Minicourse 2: Asymptotic Techniques for AofA

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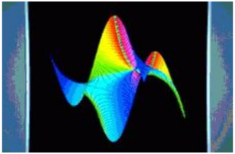


AofA'08, Maresias, Brazil
Sunday 8:30–10:30 (!)

I Introduction

Overview of the 3 Minicourses

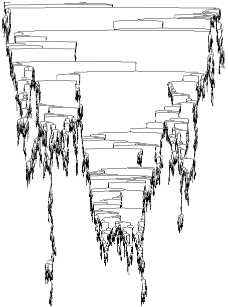
Combinatorial Structure

↓ Combinatorics (MC1)↓

Generating Functions

$$F(z) = \sum_{n \geq 0} f_n z^n$$

Example: binary trees



$$B(z) = z + B^2(z)$$

Overview of the 3 Minicourses

Combinatorial Structure

↓ Combinatorics (MC1)↓

Generating Functions

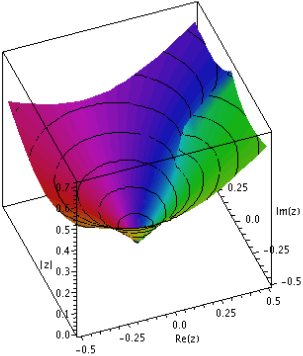
$$F(z) = \sum_{n\geq 0} f_n z^n$$

↓ Complex Analysis (MC2)↓

Asymptotics

$f_n \sim \dots, n \to \infty.$

Example: binary trees



$$B_n \sim \frac{4^{n-1} n^{-3/2}}{\sqrt{\pi}}$$

Overview of the 3 Minicourses

Combinatorial Structure

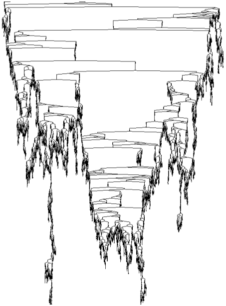
+ parameter

↓ Combinatorics (MC1)↓

Generating Functions

$$F(z,u)=\sum_{n\geq 0} f_{n,k}u^k z^n$$

Example: path length in binary trees



$$B(z,u)=\sum_{t\in T} u^{\mathrm{pl}(t)}z^{|t|}$$

$$= z + B^2(zu,u)$$

$$P(z) := \frac{\partial}{\partial u}B(z,u)\Big|_{u=1}$$

Overview of the 3 Minicourses

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+ parameter

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Generating Functions

$$F(z)=\sum_{n\geq 0} f_n z^n$$

↓ Complex Analysis (MC2)↓

Asymptotics

$f_n \sim \dots, n \to \infty.$

Example: path length in binary trees

$$B_n = \frac{4^{n-1}n^{-3/2}}{\sqrt{\pi}}\left(1+\frac{3}{8n}+\cdots\right),$$

$$P_n = 4^{n-1}(1-\frac{1}{\sqrt{\pi n}}+\cdots),$$

$$\frac{P_n}{nB_n}=\sqrt{\pi n}-1+\cdots.$$

Also, variance and higher moments

Overview of the 3 Minicourses

Combinatorial Structure
+ parameter
↓ Combinatorics (MC1)↓

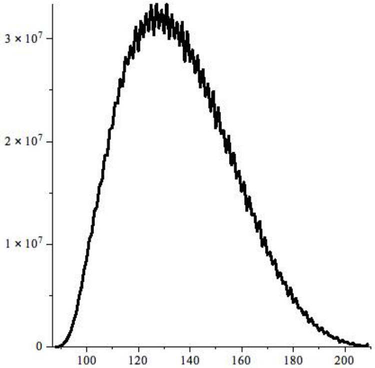
Generating Functions

$$F(z,u)=\sum_{n\geq 0} f_{n,k} u^k z^n$$

↓ Multivariate Analysis (MC3)↓
Distribution

$$f_{n,k}\sim \dots, n\to\infty.$$

Example: path length in binary trees



Examples for this Course

- Conway's sequence: 1, 11, 21, 1211, 111221, 312211,...

$$\ell_n \simeq 2.042160077\rho^n, \qquad \rho \simeq 1.3035772690343$$

ρ root of a polynomial of degree 71.

- Catalan numbers (binary trees): 1, 1, 2, 5, 14, 42, 132,...

$$B_n \sim \frac{1}{\sqrt{\pi}} \frac{4^n}{n^{3/2}}$$

- Cayley trees (T=Prod(Z,Set(T))): 1, 2, 9, 64, 625, 7776,...

$$\frac{T_n}{n!} \sim \frac{e^n}{\sqrt{2\pi} n^{3/2}}$$

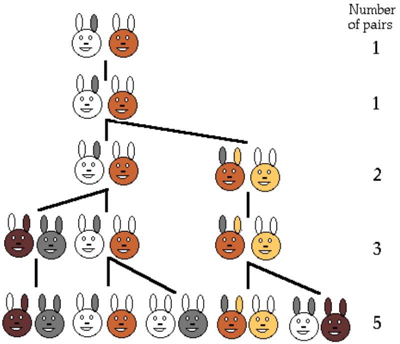
- Bell numbers (set partitions): 1, 1, 2, 5, 15, 52, 203, 877,...

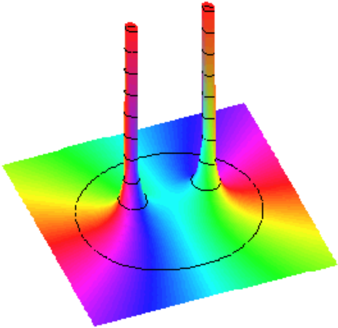
$$\log \frac{B_n}{n!} \sim -n \log\log n$$

Starting point: generating function

A Gallery of Combinatorial Pictures

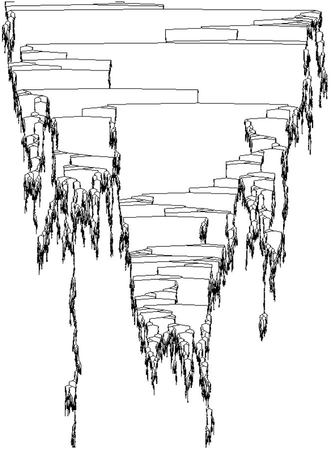
Fibonacci Numbers: $$\frac{1}{1-z-z^2} = 1 + z + 2z^2 + 3z^3 + 5z^4 + \cdots$$

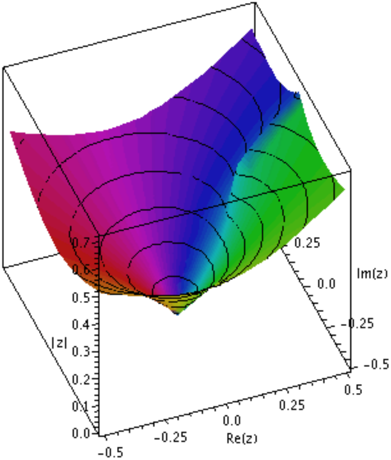




A Gallery of Combinatorial Pictures

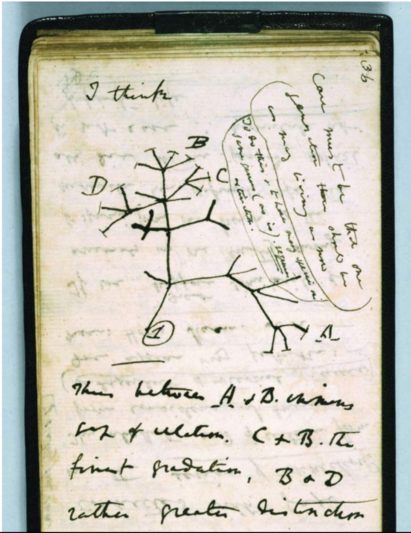
Binary Trees: $\frac{1-\sqrt{1-4z}}{2} = z + z^2 + 2z^3 + 5z^4 + 14z^5 + \cdots$

