Boltzman Sampling and Properties of Trees

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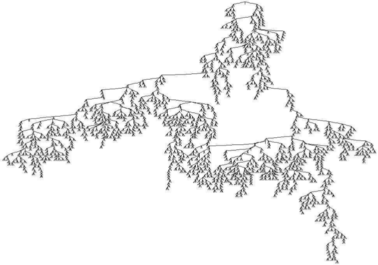
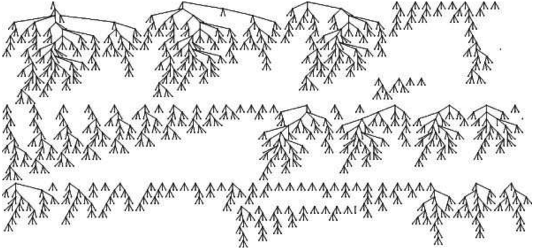
Same distribution & same properties

Subtrees of a **large** uniform random tree





Random trees independently generated with a singular Boltzmann sampler

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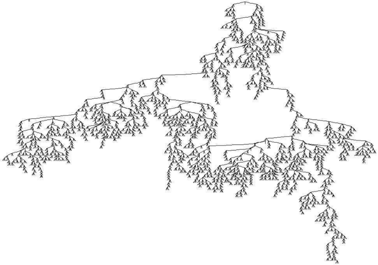

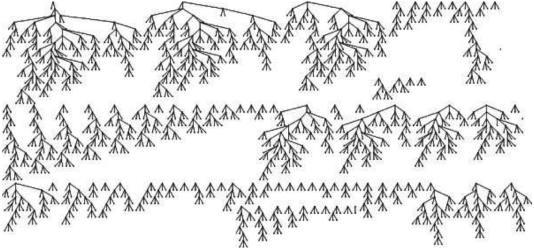

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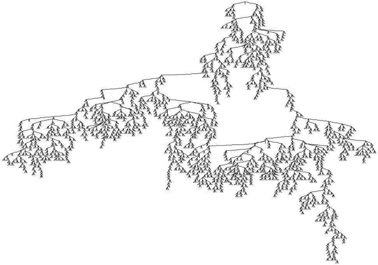
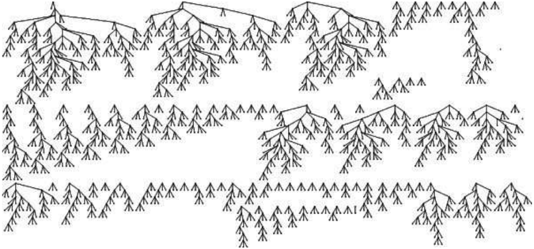




Random trees independently generated with a singular Boltzmann sampler

Illustration : size in ternary trees

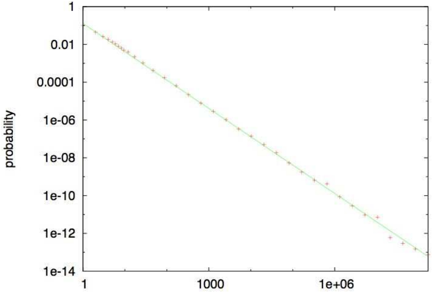
Ternary trees $T(z) = 1 + zT^3(z)$

$$T(z) = 3/2 - \sqrt{3}/2\sqrt{1 - z/\rho} + \dots \qquad \rho = 4/27$$

Singular Boltzmann sampler ≡ branch with probability 1/3

$$\Pr(|A| = k) = \frac{T_k \rho^k}{T(\rho)} \sim Ck^{-3/2}$$

Power law



Boltzmann generation

P. Duchon, P. Flajolet, G. Louchard, and G. Schaeffer

Boltzmann model parameter x

Combinatorial class $\mathcal{A}$, g. f.
$A(z) = \sum a_n z^n$
$\forall A \in \mathcal{A},\ \Pr_x(A) = \frac{x^{|A|}}{A(x)}$

- uniform random sampling
- but result size is random :

$\Pr_x(|A| = n) = \frac{a_n x^n}{A(x)}$

- Probabilistic algorithm
- combinatorial constr.
- linear time

construction	sampler with parameter x
$\mathcal{C} = \mathcal{I}$	$\Gamma C(x) := \varepsilon$
$\mathcal{C} = \mathcal{Z}$	$\Gamma C(x) := z$
$\mathcal{C} = \mathcal{A} + \mathcal{B}$	$\Gamma C(x) :=$ Bern $\frac{A(x)}{C(x)} \Longrightarrow \Gamma A(x) \mid \Gamma B(x)$
$\mathcal{C} = \mathcal{A} \times \mathcal{B}$	$\Gamma C(x) := \langle\ \Gamma A(x)\ ;\ \Gamma B(x)\ \rangle$
$\mathcal{C} = \textsc{Seq}(\mathcal{A})$	$\Gamma C(x) :=$ Geom $A(x) \Longrightarrow \Gamma A(x)$
...	...

