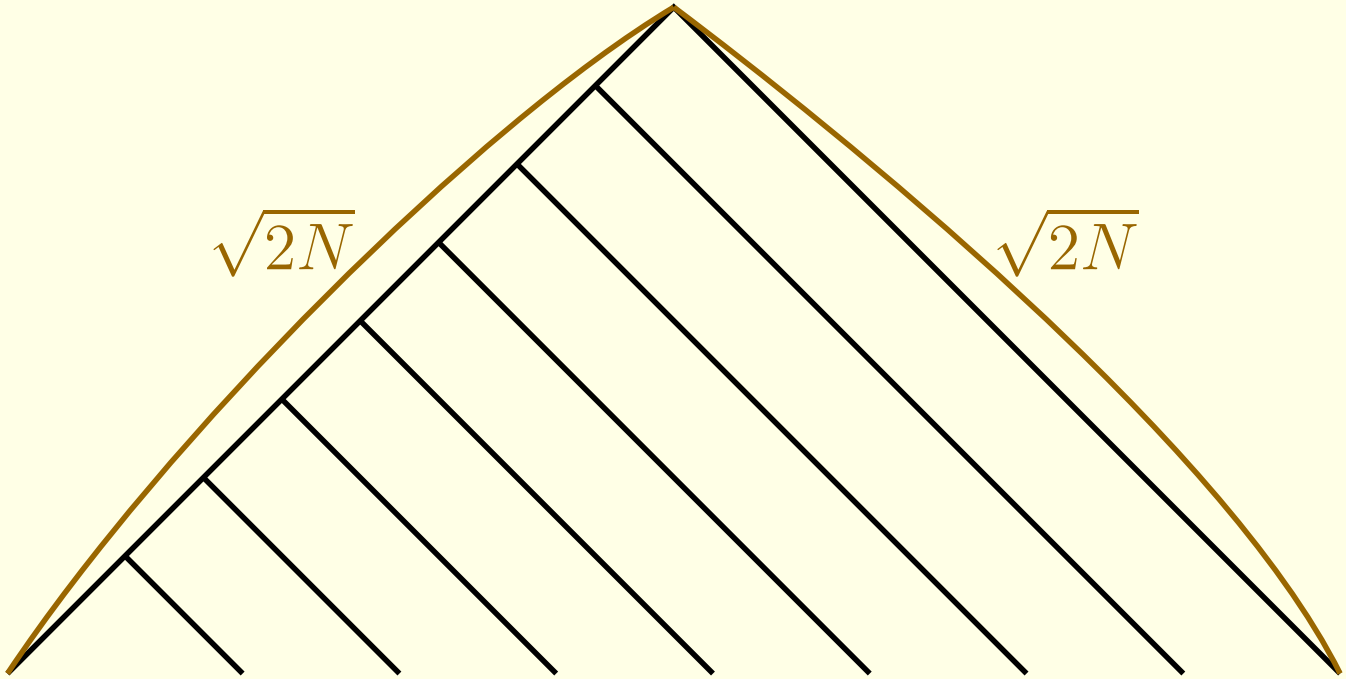
The greedy orbit decomposition of f is $C_0, C_1, ..., C_{\ell-1}$ defined as follows:

C_0 is the maximal length orbit in V and

C_i is the maximal length orbit in $V - C_0 \cup \cdots \cup C_{i-1}$ for $i > 0$.

The worst case

The maximal number of descents for greedy orbit decomposition is about $\sqrt{2N}$.



The average case

The experiment in Tsaban 2007 indicates that the expected number of descents for the greedy orbit decomposition is about

$$\frac{\log_2 N}{5}$$

for random functions and random points.

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2. "Finding an approach which reduces the average number of descents seems to be of great practical interest." That is to find a θ which minimizes

$$\sum_{x \in V} D_\theta(x).$$

New Results

Orbit: graphical point of view

The graph of a function is a disjoint union of components of graph with one cycle.

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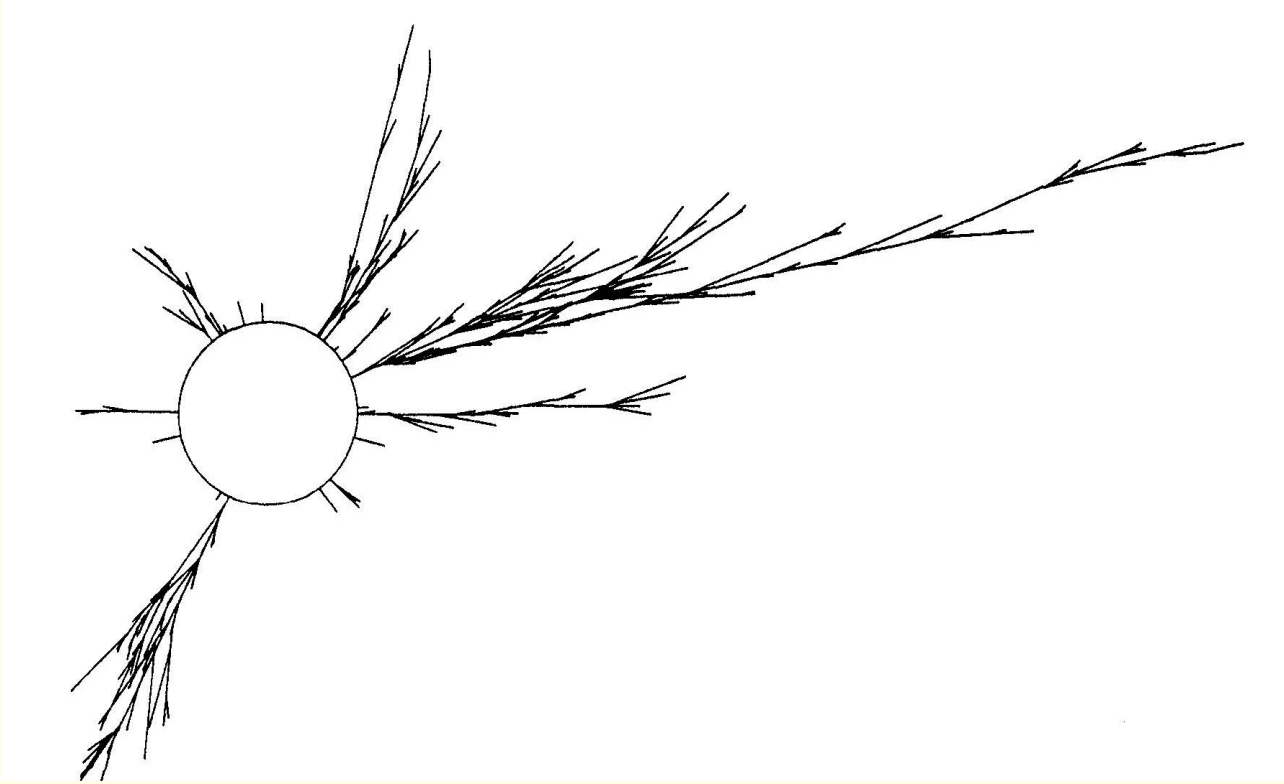
The first orbit in an original component ends on the circle. When we erase an orbit from a component, the remainder is a disjoint union of components of rooted tree.