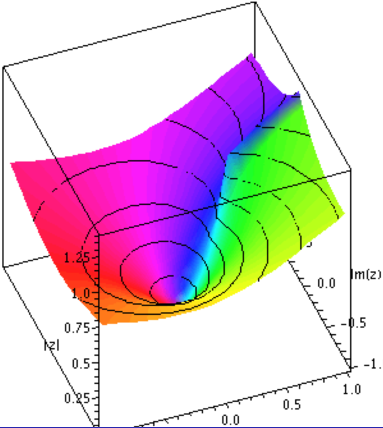




A Gallery of Combinatorial Pictures

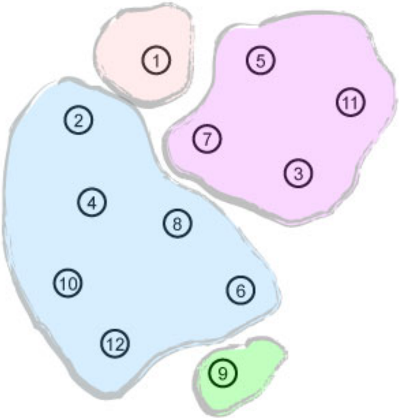
Cayley Trees: $T(z) = z\exp(T(z)) = z + 2\frac{z}{2!} + 9\frac{z}{3!} + 64\frac{z}{4!} + \cdots$

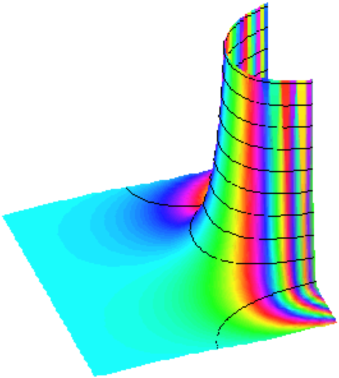




A Gallery of Combinatorial Pictures

Set Partitions: $\exp(\exp(z)-1) = 1 + 1\frac{z}{1!} + 2\frac{z^2}{2!} + 5\frac{z^3}{3!} + 15\frac{z^4}{4!} + \cdots$





II Mini-minicourse in complex analysis

Basic Definitions and Properties

Definition

$f : D \subset \mathbb{C} \to \mathbb{C}$ is analytic at x_0 if it is the sum of a power series in a disc around x_0.

Proposition

- f, g analytic at x_0, then so are $f + g$, $f \times g$ and f'.
- g analytic at x_0, f analytic at $g(x_0)$, then $f \circ g$ analytic at x_0.

Same def. and prop. in several variables.

Examples

f	analytic at 0?	why
polynomial	Yes	
$\exp(x)$	Yes	$1 + x + x^2/2! + \cdots$
$\frac{1}{1-x}$	Yes	$1 + x + x^2 + \cdots \quad (\lvert x\rvert < 1)$
$\log\frac{1}{1-x}$	Yes	$x + x^2/2 + x^3/3 \cdots \quad (\lvert x\rvert < 1)$
$\frac{1-\sqrt{1-4x}}{2x}$	Yes	$1 + \cdots + \frac{1}{k+1}\binom{2k}{k}x^k + \cdots \quad (\lvert x\rvert < 1/4)$;
$\frac{1}{x}$	No	infinite at 0
$\log x$	No	derivative not analytic at 0
$\sqrt{x}$	No	derivative infinite at 0

Combinatorial Generating Functions I

Proposition (Labeled)

The labeled structures obtained by iterative use of

$$\textsc{Seq}, \textsc{Cyc}, \textsc{Set}, +, \times \textit{ starting with } 1, \mathcal{Z}$$

have exponential generating series that are analytic at 0.

Recall Translation Table (MC1)

A + B	$A(z) + B(z)$
A × B	$A(z) \times B(z)$
$\textsc{Seq}$(C)	$\frac{1}{1-C(z)}$
$\textsc{Cyc}$(C)	$\log \frac{1}{1-C(z)}$
$\textsc{Set}$(C)	$\exp(C(z))$

Proof by induction.

$+, \times$, and composition with $\frac{1}{1-x}$, $\log \frac{1}{1-x}$, $\exp(x)$. □

Combinatorial Generating Functions II

Proposition (Unlabeled)

The unlabeled structures obtained by iterative use of

$\textsc{Seq}, \textsc{Cyc}, \textsc{PSet}, \textsc{MSet}, +, \times$ *starting with* $1, \mathcal{Z}$

have ordinary generating series that are analytic at 0.

Proof by induction.

Recall Translation Table (MC1)

A + B	$A(z) + B(z)$	easy
A × B	$A(z) \times B(z)$	easy
$\textsc{Seq}$(C)	$\frac{1}{1-C(z)}$	easy
$\textsc{PSet}$(C)	$\exp(C(z) - \frac{1}{2}C(z^2) + \frac{1}{3}C(z^3) - \cdots)$	?
$\textsc{MSet}$(C)	$\exp(C(z) + \frac{1}{2}C(z^2) + \frac{1}{3}C(z^3) + \cdots)$	?
$\textsc{Cyc}$(C)	$\sum_{k\geq 1} \frac{\phi(k)}{k} \log \frac{1}{1-C(z^k)}$	?

Combinatorial Generating Functions II

Proposition (Unlabeled)

The unlabeled structures obtained by iterative use of

$$\textsc{Seq}, \textsc{Cyc}, \textsc{PSet}, \textsc{MSet}, +, \times \text{ starting with } 1, \mathcal{Z}$$

have ordinary generating series that are analytic at 0.

Proof by induction.

- $\textsc{MSet}(C)$: by induction, there exists $K > 0$, $\rho \in (0,1)$, s.t. $|C(z)| < K|z|$ for $|z| < \rho$. Then
$C(z) + \frac{1}{2}C(z^2) + \frac{1}{3}C(z^3) + \cdots < K \log \frac{1}{1-|z|}, \quad |z| < \rho.$
Uniform convergence ⇒ limit analytic (Weierstrass).
- $\textsc{PSet}$, $\textsc{Cyc}$: similar.

□

Analytic Continuation & Singularities

Definition

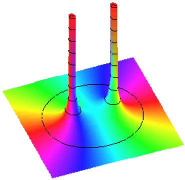
Analytic on a region ($=$ connected, open, $\neq \emptyset$): at each point.

Proposition

$R \subset S$ regions. f analytic in R. There is at most one analytic function in S equal to f on R (the analytic continuation of f to S).

Definition

- Singularity: a point that cannot be reached by analytic continuation;
- Polar singularity α: isolated singularity and $(z-\alpha)^m f$ analytic for some $m \in \mathbb{N}$;
- residue at a pole: coefficient of $(z-\alpha)^{-1}$;
- f meromorphic in R: only polar singularities.



