FIND REVISITED

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Stochastic fixed points for processes.
Setting: On D

$$X \stackrel{\mathcal{D}}{=} \sum_i A_i X_i \circ B_i + C.$$

$((A_i, B_i)_i, C), X_j, j \in I\!N$ independent, $X_j \sim X$.

Why another talk on Find?
– Toy example
– Introduction of FIND.
– Runtime analysis of Gruebel-Roesler.
– Problem of 2- and 3-version.
– FIND-process as fixed point.
– Knof's approach via finite marginals.
– Solution via Weighted Branching Process.

FIND or QUICKSELECT

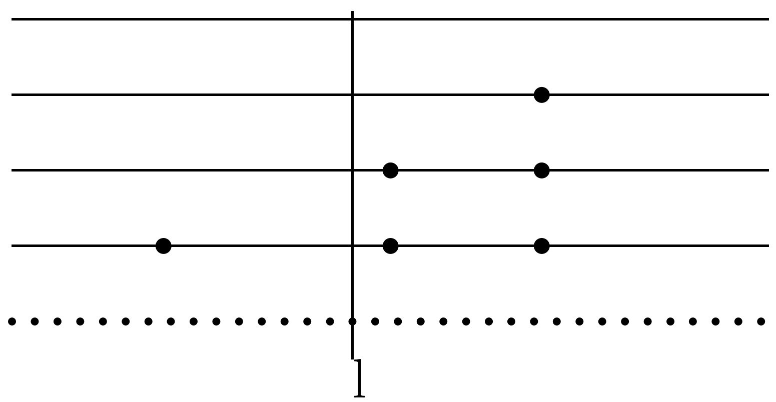
Algorithm: FIND or QUICKSELECT
and Algorithm 63, PARTITION, Hoare 61,

– **Input:** Set S of n different reals.

– **Output:** l-th smallest in S

– **Procedure:** Divide and conquer FIND(S, l):

Choose random pivot within S
Split S into $S_<, \{p\}, S_>$
Continue with $S_<$ or $\{p\}$ or $S_>$.



internal versus external randomness
FIND(S, l)=FIND$(|S|, l)$

ANALYSIS

Running time is proportional to
$X_n(l)$ = number of comparisons for FIND(n, l)

Key equation:

$$X_n(l) = n - 1 + \mathbb{1}_{U_n > l} X_{U_n - 1}(l) + \mathbb{1}_{U_n < l} \overline{X}_{n - U_n}(l - U_n)$$

$X, \overline{X}, U_n$ independent,
U_n uniformly on $\{1, 2, \ldots, n\}$, $\qquad X \sim \overline{X}$
Notice U_n is rank of pivot, (= position after comparisons).

DIVERSES

best case: $X_n(l) = n - 1$
worst case: $X_n(l) = \frac{n(n-1)}{2}$
average: H_n harmonic numbers

$$EX_n(l) = 2((n+1)H_n - (n+3-k)H_{n+1-l} - (l+2)H_l + n + 3)$$

$$\lim_{\frac{l}{n}\to t} \frac{EX_n(l)}{n} = 2 - 2t\ln t - 2(1-t)\ln(1-t)$$

Knuth 71

variance: explicit Kirschenhofer, Prodinger 98,

tail: $\frac{X_n(l)}{n} = C$, Devroye 84,

$$E(C) \; \leq \; 4 \qquad E(C^k) = O(1)$$

$$P(C \geq c) \; \leq \; \left(\frac{3}{4}\right)^{(1+o(1))c}$$

RECALLS

Recalls:
$R_n(l)$ = number of recalls for FIND(n, l).

$$R_n(l) \stackrel{\mathcal{D}}{=} \mathbb{1}_{U>l} R_{U-1}(l) + \mathbb{1}_{U<l} \overline{R}_{n-V}(l-V) + 1$$

$$ER_n(l) = H_l + H_{n-l} - 1$$

variance $R_n(l) = ...$
MAPLE, Kirschenhofer, Prodinger 1998
Grübel 2003

JOINT AVERAGES

Joint averages:
$X_n(J)$ = number for FIND(n, J)
Grand averages: Mahmoud et al 95

$$X_n^{(p)} := \frac{1}{\binom{n}{p}} \sum_{|J|=p} X_n(J)$$

Analog $R_n(J)$ and $R_n^{(p)}$.
First and second moment, Prodinger 95

DISTRIBUTION

Normalization:

$$Y_n(\frac{l}{n}) = \frac{X_n(l)}{n} \qquad Y_n : [0,1] \to I\!R, \text{ linear}$$

Theorem Grübel-Rösler 96
For FIND 2-version exists versions

$$Y_n \to_n Z$$

pointwise in $D[0,1]$ and Skorodhod metric.
Not true for 3 version!