Typical graph of function

a giant component

rendered by Quisquater and Delescaille 1988.



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Erase an orbit from each component each time.

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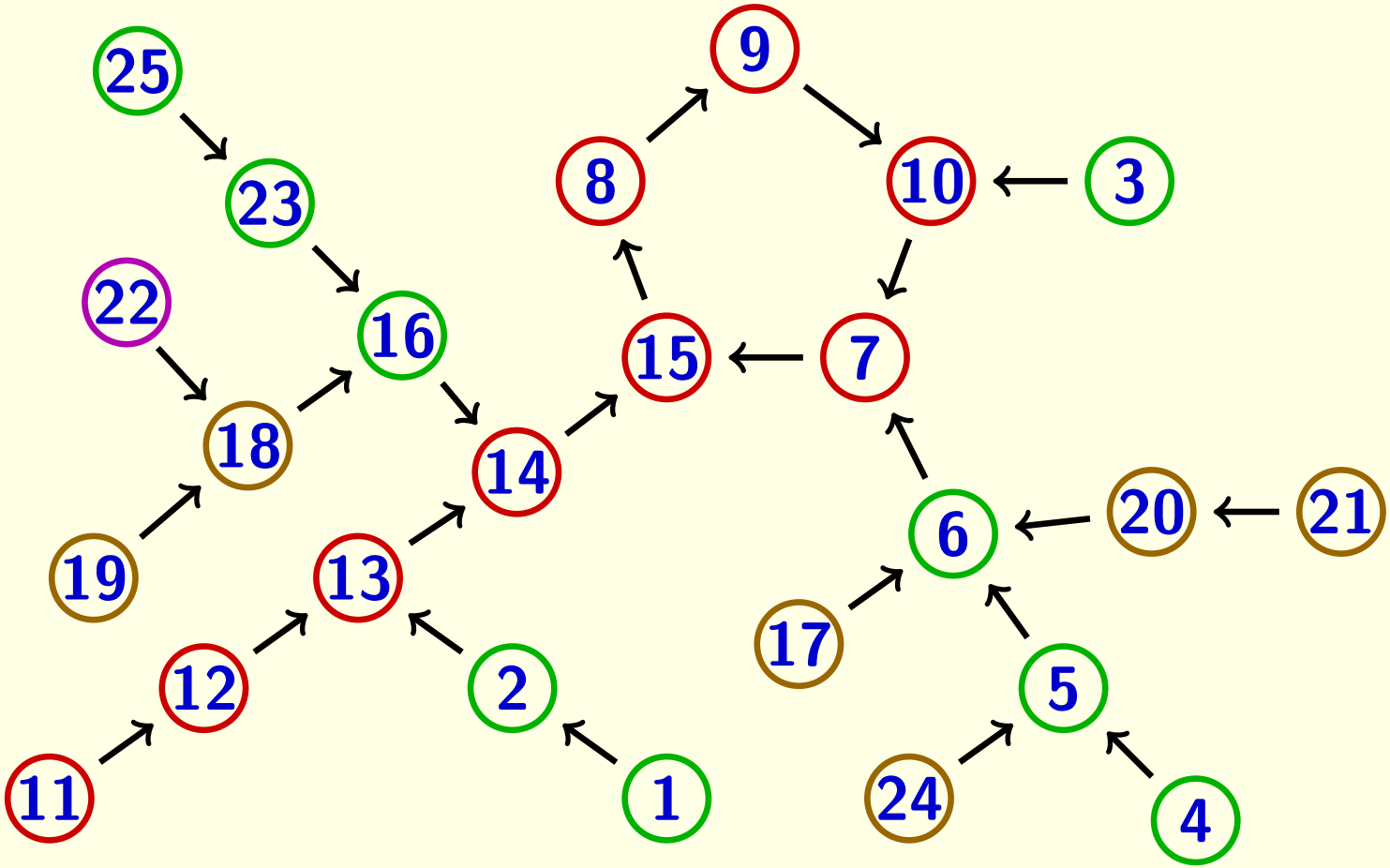
The number of descents only depends on the decomposition of a graph (independent of the order of components).