Combinatorial Examples

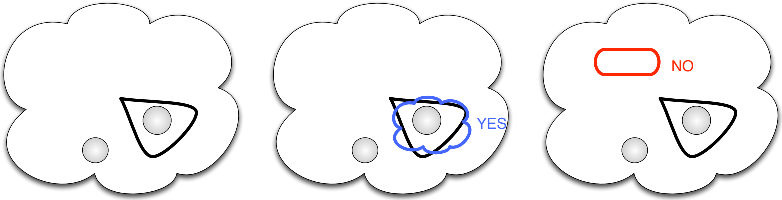
Structure	GF	Sings	Mero. in $\mathbb{C}$
Set	$\exp(z)$	none	Yes
Set Partitions	$\exp(e^z - 1)$	none	Yes
Sequence	$\frac{1}{1-z}$	1	Yes
Bin Seq. no adj.0	$\frac{1+z}{1-z-z^2}$	$\phi, -1/\phi$	Yes
Derangements	$\frac{e^{-z}}{1-z}$	1	Yes
Rooted plane trees	$\frac{1-\sqrt{1-4z}}{2z}$	$1/4$	No
Integer partitions	$\prod_{k\geq 1} \frac{1}{1-z^k}$	roots of 1	No
Irred. pols over $\mathbb{F}_q$	$\sum_{r\geq 1} \frac{\mu(r)}{r} \ln \frac{1}{1-qz^r}$	roots of $\frac{1}{q}$	No
Exercise: Bernoulli nbs	$\frac{z}{\exp(z)-1}$	?	?

Integration of Analytic Functions

Theorem

f analytic in a region R, Γ_1 and Γ_2 two closed curves that are homotopic wrt R (= can be deformed continuously one into the other) then

$$\int_{\Gamma_1} f = \int_{\Gamma_2} f.$$

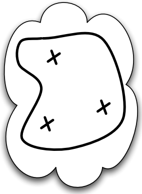


Residue Theorem: from Global to Local

Corollary

f meromorphic in a region R, Γ a closed path in $\mathbb{C}$ encircling the poles $\alpha_1, \dots, \alpha_m$ of f once in the positive sense. Then

$$\int_\Gamma f = 2\pi i \sum_j \operatorname{Res}(f; \alpha_j).$$



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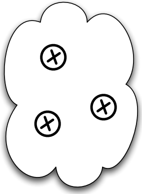


Proof.

- $g_j := P_j(z)/(z-\alpha_j)^{m_j}$ polar part at α_j;
- $h := f - (g_1 + \dots + g_m)$ analytic in R;
- Γ homotopic to a point in $R \Rightarrow \int_\Gamma h = 0$;
- Γ homotopic to a circle centered at α_j in $R \setminus \{\alpha_j\}$;
- $\displaystyle\int_\Gamma (z-\alpha_j)^m \, dz = i \int_0^{2\pi} r^{m+1} e^{i(m+1)\theta} \, d\theta = \begin{cases} 2\pi i & m = -1, \\ 0 & \text{otherwise.} \end{cases}$

Cauchy's Coefficient Formula

Corollary

If $f = a_0 + a_1 z + \dots$ *is analytic in* $R \ni 0$ *then*

$$a_n = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z^{n+1}}\, dz$$

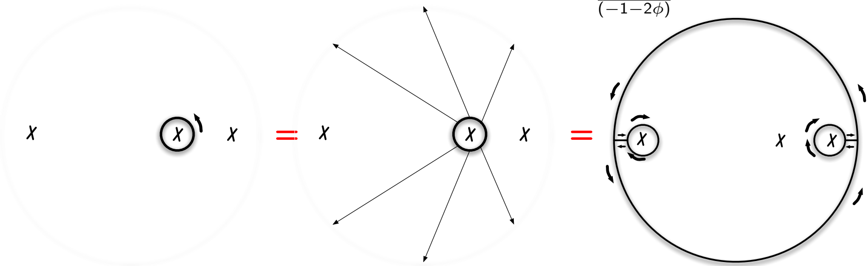
for every closed Γ *in* R *encircling 0 once in the positive sense.*

Proof.

$f(z)/z^{n+1}$ meromorphic in R, pole at 0, residue a_n. □

Coefficients of Rational Functions by Complex Integration

$$2\pi i F_n = \int_\Gamma \underbrace{\frac{z^{-n-1}}{1-z-z^2}}_{g(z)} dz = \Big(\int_{|z|=R} g - \underbrace{\int_\phi g}_{\frac{\phi^{-n-1}}{(-1-2\phi)}} - \underbrace{\int_{\bar\phi} g}_{\text{idem}}\Big)$$



When $|z| = R$, $|g(z)| \le \dfrac{R^{-n-1}}{R^2 - R - 1} \Rightarrow 2\pi R\,|g(z)| \to 0, \quad R \to \infty.$

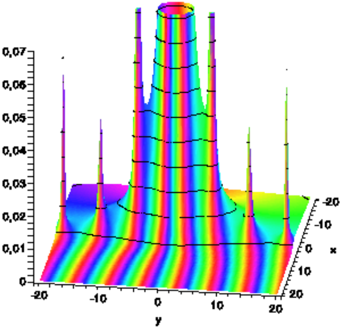
Conclusion: $F_n = \dfrac{\phi^{-n-1}}{1+2\phi} + \dfrac{\bar\phi^{-n-1}}{1+2\bar\phi}.$

III Dominant Singularity

Cauchy's Formula

$$[z^n]f(z) = \frac{1}{2\pi i}\oint \frac{f(z)}{z^{n+1}}\,dz$$

$$[z^2]\frac{z}{e^z-1} = \frac{1}{12}$$



As n increases, the smallest singularities dominate.

Exponential Growth

Definition

Dominant singularity: singularity of minimal modulus.

Theorem

$f = a_0 + a_1 z + \cdots$ analytic at 0;
R modulus of its dominant singularities, then

$$a_n = R^{-n}\theta(n), \qquad \limsup_{n\to\infty} |\theta(n)|^{1/n} = 1.$$

Proof (Idea).

1. integrate on circle of radius $R - \epsilon \Rightarrow |a_n| \leq C(R-\epsilon)^{-n}$;
2. if $(R+\epsilon)^{-n} \leq K a_n$, then convergence on a larger disc.

□

General Principle for Asymptotics of Coefficients

$$[z^n]f(z) = \frac{1}{2\pi i} \oint \frac{f(z)}{z^{n+1}}\,dz$$

Singularity of smallest modulus $\longrightarrow$ exponential growth

Local behaviour $\longrightarrow$ sub-exponential terms

Algorithm

1. Locate dominant singularities
2. Compute local expansions
3. Transfer

Rational Functions

Dominant singularities: roots of denominator of smallest modulus.

Conway's sequence:

1, 11, 21, 1211, 111221,. . .

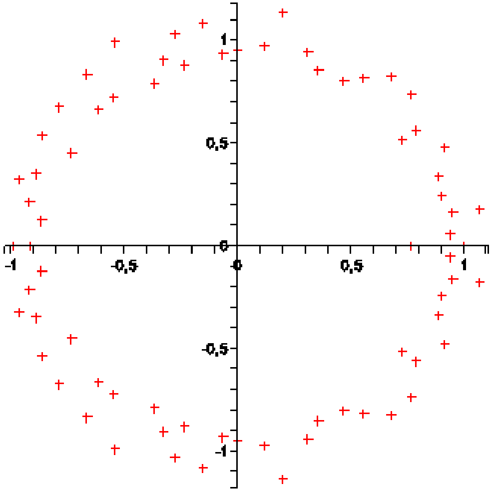
Generating function:

$f(z) = \frac{P(z)}{Q(z)}$
with deg $Q = 72$.