3-version $\Leftrightarrow$ split into $S_<, \{p\}, S_>$
2-version $\Leftrightarrow$ split into $S_<, S_\geq$
Skorodhod metric d on cadlad functions

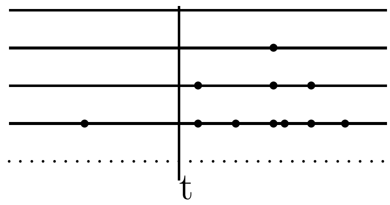
$$d(f,g) = \inf\{\epsilon \mid \exists \lambda \ \|\lambda - id\|_\infty, \|f - g \circ \lambda\|_\infty < \epsilon\}$$

where exists $\lambda : [0,1] \to [0,1]$ one-one and increasing.

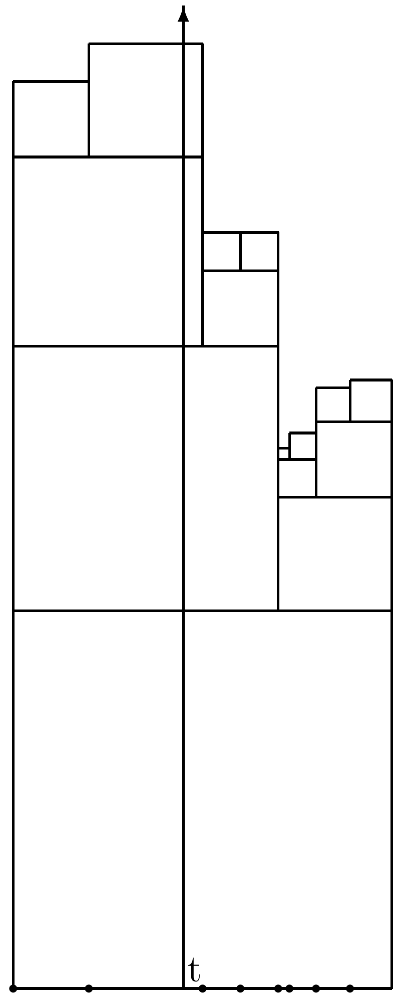
Intuition: Always $Y_n \to_n Z$ in distribution.
But: (D,d) not complete.

FIND PROCESS

Characterize **Find** process $(Z(t))_{t\in[0,1]}$

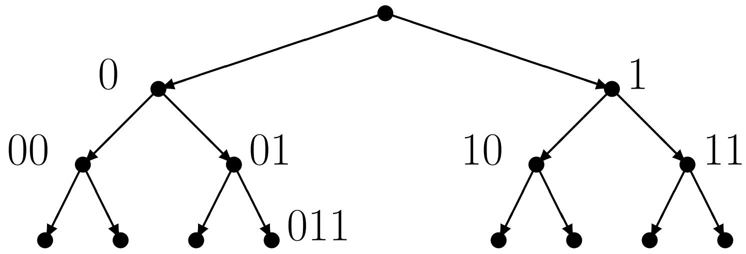


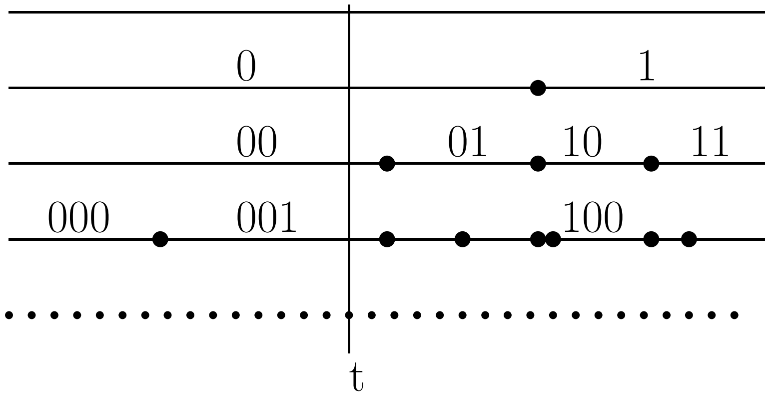
$Z(t)$ = sum over the length of intervals containing t