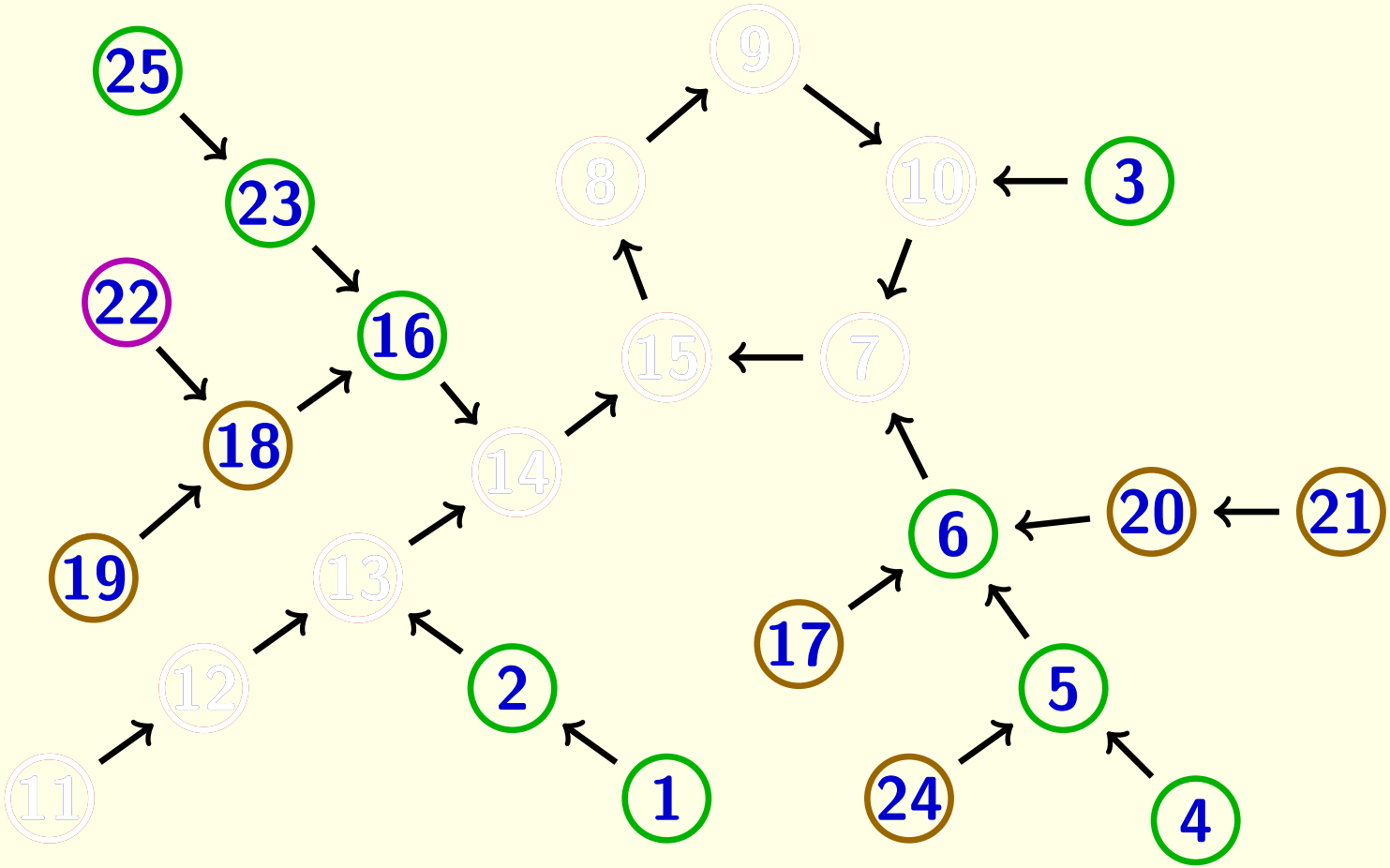
Example for numbers of descents

red: 0



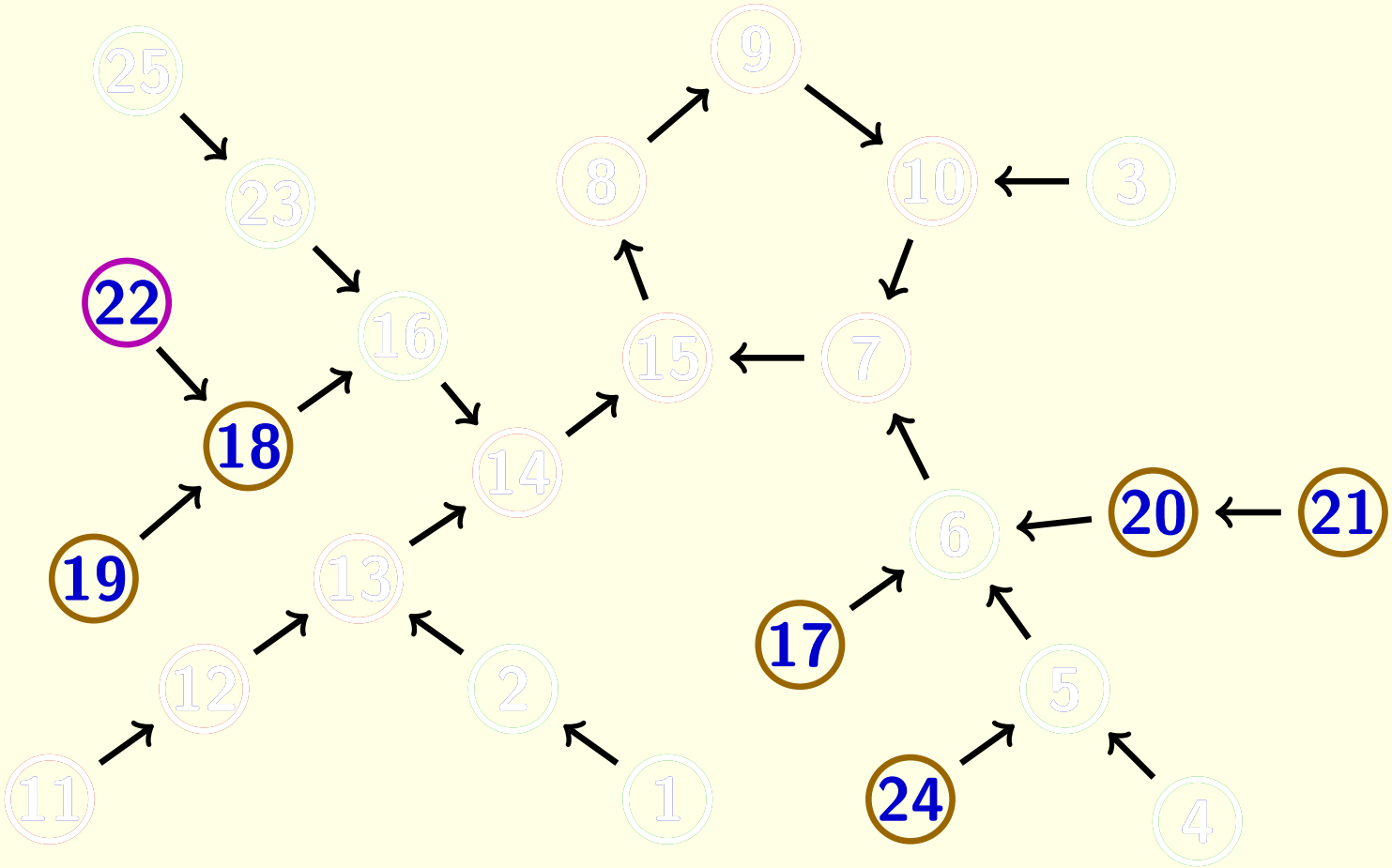
Example for numbers of descents

red: 0 **green**: 1



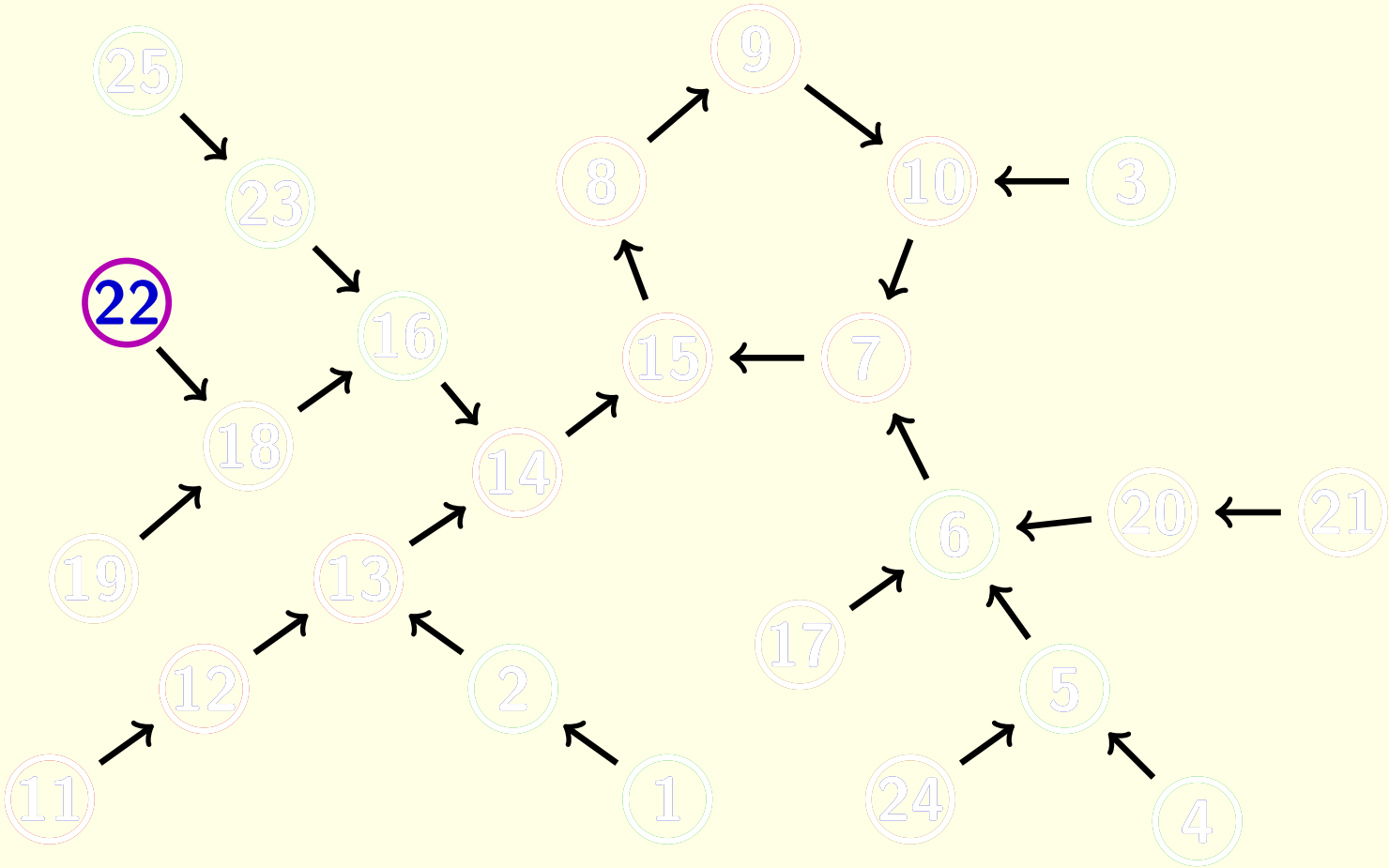
Example for numbers of descents

red: 0 **green**: 1 **brown**: 2



Example for numbers of descents

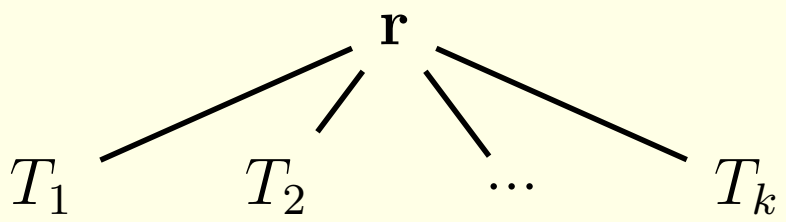
red: 0 **green: 1** **brown: 2** **purple: 3**



Now, the problem is to find a decomposition of a tree that minimizes the number of descents.