$\delta(f) \simeq 0.7671198507$,

$\rho \simeq 1.3035772690343$,

$\ell_n \simeq 2.042160077\rho^n$



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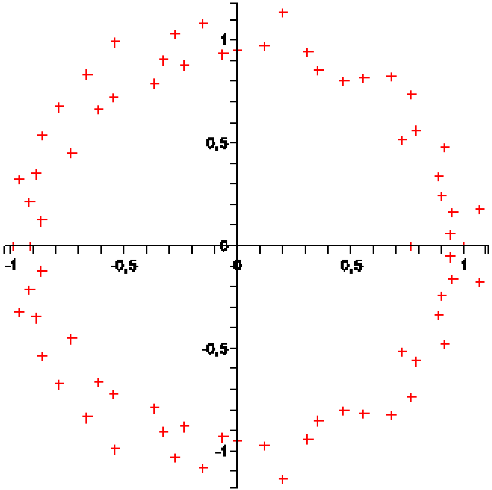

Iterative Generating Functions

Algorithm Dominant Singularity

Function F	Dom. Sing. $\delta(F)$
$\exp(f)$	$\delta(f)$
$1/(1-f)$	$\min(\delta(f), \{z \mid f(z)=1\})$
$\log(1/(1-f))$	idem
fg, $f+g$	$\min(\delta(f), \delta(g))$
$f(z)+\frac{1}{2}f(z^2)+\frac{1}{3}f(z^3)+\cdots$	$\min(\delta(f), 1)$.

Iterative Generating Functions

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Note: f has coeffs $\geq 0 \Rightarrow \min(\delta(f), \{z \mid f(z)=1\}) \in \mathbb{R}^+$.

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Pringsheim's Theorem

f analytic with nonnegative Taylor coefficients has its radius of convergence for dominant singularity.

Iterative Generating Functions

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$f(z)+\frac{1}{2}f(z^2)+\frac{1}{3}f(z^3)+\cdots$	$\min(\delta(f), 1)$.

Exercise

Dominant singularity of $\frac{1}{2}\left(1-\sqrt{1-4\log\left(\frac{1}{1-\log\frac{1}{1-z}}\right)}\right)$.

(Binary trees of cycles of cycles)

Implicit Functions

Proposition (Implicit Function Theorem)

The equation

$$\mathbf{y} = \mathbf{f}(z, \mathbf{y})$$

admits a solution $\mathbf{y} = \mathbf{g}(z)$ that is analytic at z_0 when

- $\mathbf{f}(z, \mathbf{y})$ is analytic in $1 + n$ variables at $(z_0, \mathbf{y_0}) := (z_0, \mathbf{g}(z_0))$,
- $\mathbf{f}(z_0, \mathbf{y_0}) = \mathbf{y_0}$ and $\det |I - \partial \mathbf{f} / \partial \mathbf{y}| \neq 0$ at $(z_0, \mathbf{y_0})$.

Example (Cayley Trees: $T = z \exp(T)$)

1. Generating function analytic at 0;
2. potential singularity when $1 - z \exp(T) = 0$, whence $T = 1$, whence $z = e^{-1}$.

More generally, solutions of combinatorial systems are analytic.

Implicit Functions

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Example (Cayley Trees: $T = z\exp(T)$)

1. Generating function analytic at 0;
2. potential singularity when $1 - z\exp(T) = 0$, whence $T = 1$, whence $z = e^{-1}$.

Exercises

1. Binary trees;
2. $T(z) \underset{z \to e^{-1}}{\sim} ?$.

More generally, solutions of combinatorial systems are analytic.

IV Singularity Analysis

General Principle for Asymptotics of Coefficients

$$[z^n]f(z) = \frac{1}{2\pi i}\oint \frac{f(z)}{z^{n+1}}\,dz$$

Singularity of smallest modulus $\longrightarrow$ exponential growth

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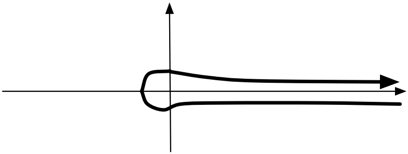
The Gamma Function

- **Def.** Euler's integral: $\Gamma(z) := \int_0^{+\infty} t^{z-1} e^{-t}\,dt$;
- **Recurrence:** $\Gamma(z+1) = z\Gamma(z)$ (integration by parts);
- **Reflection formula:** $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$;
- **Hankel's loop formula:** $\dfrac{1}{\Gamma(z)} = \dfrac{1}{2\pi i}\displaystyle\int_{(0)}^{+\infty} (-t)^{-z} e^{-t}\,dt.$



Idea for the last one:
$\int_0^{+\infty} (e^{-\pi i})^{-z} t^{-z} e^{-t}\,dt - \int_0^{+\infty} (e^{\pi i})^{-z} t^{-z} e^{-t}\,dt.$

Basic Transfer Toolkit

Singularity Analysis Theorem [Flajolet-Odlyzko]

1. If f is analytic in $\Delta(\phi, R)$, and

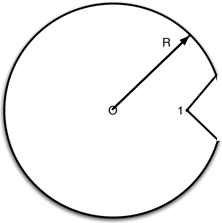
$$f(z) \underset{z\to 1}{=} O\left((1-z)^{-\alpha} \log^{\beta} \frac{1}{1-z}\right),$$

then $[z^n]f(z) \underset{n\to\infty}{=} O(n^{\alpha-1} \log^{\beta} n)$.

2. $$[z^n](1-z)^{-\alpha} \underset{n\to\infty}{=} \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + \sum_{k\geq 1} \frac{e_k(\alpha)}{n^k}\right),$$

$\alpha \in \mathbb{C} \setminus \mathbb{Z}^-$, $e_k(\alpha)$ polynomial;

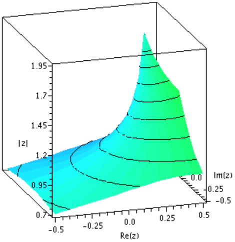
3. similar result with a $\log^{\beta}(1/(1-z))$.



$\Delta(\phi, R)$

Example: Binary Trees

$$B(z) = \frac{1-\sqrt{1-4z}}{2z}$$



1. Dominant singularity: $1/4$;
2. Local expansion: $B = 2 - 2\sqrt{1-4z} + 2(1-4z) + O((1-4z)^{3/2})$;
3. $O((1-4z)^{3/2})) \to O(4^n n^{-5/2})$;
4. $-2\sqrt{1-4z} \to \frac{4^n}{\sqrt{\pi} n^{3/2}} + \star \frac{4^n}{n^{5/2}} + \cdots$.

Conclusion: $B_n = \frac{4^n}{\sqrt{\pi} n^{3/2}} + O(4^n n^{-5/2})$.

Exercise

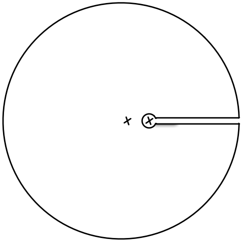
Cayley trees.