Tree sampling

Recursive structures $T(z) = z\Phi(T(z)), \quad \rho > 0$

- $T(z) = \tau - h\sqrt{1 - z/\rho} + g(1 - z/\rho) + O(1 - z/\rho)^{3/2}$
  $T_n \sim \frac{h}{2\sqrt{\pi}}\rho^{-n}n^{-3/2}$
- derivative $T'(z) = \frac{h}{2\rho\sqrt{1-z/\rho}} + \ldots$ and $T'_n \sim nT_n/\rho$
- bivariate g.f. $T(z,u) = \sum T_{n,k}u^k z^n$

Singular Boltzman free generation

- $\forall A \in \mathcal{T},\ \Pr_\rho(A) = \frac{\rho^{|A|}}{T(\rho)}$ $\mathbb{E}(A)$ infinite
- parameter $\Omega : \mathcal{T} \to \mathbb{N}$
  $\Pr(\Omega = k) = \sum_n \frac{T_{n,k}}{T_n} \times \frac{T_n \rho^n}{T(\rho)} = \frac{[u^k]T(\rho,u)}{T(\rho)}$

Atomic probability of trees

Proposition

$\mathcal{T}$ simple family of trees ; $T(z) = z\phi(T(z))$, Rdc ρ.
The distribution of subtrees in a uniform random tree of size n is asymptotically equivalent to the atomic distribution of trees in a singular Boltzmann sampling :

$$\forall A \in \mathcal{T},\ \frac{1}{nT_n} \times 0cc(A, \mathcal{T}_n) \rightarrow_{n\rightarrow\infty} \Pr(A, \Gamma_\rho)$$

Proof.

$$Occ_A(z) \quad \equiv \quad \sum 0cc(A,T)\, z^{|T|} = \sum 0cc(A,\mathcal{T}_n)\, z^n = \frac{zT'(z)}{T(z)} z^{|A|}$$

Thus the probability that A appears as a subtrees in a tree of size n is

$$\Pr(A,n) = \frac{1}{nT_n}[z^n]\frac{zT'(z)}{T(z)} z^{|A|} \rightarrow_{n\rightarrow\infty} \frac{\rho^{|A|}}{T(\rho)}$$

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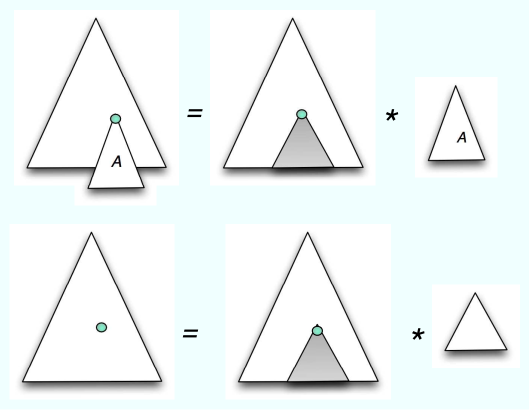
properties of subtrees

$$Occ_A(z) = \frac{zT'(z)}{T(z)} z^{|A|}$$



Parameters of trees

Corollary

$\mathcal{T}$ simple family of trees, Ω: $\mathcal{T} \to \mathbb{N}$ parameter.
Distribution of Ω on subtrees of a uniform random tree size n
asymptotically equivalent to
distrib. of Ω on trees generated by singular Boltzmann sampler.

Proof.

$$\Lambda_k(z) = \sum_{A;\Omega(A)=k} z^{|A|} = \sum_n T_{n,k} z^n = [u^k]T(z,u), \quad \Lambda_k(z) \to_{z\to\rho} \Lambda_k(\rho) < \infty$$

$$S\Lambda_k(z) = \sum 0cc(\Omega_k, T)\, z^{|T|} = \sum S_{n,k} z^n = \frac{zT'(z)}{T(z)} \Lambda_k(z)$$

Probability random subtree parameter k, in a random tree of size n :

$$\Pr(\Omega_k, n) = \frac{1}{nT_n}[z^n]S\Lambda_k(z) \to_{n\to\infty} \frac{S\Lambda_k(\rho)}{T(\rho)}$$

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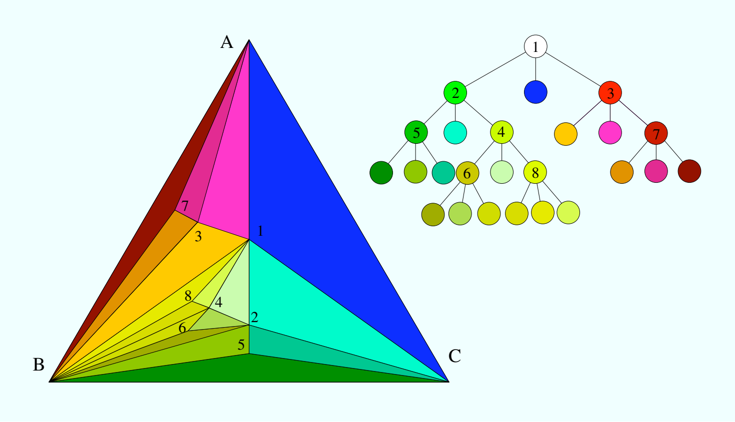
RANS and Ternary trees

Darrasse, S. AofA07

Random Appollonian Networks Structures $\longleftrightarrow$ Ternary trees



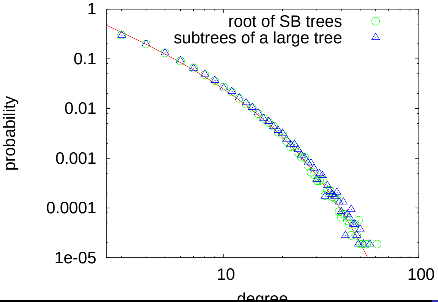
Degree of a vertex in a RANS $\longleftrightarrow$ size of binary subtrees

Degrees in RANS

Choose a random vertex in a random RANS of size n ; degree ?
Choose a random node in a random TT of size n ; size of BT ?

Theorem

The limiting distribution of degrees in RANS follows a power law with an exponential cutoff : $\Pr(D = k) \sim C\, \beta^k\ k^{-3/2}$, $\beta = \frac{8}{9}$



Simulation

- Root degree in 100.000 trees generated with free Singular Boltzmann
- Root degree in all subtrees of a uniform tree of size 100.000