Bottom-up construction

Lemma 1 **Let T be a rooted tree:**



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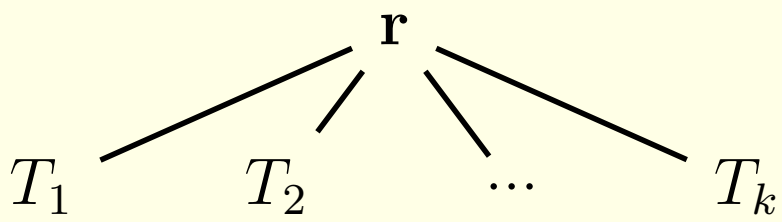
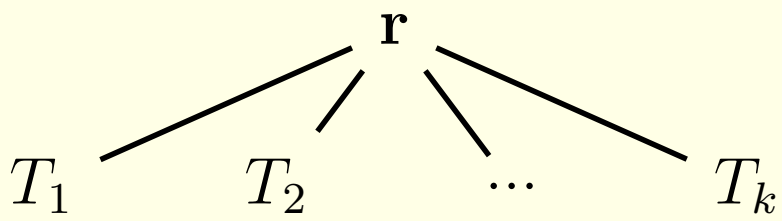


Give each T_i a decomposition. Let $D_{old}(x)$ be the number of descents of x.

Now, extend the decomposition to T: r and the root of T_a in a same orbit. Then $D_{new}(\mathrm{r}) = 0$ and

$$D_{new}(x) = \begin{cases} D_{old}(x), & \text{if } x \in T_a, \\ D_{old}(x) + 1, & \text{otherwise.} \end{cases}$$

Which one is the best connection?



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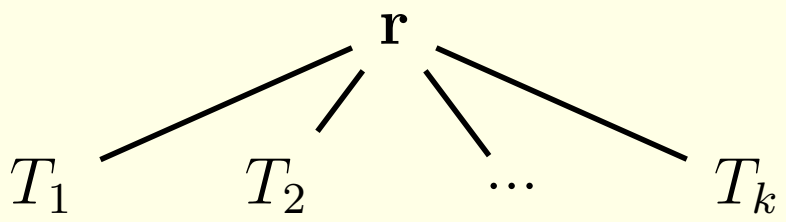

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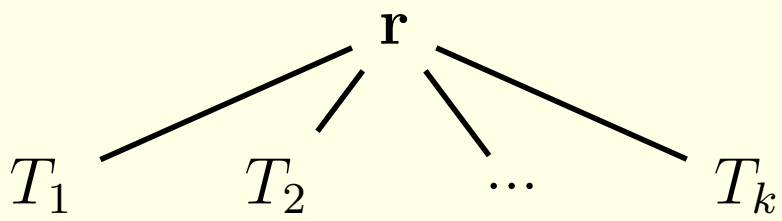

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The function characterized by above rule has been studied in other areas.

The Horton-Strahler number

It's originally used to classify river system. Later also appeared in computer science as register function.

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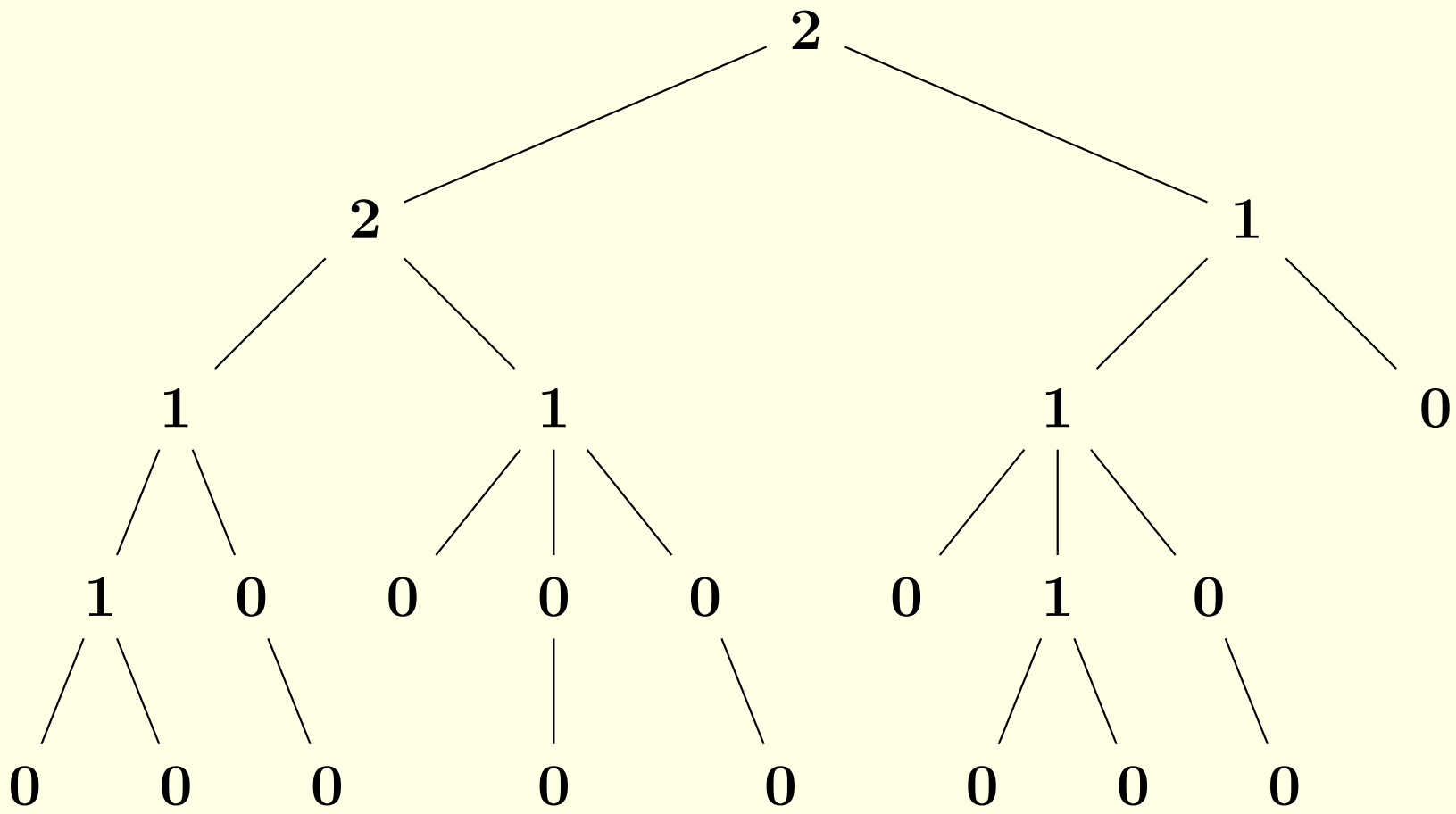
It's originally used to classify river system. Later also appeared in computer science as register function.