Proof of the Singularity Analysis Theorem I

Part I. Scale

2. $[z^n](1-z)^{-\alpha} \underset{n\to\infty}{=} \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1+\sum_{k\geq 1}\frac{e_k(\alpha)}{n^k}\right)$,

   $\alpha \in \mathbb{C} \setminus \mathbb{Z}^-$, $e_k(\alpha)$ polynomial;



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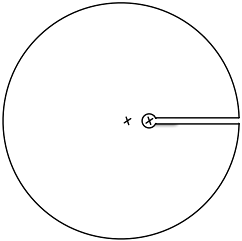

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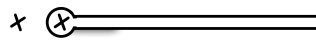

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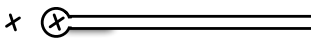

1. On the almost full circle, $f(z)/z^{n+1}$ small: $O(R^{-n})$;
2. Extending the rest to a full Hankel contour changes the integral by $O(R^{-n})$;
3. On this part, change variable: $z := 1 + t/n$

$$[z^n](1-z)^{-\alpha} = \frac{1}{2\pi i}\int_{(0)}^{+\infty}\left(-\frac{t}{n}\right)^{-\alpha-1}\left(1+\frac{t}{n}\right)^{-n-1} dt + O(R^{-n}).$$

Recognize 1/Γ?

Proof of the Singularity Analysis Theorem I

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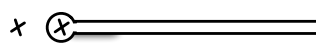

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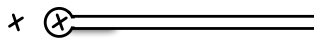

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4. $$\left(1+\frac{t}{n}\right)^{-n-1} = e^{-(n+1)\log(1+\frac{t}{n})} = e^{-t}\left(1+\frac{t^2-2t}{2n}+\cdots\right);$$
5. Integrate termwise (+ uniform convergence).

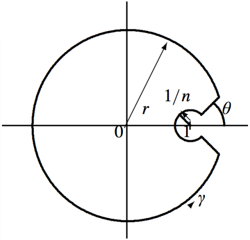
Proof of the Singularity Analysis Theorem II

Part II. $O()$

1. If f is analytic in $\Delta(\phi, R)$, and

$$f(z) \underset{z\to1}{=} O\left((1-z)^{-\alpha}\log^{\beta}\frac{1}{1-z}\right),$$

then $[z^n]f(z) \underset{n\to\infty}{=} O(n^{\alpha-1}\log^{\beta} n)$.



inside $\Delta(\phi, R)$

Easier than previous part:

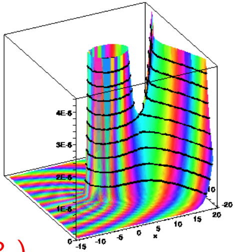
1. Outer circle: r^{-n};
2. Inner circle: use hypothesis and simple bounds;
3. Segments: the key is that $(1 + t\cos\theta/n)^{-n}$ converges to e^t, which is sufficient.

V Saddle-Point Method

Functions with Fast Singular Growth

(Functions with fast singular growth)

$$[z^n]f(z) = \frac{1}{2\pi i}\oint \underbrace{\frac{f(z)}{z^{n+1}}}_{=:\exp(h(z))} dz$$



1. **Saddle-point equation**: $h'(R_n) = 0$ i.e. $R_n \frac{f'(R_n)}{f(R_n)} - 1 = n$
2. **Change of variables:** $h(z) = h(\rho) - u^2$
3. **Termwise integration:**

$$f_n \approx \frac{f(R_n)}{R_n^{n+1}\sqrt{2\pi h''(R_n)}}$$

4. **Sufficient conditions:** Hayman (1st order), Harris & Schoenfeld, Odlyzko & Richmond, Wyman.

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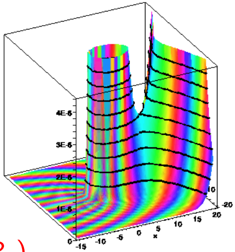

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Exercise

Stirling's formula ($f = \exp$).

4. **Sufficient conditions:** Hayman (1st order), Harris & Schoenfeld, Odlyzko & Richmond, Wyman.

Hayman admissibility

A set of analytic conditions and easy-to-use sufficient conditions.

Theorem

Hyp. *f, g admissible, P polynomial*

1. $\exp(f)$, *fg and $f + P$ admissible.*
2. $\operatorname{lc}(P) > 0 \Rightarrow fP$ *and* $P(f)$ *admissible.*
3. *if e^P has ultimately positive coefficients, it is admissible.*

Example

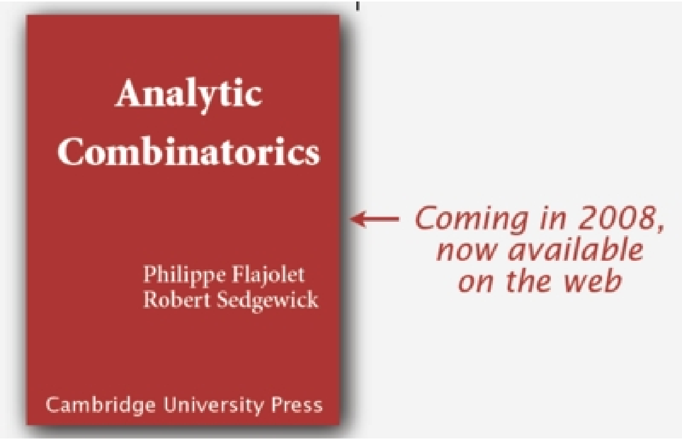
- sets ($\exp(z)$),
- involutions ($\exp(z + z^2/2)$),
- set partitions ($\exp(\exp(z) - 1)$).

VI Conclusion

Summary

- Many generating functions are analytic;
- Asymptotic information on their coefficients can be extracted from their singularities;
- Starting from bivariate generating functions gives asymptotic averages or variances of parameters;
- A lot of this can be automated.

Want More Information?



Algolib can be downloaded from http://algo.inria.fr

```
> libname:="/Users/salvy/lib/maple/Algolib/11",libname:
```

Dominant singularity

Rational generating functions

Fibonacci

```
> infsing(1/(1-z-z^2),z);
```

$$\left[\left[-\frac{1}{2}+\frac{1}{2}\sqrt{5}\right], \text{polar}, \mathit{false}\right]$$

asymptotic behaviour:

```
> equivalent(1/(1-z-z^2),z,n);
```

$$\frac{\left(\mathrm{e}^{-n}\right)^{-\ln(2)+\ln\left(-1+\sqrt{5}\right)}}{-\frac{1}{2}+\frac{1}{2}\sqrt{5}+2\left(-\frac{1}{2}+\frac{1}{2}\sqrt{5}\right)^{2}}+\mathrm{O}\left(\frac{\left(\mathrm{e}^{-n}\right)^{-\ln(2)+\ln\left(-1+\sqrt{5}\right)}}{n}\right)$$

```
> read "conway.mpl";
```