$Z(t)$ = height at t

RANDOM BINARY NUMBERS





$$[0,1) \ni \text{ value } t \leftrightarrow (t_1, t_2, \ldots) \text{ path } \in \{0,1\}^{\mathbb{N}}$$

$$Z_n(t) := 1 + \sum_{i=1}^{n} L(t_1, ..., t_i)$$

FIXED POINT EQUATION

Forward view and backward view

Forward view

$$Z_{n+1}(t) = Z_n(t) + L(t_1, \ldots, t_{n+1}).$$

$$Z_n(t) \nearrow_n Z(t)$$

Backward view

$$Z_{n+1}(t) \stackrel{\mathcal{D}}{=} 1 + 1\!\!1_{U>t} U Z_n(\frac{t}{U}) + 1\!\!1_{U\le t}(1-U)\overline{Z}_n(\frac{t-U}{1-U})$$

= with suitable choosen versions

$$(Z_{n+1}(t))_t = \left(1 + 1\!\!1_{U>t} U Z_n^{(0)}(\frac{t}{U}) + 1\!\!1_{U\le t}(1-U)\overline{Z}_n^{(1)}(\frac{t-U}{1-U})\right)_t$$

Fixed point equation

$$(Z(t))_t \stackrel{\mathcal{D}}{=} \left(1 + 1\!\!1_{t<U} U Z(\frac{t}{U}) + 1\!\!1_{U\le t}(1-U)\overline{Z}(\frac{t-U}{1-U})\right)_t$$

$Z, \overline{Z}, U$ independent, U uniform distributed.

Compare to key equation for Y_n

$$\begin{array}{ccccccccc} Y_n(\frac{l}{n}) & \stackrel{\mathcal{D}}{=} & 1\!\!1_{\frac{U_n}{n}>\frac{l}{n}} \frac{U_n}{n} Y_{U_n}(\frac{k}{U_n}) & + & 1\!\!1_{\frac{U_n}{n}<\frac{l}{n}} \frac{n-U_n}{n} \overline{Y}_{n-U_n}(\frac{l-U_n}{n-U_n}) & + & \frac{n-1}{n} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Z(t) & \stackrel{\mathcal{D}}{=} & 1\!\!1_{U>t} U Z(\frac{t}{U}) & + & 1\!\!1_{U\le t}(1-U)\overline{Z}(\frac{t-U}{1-U}) & + & 1 \end{array}$$

FIND MOMENTS

Expectation: $a(t) = EZ(t)$

$$a(t) = \int_t^1 ua(\frac{t}{u})du + \int_0^t (1-u)a(\frac{t-u}{1-u})du + 1$$

Fixed point equation

Rate

$$l_1\left(\frac{X_{n,l}^k}{n}, Z\left(\frac{l-1/2}{n}\right)\right) \sim \ln \frac{l^5(n-l+1)^5}{n^2}$$

Kodaj and Mori 1997 in Wasserstein metric

$$l_1(\mu,\nu) = \inf \|X-Y\|_1 = \int |F_\mu(x) - F_\nu(x)|dx$$

(Orlicz distance, limiting process)

HIGHER MOMENTS

Higher moments:

$$\lim_{n\to\infty} E\frac{X_n^k(l_n)}{n} = E(Z^k(t)) =: m_k(t)$$

$\frac{l_n}{n} \to_n t \in [0,1]$

m_k as **fixed point**, Paulsen 1997

$$K_k(f)(t) := \int_t^1 u^k f(\frac{t}{u})du + \int_0^t (1-u)^k f(\frac{t-u}{1-u})du + b_k(t)$$

$$b_k(t) := \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} m_{k-i}(t).$$

m_k is unique fixed point of K_k

$$K_k(m_k) = m_k$$

$$D^{k+2}m_k(t) = \left(\frac{1}{1-t} - \frac{1}{t}\right) D^{k+1}m_k(t) + D^{k+2}b_k(t)$$

D differentiation t.