It was defined on binary trees. Now, we extend the definition to rooted trees with branching factor ≥ 1:

$$S(\mathbf{r}) = \begin{cases} 0 & \textbf{if } \mathrm{r} \textbf{ has no child,} \\ M(\mathbf{r}) & \textbf{if only one child with } S(\mathbf{u}) = M(\mathbf{r}), \\ M(\mathbf{r}) + 1 & \textbf{otherwise,} \end{cases}$$

where $M(\mathbf{r}) = \max\{S(\mathbf{u}) : \mathbf{u} \textbf{ is a child of } \mathrm{r}\}$.

Example of the Horton-Strahler number



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Moreover, the least maximal number of descents is equal to the largest Horton-Strahler number in the tree.

Horton-Strahler number for binary tree

Let T be a binary tree with N nodes. Let S_N be the Horton-Strahler number of the root of T.

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Recently, Auber et al. (2004) studied a different extension.

The worst case

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Let d be the number of levels. Then the maximal number of descents is $d-1$ and the total number of nodes is $2^d - 1 = N$.

Thus, generally, the least maximal number of descents is bounded by $\log_2(N+1) - 1$.

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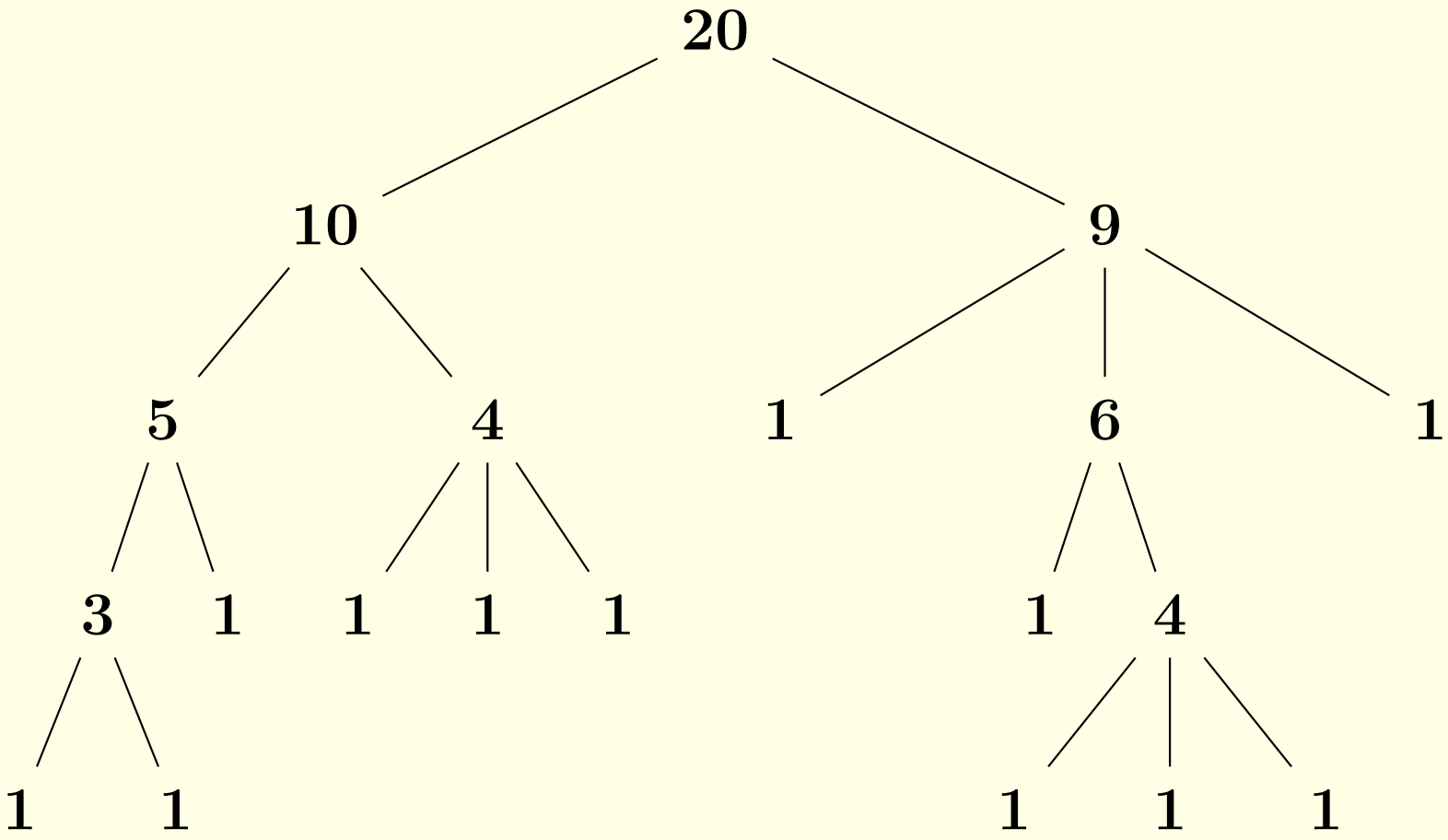
Orbit decomposition:

Compute for each node the number of subtree size.