$$\begin{aligned}
\mathit{GFconway} := {} & -\big(-1+z^{2}-z+z^{3}+12z^{78}+6z^{11}-20z^{30}-30z^{29}+z^{4}-20z^{73}+18z^{76} \\
& -4z^{69}+18z^{74}+31z^{71}-4z^{68}-z^{18}+3z^{19}-36z^{24}+58z^{27}+13z^{22}+8z^{12}-4z^{17} \\
& -23z^{31}+15z^{70}-6z^{23}-20z^{25}+8z^{21}-z^{13}-z^{16}+6z^{20}-6z^{9}-18z^{77}-5z^{14} \\
& -18z^{75}-22z^{72}-4z^{15}+45z^{55}-11z^{63}+41z^{62}+54z^{61}-56z^{60}+15z^{58}-44z^{59} \\
& -27z^{57}+62z^{66}-21z^{64}-19z^{67}-50z^{65}+34z^{28}+z^{5}-4z^{8}+35z^{32}+7z^{38}+12z^{36} \\
& -79z^{39}+107z^{43}+8z^{35}-13z^{40}+38z^{49}+16z^{41}-z^{26}+z^{7}-64z^{52}-15z^{56} \\
& +89z^{53}-25z^{50}-8z^{54}+126z^{48}-26z^{34}-9z^{33}+42z^{37}-39z^{47}-32z^{46}-66z^{51} \\
& -33z^{45}+14z^{42}-65z^{44}\big)\big/\big((z-1)\,(-1+z^{2}+2z^{3}+z^{11}-8z^{30}-6z^{29}+z^{4}+6z^{69} \\
& +6z^{71}-12z^{68}+3z^{18}+2z^{19}+3z^{24}+8z^{27}-z^{22}+z^{12}+10z^{17}+5z^{31}-3z^{70} \\
& -9z^{23}+7z^{25}-6z^{21}-2z^{13}+2z^{16}-6z^{20}+z^{9}-5z^{14}-3z^{15}+7z^{55}-5z^{62}+2z^{61} \\
& +4z^{60}-2z^{58}+12z^{59}-7z^{57}-7z^{66}-z^{64}+z^{10}+4z^{67}+7z^{65}-10z^{28}-2z^{5}+z^{8} \\
& +12z^{32}-10z^{38}-z^{36}-z^{39}+3z^{43}-7z^{35}+6z^{40}+2z^{41}+8z^{26}-z^{7}-3z^{52}-12z^{56} \\
& +4z^{53}+7z^{50}+10z^{54}+8z^{48}+7z^{34}-7z^{33}+3z^{37}-14z^{47}+3z^{46}-9z^{51}-9z^{45} \\
& +10z^{42}-2z^{44}-2z^{6})\big)
\end{aligned}$$

```
> infsing(GFconway,z);
```

$$\begin{aligned}
& [[\mathrm{RootOf}\big(-1+_Z^{2}+2_Z^{3}+_Z^{11}-8_Z^{30}-6_Z^{29}+_Z^{4}+6_Z^{69}+6_Z^{71}-12_Z^{68} \\
& +3_Z^{18}+2_Z^{19}+3_Z^{24}+8_Z^{27}-_Z^{22}+_Z^{12}+10_Z^{17}+5_Z^{31}-3_Z^{70} \\
& -9_Z^{23}+7_Z^{25}-6_Z^{21}-2_Z^{13}+2_Z^{16}-6_Z^{20}+_Z^{9}-5_Z^{14}-3_Z^{15} \\
& +7_Z^{55}-5_Z^{62}+2_Z^{61}+4_Z^{60}-2_Z^{58}+12_Z^{59}-7_Z^{57}-7_Z^{66}-_Z^{64} \\
& +_Z^{10}+4_Z^{67}+7_Z^{65}-10_Z^{28}-2_Z^{5}+_Z^{8}+12_Z^{32}-10_Z^{38}-_Z^{36}-_Z^{39}
\end{aligned}$$

$$+ 3\,\_Z^{43} - 7\,\_Z^{35} + 6\,\_Z^{40} + 2\,\_Z^{41} + 8\,\_Z^{26} - \_Z^{7} - 3\,\_Z^{52} - 12\,\_Z^{56} + 4\,\_Z^{53}$$
$$+ 7\,\_Z^{50} + 10\,\_Z^{54} + 8\,\_Z^{48} + 7\,\_Z^{34} - 7\,\_Z^{33} + 3\,\_Z^{37} - 14\,\_Z^{47} + 3\,\_Z^{46} - 9\,\_Z^{51}$$
$$- 9\,\_Z^{45} + 10\,\_Z^{42} - 2\,\_Z^{44} - 2\,\_Z^{6}, 0.7671198507 \big) \big], \mathrm{polar}, \mathit{false} \big]$$

It's the root of this polynomial that is approximately 0.767. We call it α later.

```
> alias(alpha=%[1][1]):
```

asymptotic behaviour of the coefficients:

```
> equivalent(GFconway,z,n);
```

$$-\Big(\big(-1 - 64\,\alpha^{52} - 4\,\alpha^{15} - 26\,\alpha^{34} + 38\,\alpha^{49} + 16\,\alpha^{41} - \alpha^{26} - 66\,\alpha^{51} - 33\,\alpha^{45} + 41\,\alpha^{62}$$
$$+ 12\,\alpha^{36} + 58\,\alpha^{27} + 42\,\alpha^{37} - 39\,\alpha^{47} - 32\,\alpha^{46} + \alpha^{7} - \alpha^{16} + 13\,\alpha^{22} - \alpha^{13} - 9\,\alpha^{33}$$
$$+ 31\,\alpha^{71} - 4\,\alpha^{68} - 50\,\alpha^{65} - \alpha^{18} + \alpha^{5} - 4\,\alpha^{8} + 35\,\alpha^{32} + 54\,\alpha^{61} - 56\,\alpha^{60} + 6\,\alpha^{20}$$
$$- 4\,\alpha^{17} - 23\,\alpha^{31} - 13\,\alpha^{40} - 65\,\alpha^{44} + 62\,\alpha^{66} - 22\,\alpha^{72} + 7\,\alpha^{38} + 34\,\alpha^{28} - 25\,\alpha^{50}$$
$$- 6\,\alpha^{23} - 20\,\alpha^{25} + 8\,\alpha^{21} + 8\,\alpha^{12} - 21\,\alpha^{64} - 19\,\alpha^{67} + 15\,\alpha^{70} + 15\,\alpha^{58} + 3\,\alpha^{19} - 36\,\alpha^{24}$$
$$- 15\,\alpha^{56} + 89\,\alpha^{53} - 44\,\alpha^{59} - 27\,\alpha^{57} - 8\,\alpha^{54} + 126\,\alpha^{48} + 45\,\alpha^{55} + 14\,\alpha^{42} - 79\,\alpha^{39}$$
$$+ 18\,\alpha^{74} + \alpha^{2} + \alpha^{3} + 12\,\alpha^{78} - 20\,\alpha^{73} - \alpha - 5\,\alpha^{14} - 18\,\alpha^{75} - 11\,\alpha^{63} - 6\,\alpha^{9} - 18\,\alpha^{77}$$
$$+ 18\,\alpha^{76} - 30\,\alpha^{29} + \alpha^{4} - 20\,\alpha^{30} + 6\,\alpha^{11} + 107\,\alpha^{43} + 8\,\alpha^{35} - 4\,\alpha^{69}\big)\,\big(4\,\alpha^{52} + 2\,\alpha^{15}$$
$$- 7\,\alpha^{34} + 7\,\alpha^{49} + 10\,\alpha^{41} + 8\,\alpha^{26} - 3\,\alpha^{51} + 3\,\alpha^{45} + 3\,\alpha^{36} - 10\,\alpha^{27} - 10\,\alpha^{37} + 8\,\alpha^{47}$$
$$- 14\,\alpha^{46} + \alpha^{7} + 10\,\alpha^{16} - 9\,\alpha^{22} - 5\,\alpha^{13} + 7\,\alpha^{33} + 6\,\alpha^{68} - 7\,\alpha^{65} + 2\,\alpha^{18} - 2\,\alpha^{5} + \alpha^{8}$$
$$- 7\,\alpha^{32} - 5\,\alpha^{61} + 2\,\alpha^{60} - 6\,\alpha^{20} + 3\,\alpha^{17} + 12\,\alpha^{31} + 2\,\alpha^{40} - 9\,\alpha^{44} + 4\,\alpha^{66} - \alpha^{38}$$
$$- 6\,\alpha^{28} - 9\,\alpha^{50} + 3\,\alpha^{23} + 8\,\alpha^{25} - \alpha^{21} - 2\,\alpha^{12} + 7\,\alpha^{64} - 12\,\alpha^{67} + 6\,\alpha^{70} + 12\,\alpha^{58}$$
$$- 6\,\alpha^{19} + 7\,\alpha^{24} - 7\,\alpha^{56} + 10\,\alpha^{53} + 4\,\alpha^{59} - 2\,\alpha^{57} + 7\,\alpha^{54} - 12\,\alpha^{55} + 3\,\alpha^{42} + 6\,\alpha^{39}$$
$$+ 2\,\alpha^{2} + \alpha^{3} + \alpha - 3\,\alpha^{14} - \alpha^{63} + \alpha^{9} - 8\,\alpha^{29} - 2\,\alpha^{4} + 5\,\alpha^{30} + \alpha^{11} - 2\,\alpha^{43} - \alpha^{35}$$
$$- 3\,\alpha^{69} - \alpha^{6} + \alpha^{10}\big)^{-\ln(e^{-n})}\Big) \Big/ \Big((\alpha - 1)\,\big(156\,\alpha^{52} + 45\,\alpha^{15} - 238\,\alpha^{34} - 82\,\alpha^{41}$$
$$- 208\,\alpha^{26} + 459\,\alpha^{51} + 405\,\alpha^{45} + 310\,\alpha^{62} + 36\,\alpha^{36} - 216\,\alpha^{27} - 111\,\alpha^{37} + 658\,\alpha^{47}$$
$$- 138\,\alpha^{46} + 7\,\alpha^{7} - 32\,\alpha^{16} + 22\,\alpha^{22} + 26\,\alpha^{13} + 231\,\alpha^{33} - 426\,\alpha^{71} + 816\,\alpha^{68} - 455\,\alpha^{65}$$
$$- 54\,\alpha^{18} + 10\,\alpha^{5} - 8\,\alpha^{8} - 384\,\alpha^{32} - 122\,\alpha^{61} - 240\,\alpha^{60} + 120\,\alpha^{20} - 170\,\alpha^{17} - 155\,\alpha^{31}$$
$$- 240\,\alpha^{40} + 88\,\alpha^{44} + 462\,\alpha^{66} + 380\,\alpha^{38} + 280\,\alpha^{28} - 350\,\alpha^{50} + 207\,\alpha^{23} - 175\,\alpha^{25}$$
$$+ 126\,\alpha^{21} - 12\,\alpha^{12} + 64\,\alpha^{64} - 268\,\alpha^{67} + 210\,\alpha^{70} + 116\,\alpha^{58} - 38\,\alpha^{19} - 72\,\alpha^{24}$$

$$+ 672\,\alpha^{56} - 212\,\alpha^{53} - 708\,\alpha^{59} + 399\,\alpha^{57} - 540\,\alpha^{54} - 384\,\alpha^{48} - 385\,\alpha^{55} - 420\,\alpha^{42} + 39\,\alpha^{39} - 2\,\alpha^{2} - 6\,\alpha^{3} + 70\,\alpha^{14} - 9\,\alpha^{9} + 174\,\alpha^{29} - 4\,\alpha^{4} + 240\,\alpha^{30} - 11\,\alpha^{11} - 129\,\alpha^{43} + 245\,\alpha^{35} - 414\,\alpha^{69} + 12\,\alpha^{6} - 10\,\alpha^{10})\Big) + \mathrm{O}\Big(\frac{1}{n}\big(4\,\alpha^{52} + 2\,\alpha^{15} - 7\,\alpha^{34} + 7\,\alpha^{49} + 10\,\alpha^{41} + 8\,\alpha^{26} - 3\,\alpha^{51} + 3\,\alpha^{45} + 3\,\alpha^{36} - 10\,\alpha^{27} - 10\,\alpha^{37} + 8\,\alpha^{47} - 14\,\alpha^{46} + \alpha^{7} + 10\,\alpha^{16} - 9\,\alpha^{22} - 5\,\alpha^{13} + 7\,\alpha^{33} + 6\,\alpha^{68} - 7\,\alpha^{65} + 2\,\alpha^{18} - 2\,\alpha^{5} + \alpha^{8} - 7\,\alpha^{32} - 5\,\alpha^{61} + 2\,\alpha^{60} - 6\,\alpha^{20} + 3\,\alpha^{17} + 12\,\alpha^{31} + 2\,\alpha^{40} - 9\,\alpha^{44} + 4\,\alpha^{66} - \alpha^{38} - 6\,\alpha^{28} - 9\,\alpha^{50} + 3\,\alpha^{23} + 8\,\alpha^{25} - \alpha^{21} - 2\,\alpha^{12} + 7\,\alpha^{64} - 12\,\alpha^{67} + 6\,\alpha^{70} + 12\,\alpha^{58} - 6\,\alpha^{19} + 7\,\alpha^{24} - 7\,\alpha^{56} + 10\,\alpha^{53} + 4\,\alpha^{59} - 2\,\alpha^{57} + 7\,\alpha^{54} - 12\,\alpha^{55} + 3\,\alpha^{42} + 6\,\alpha^{39} + 2\,\alpha^{2} + \alpha^{3} + \alpha - 3\,\alpha^{14} - \alpha^{63} + \alpha^{9} - 8\,\alpha^{29} - 2\,\alpha^{4} + 5\,\alpha^{30} + \alpha^{11} - 2\,\alpha^{43} - \alpha^{35} - 3\,\alpha^{69} - \alpha^{6} + \alpha^{10}\big)^{n}\Big)$$

Numerical value:

```
> evalf(%);
```

$$2.042160079\; 1.303577270^{-1.\ln(e^{-1.n})} + \mathrm{O}\left(\frac{1.303577270^{n}}{n}\right)$$

```
> map(simplify,%) assuming n::posint;
```

$$2.042160079\, e^{0.2651122315\,n} + \mathrm{O}\left(\frac{e^{0.2651122315\,n}}{n}\right)$$

Meromorphic functions

Derangements

```
> derangements:={S=Set(Cycle(Z,card>1))};
```

$$derangements := \{S = Set(Cycle(Z, 1 < card))\}$$

```
> combstruct[gfsolve](derangements,labelled,z);
```

$$\left\{Z(z) = z, S(z) = e^{\ln\left(\frac{1}{1-z}\right) - z}\right\}$$

```
> der:=simplify(subs(%,S(z)));
```

$$der := -\frac{e^{-z}}{z-1}$$

```
> infsing(der,z);
```

$$[[1], \text{polar}, \mathit{false}]$$

asymptotic number:

```
> equivalent(der,z,n);
```

$$e^{-1} + \mathrm{O}\left(\frac{1}{n}\right)$$

Surjections

```
> surjections:={S=Sequence(Set(Z,card>0))};
```

$$surjections := \{S = Sequence(Set(Z, 0 < card))\}$$

```
> combstruct[gfsolve](surjections,labelled,z);
```

$$\left\{Z(z) = z, S(z) = -\frac{1}{-2 + e^z}\right\}$$

```
> surj:=subs(%,S(z));
```

$$surj := -\frac{1}{-2 + e^z}$$

```
> infsing(surj,z);
```

$$[[\ln(2)], \text{polar}, false]$$

asymptotic number

```
> equivalent(surj,z,n);
```

$$\frac{1}{2}\frac{(e^{-n})^{\ln(\ln(2))}}{\ln(2)} + O\left(\frac{(e^{-n})^{\ln(\ln(2))}}{n}\right)$$

```
> map(simplify,%) assuming n::posint;
```