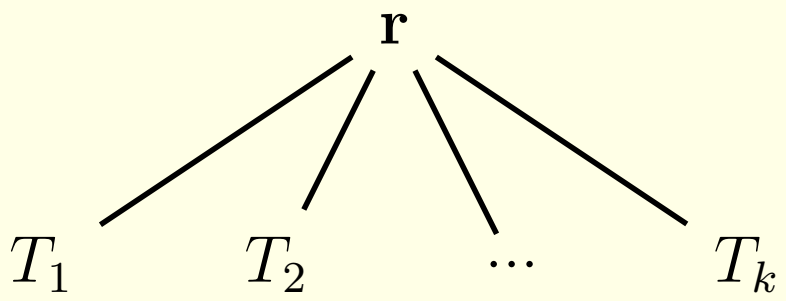
For each component, starting from the root we choose the path of the orbit through the nodes with the largest number of subtree size.

Example **The number is the size of the subtree rooted at the node.**



Compare to greedy orbit decomposition

The main difference is the rule to connect the nodes and their children:



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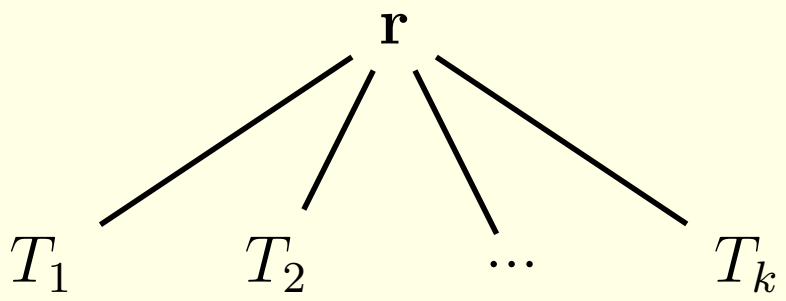

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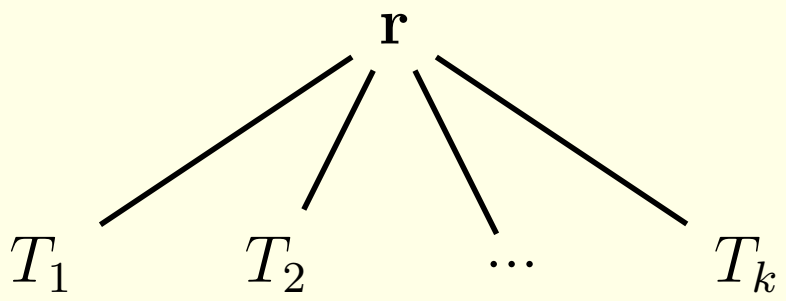

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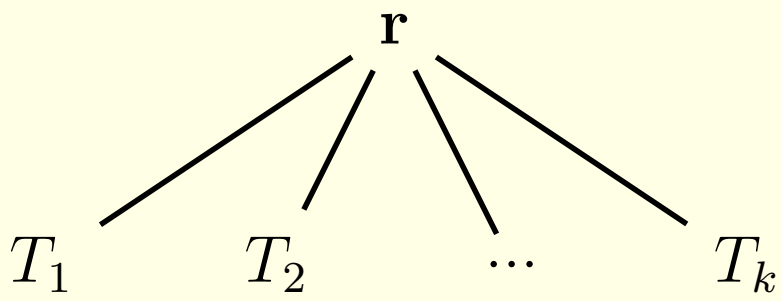

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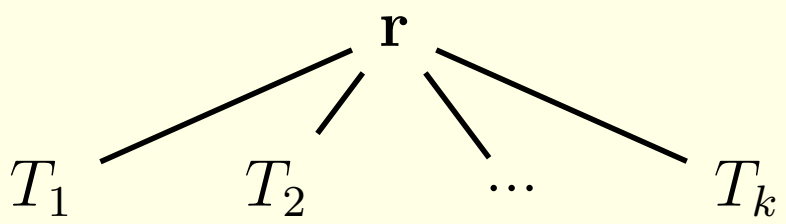
least avg the heaviest subtree (with most nodes)

Theorem 2

Let T be a rooted tree with N nodes:



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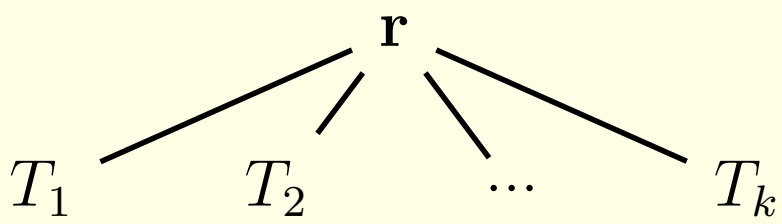

Let m_i be the total number of descents in T_i and n_i be the total number of nodes in T_i.