$$\frac{1}{2}\ln(2)^{-1-n} + O\left(\frac{\ln(2)^{-n}}{n}\right)$$

Bernoulli numbers

```
> infsing(z/(exp(z)-1),z);
```

$$[[-2\,I\,\pi, 2\,I\,\pi], \text{polar}, false]$$

Iterative constructions

Binary trees of cycles of cycles

```
> btcc:={S=Union(CC,Prod(S,S)),CC=Cycle(Cycle(Z))};
```

$$btcc := \{S = Union(CC, Prod(S, S)), CC = Cycle(Cycle(Z))\}$$

```
> combstruct[gfsolve](btcc,labelled,z);
```

$$\left\{Z(z) = z, S(z) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\ln\left(-\frac{1}{-1 + \ln\left(-\frac{1}{z-1}\right)}\right)}, CC(z) = \ln\left(-\frac{1}{-1 + \ln\left(-\frac{1}{z-1}\right)}\right)\right\}$$

```
> btcc:=subs(%,S(z));
```

$$btcc := \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\ln\left(-\frac{1}{-1 + \ln\left(-\frac{1}{z-1}\right)}\right)}$$

```
> infsing(btcc,z);
```

$$\left[\left[\frac{e^{\frac{e^{1/4}-1}{e^{1/4}}}-1}{e^{\frac{e^{1/4}-1}{e^{1/4}}}}\right], algebraic, false\right]$$

Singularity analysis

Binary trees:

```
> equivalent((1-sqrt(1-4*z))/2/z,z,n,5);
```

$$\frac{\left(\frac{1}{n}\right)^{3/2}\left(e^{-n}\right)^{-2\ln(2)}}{\sqrt{\pi}}-\frac{9}{8}\,\frac{\left(\frac{1}{n}\right)^{5/2}\left(e^{-n}\right)^{-2\ln(2)}}{\sqrt{\pi}}+\frac{145}{128}\,\frac{\left(\frac{1}{n}\right)^{7/2}\left(e^{-n}\right)^{-2\ln(2)}}{\sqrt{\pi}}-\frac{1155}{1024}\,\frac{\left(\frac{1}{n}\right)^{9/2}\left(e^{-n}\right)^{-2\ln(2)}}{\sqrt{\pi}}+\frac{36939}{32768}\,\frac{\left(\frac{1}{n}\right)^{11/2}\left(e^{-n}\right)^{-2\ln(2)}}{\sqrt{\pi}}+O\left(\left(\frac{1}{n}\right)^{13/2}\left(e^{-n}\right)^{-2\ln(2)}\right)$$

Cayley trees:

```
> Cayley:={T=Prod(Z,Set(T))};
```

$$Cayley := \{T=Prod(Z, Set(T))\}$$

```
> combstruct[gfsolve](Cayley,labelled,z);
```

$$\{Z(z)=z, T(z)=-\mathrm{LambertW}(-z)\}$$

```
> equivalent(subs(%,T(z)),z,n);
```

$$\frac{1}{2}\,\frac{\sqrt{2}\,\sqrt{e}\,\sqrt{e^{-1}}\left(\frac{1}{n}\right)^{3/2}e^{n}}{\sqrt{\pi}}+O\left(\frac{e^{n}}{n^{2}}\right)$$

```
> map(simplify,%) assuming n::posint;
```

$$\frac{1}{2}\,\frac{\sqrt{2}\,e^{n}}{\sqrt{\pi}\,n^{3/2}}+O\left(\frac{e^{n}}{n^{2}}\right)$$

Binary trees of cycles of cycles

```
> equivalent(btcc,z,n):
> map(simplify,%) assuming n::posint;
```

$$\frac{1}{2}\,\frac{\left(e^{\left(e^{1/4}-1\right)e^{-1/4}}-1\right)^{\frac{1}{2}-n}e^{\frac{1}{8}\left(e^{1/4}+8\,n\,e^{1/4}-8\,n\right)e^{-1/4}}}{\sqrt{\pi}\,n^{3/2}}$$

$$+\,\mathrm{O}\left(\frac{e^{-\left(-e^{\frac{1}{4}}+1+\ln\left(e^{\left(e^{\frac{1}{4}}-1\right)e^{-\frac{1}{4}}}-1\right)e^{\frac{1}{4}}\right)e^{-\frac{1}{4}}n}}{n^{5/2}}\right)$$

Saddle-point method

Sets

```
> equivalent(exp(z),z,n);
```

$$\frac{1}{2}\,\frac{\sqrt{2}\,\sqrt{\frac{1}{n}}\,e^{n}\,n^{-n}}{\sqrt{\pi}}+\mathrm{O}\left(\left(\frac{1}{n}\right)^{3/2}e^{n}\,n^{-n}\right)$$

Involutions

```
> equivalent(exp(z+z^2/2),z,n);
```

$$\frac{1}{2}\,\frac{e^{-\frac{1}{4}}\sqrt{\frac{1}{n}}\,e^{\frac{1}{\sqrt{\frac{1}{n}}}}\sqrt{n^{-n}}}{\sqrt{\pi}\,\sqrt{e^{-n}}}+\mathrm{O}\left(\frac{e^{\frac{\cos\left(\left(\frac{1}{4}-\frac{1}{4\,\mathrm{signum}(n)}\right)\pi\right)}{\sqrt{\frac{1}{|n|}}}}\sqrt{n^{-n}}}{n\,\sqrt{e^{-n}}}\right)$$

```
> map(simplify,%) assuming n::posint;
```

$$\frac{1}{2}\,\frac{e^{-\frac{1}{4}+\sqrt{n}+\frac{1}{2}n}\,n^{-\frac{1}{2}-\frac{1}{2}n}}{\sqrt{\pi}}+\mathrm{O}\left(e^{\sqrt{n}+\frac{1}{2}n}\,n^{-1-\frac{1}{2}n}\right)$$

An example with a singularity at finite distance

```
> equivalent(exp(z/(1-z)),z,n);
```

$$\frac{1}{8}\,\frac{\sqrt{2}\,e^{-\frac{1}{2}}\,4^{3/4}\left(\frac{1}{n}\right)^{3/4}e^{\frac{2}{\sqrt{\frac{1}{n}}}}}{\sqrt{\pi}}+\mathrm{O}\left(\left(\frac{1}{n}\right)^{5/4}e^{\frac{2\cos\left(\left(\frac{1}{4}-\frac{1}{4\,\mathrm{signum}(n)}\right)\pi\right)}{\sqrt{\frac{1}{|n|}}}}\right)$$

Set partitions

```
> equivalent(exp(exp(z)-1),z,n);
```

The saddle point is , LambertW$(n+1)$

Saddle point's expansion:

$$\ln(n)-\ln(\ln(n))+\frac{\ln(\ln(n))}{\ln(n)}+\mathrm{O}\left(\frac{\ln(\ln(n))^2}{\ln(n)^2}\right)$$

$$\frac{1}{2}\,\frac{\sqrt{2}\,e^{-1}\sqrt{e^{-\_saddlepoint}}\,e^{e^{\_saddlepoint}}}{\_saddlepoint^{n}\sqrt{\pi}\,\_saddlepoint}+\mathrm{O}\left(\frac{\sqrt{e^{-\_saddlepoint}}\,e^{e^{\_saddlepoint}}}{\_saddlepoint^{2}\,\_saddlepoint^{n}}\right)$$

```
>
```