Then the best choice is the T_i with the maximal n_i and the total number of descents of T is

$$\sum_{i=1}^{k} m_i + (N - 1 - \max\{n_1, ..., n_k\}) .$$

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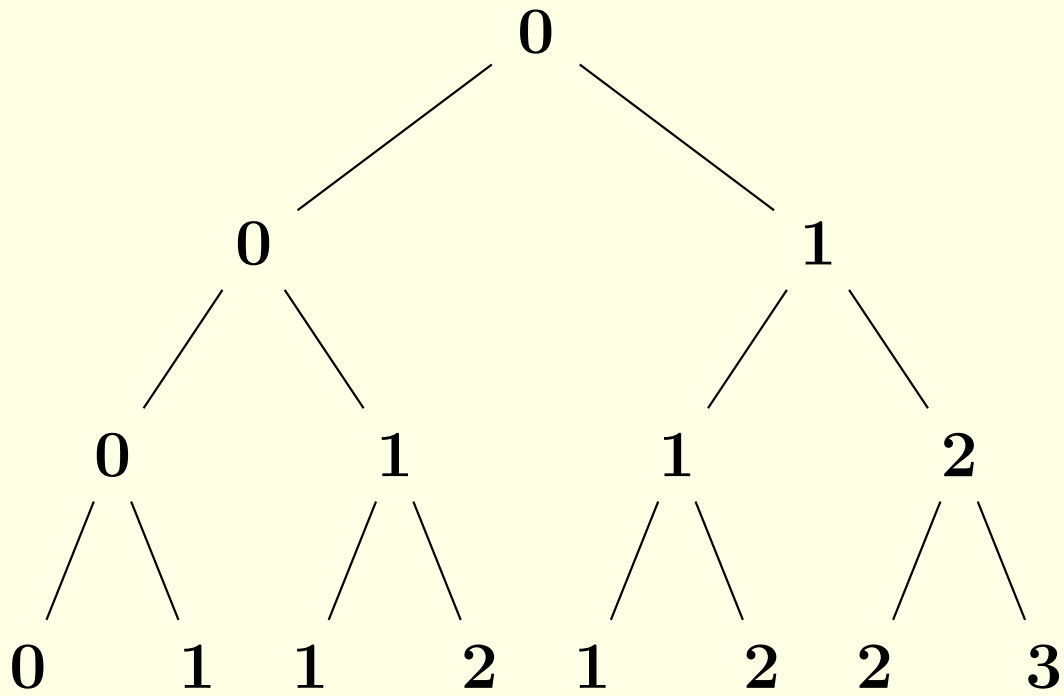
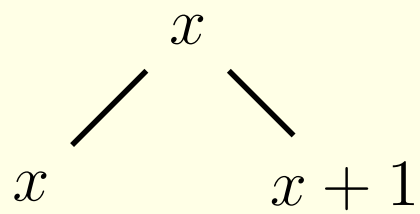


Here the number indicates the number of descents.

Average number of descents of complete binary tree

Let c_k be the total number of descents at level k.

Then $c_1 = 0$ and $c_k = 2c_{k-1} + 2^{k-2}$ for $k \geq 2$. (Why?)



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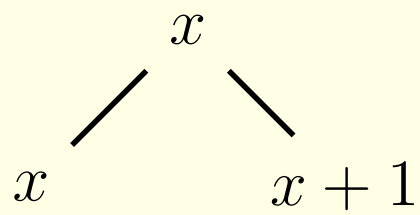

Thus $c_k = (k-1)2^{k-2}$ and the average number descents in this case is

$$\frac{\sum_{k=1}^{d}(k-1)2^{k-2}}{\sum_{k=1}^{d}2^{k-1}} = \frac{d}{2} - 1 + \frac{d}{2^d - 1},$$

where d is the number of levels.

Conclusion

1. **For any function f on $\{0, 1, ..., N-1\}$, the iteration $f^m(x)$ for each m and x can be computed in time $O(\log N)$ by using preprocessed information. The preprocess need linear space and linear time.**

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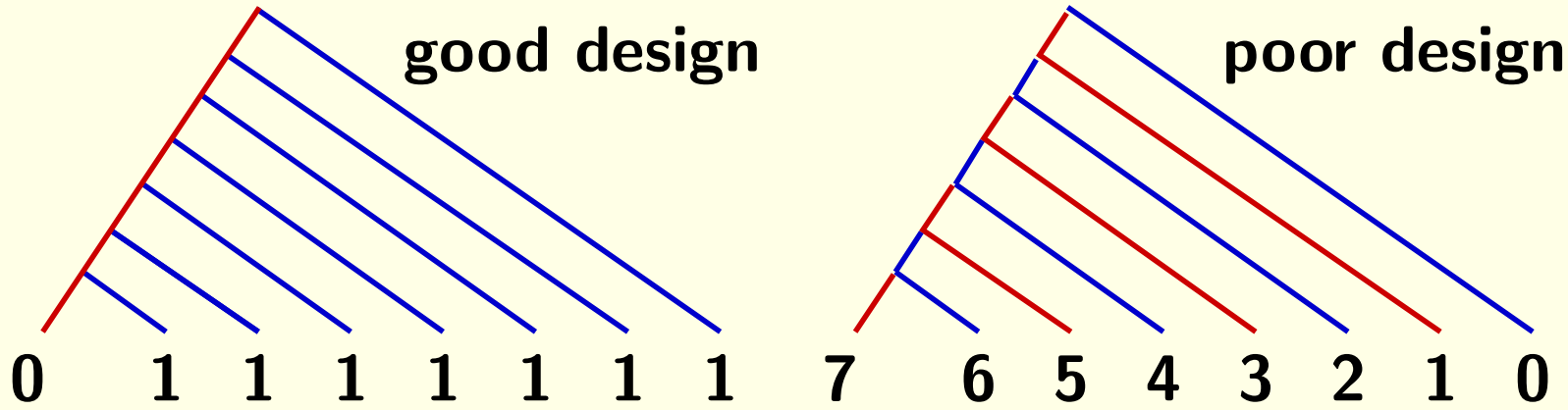
4. **Another interpretation of our work. (next page)**

Another interpretation of our work

Transportation system



Consider orbit as bus line and descent as transfer.

Our work is to design a bus system that minimizes the number of transfers.

Thank You!