Explicit solutions for m_1, m_2, More?

numerical approximations

Exponential moments exists, Devroye 84, Grübel-Rösler 96

$$\sup_t P(Z(t) \geq z) \leq e^{z+z\ln 4 - z\ln z}$$

Grübel 99, also discrete

SUPREMUM of FIND

Supremum: $\sup_l X_n(l)$
$M := \sup_t Z(t)$ satisfies fixed point equation

$$M \stackrel{\mathcal{D}}{=} 1 + (UM) \vee ((1-U)\overline{M})$$

$EM^2 < \infty$, unique solution Grübel-Rösler 96
Devroye 01

$$2.3862 < 1+2\ln 2 \leq E(M-1) \leq \frac{5}{\sqrt{2\pi}}+12\frac{12e}{5} < 8.5185$$

$$E(M-1)^k \leq 3^{k-1}k!E(M-1).$$

$$P(M \geq t) \leq Ce^{-\lambda t}.$$

$\forall \lambda > 0 \exists C \forall t.$

Lemma $M = \sup_t Z(t)$ or $M = \sup_l \frac{X_n(l)}{n}$. Then

$$P(M \geq t) \leq Ce^{-\lambda t}. \qquad (1)$$

Proof: Define

$$K(\mu) = \mathcal{L}(1 + (UX) \vee ((1-U)\overline{X})$$

If μ satisfies (1) then $K(\mu)$ satisfies (1)
K a strict contraction in l_2-metric

$$\mu, K(\mu), K^2(\mu), \ldots$$

μ satisfies the condition, all do, limit does.
Discrete analog K_n, yes q.e.d.

FIND as FIXED POINT

FIND process satisfies

$$(Z(t))_t \stackrel{\mathcal{D}}{=} (1\!\!1_{U>t} U Z_1(\frac{t}{U}) + 1\!\!1_{U\leq t}(1-U) Z_2(\frac{t-U}{1-U}) + 1)_t$$

Distribution is fixed point of map K

Contraction method:

– K contraction for suitable metric
– iterate K

Knof in thesis 2007

– On D

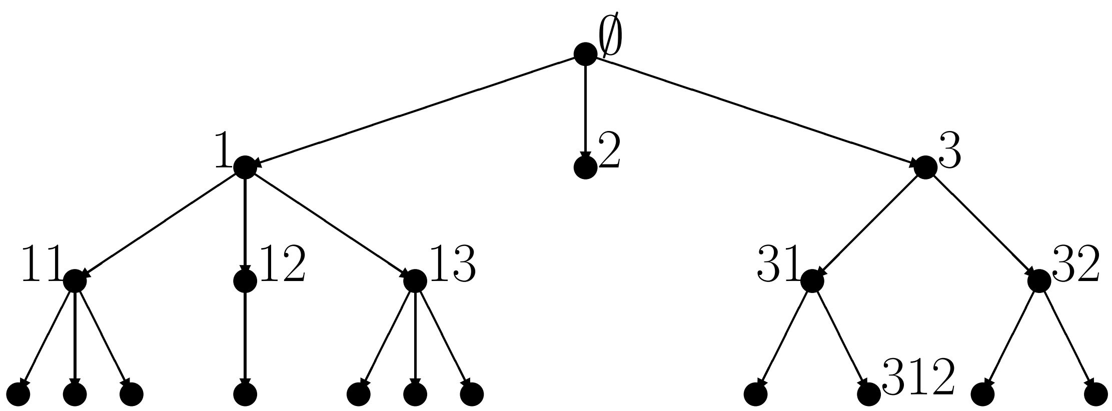
$$K(\mu) \sim \sum_i A_i X_i \circ B_i + C$$

$((A_i, B_i)_i, C), X_j, j \in \mathbb{N}$ independent, $X_j \sim \mu$.

– Suitable metric on finite dimensional marginals
– Finite dimensional marginals converge
$(K^n(\mu))_J \to_n \varphi(J)$ weakly
– $(\varphi(J))_J$ is consistent family of pm
– projective limit ν (on product space $\mathbb{R}^{[0,1]}$)
– outer $\nu^*(D) = 1$
Via Dubins-Hahn and recursive structure.
– Restrict ν to D. Done

Next aim FIND: $Y_n \to Z$.

WEIGHTED BRANCHING PROCESS



Weighted Branching Process is tupel $(V, T, (G, *))$.
– Tree $V = \cup_{n=0}^{\infty} \mathbb{N}^n$ as graph
– $(G, *)$ semigroup with grave and neutral
– edge (v, vi) carries random weight $T_i(v)$
Assumption: $T(v) = (T_i(v))_i,\ v \in V$ iid.
L_v path length to v $\quad L_{vi} = L_v * T_i(v)$.

Weighted Branching Process with utility is tupel $(V, T, U, (G, *, R))$.
– $(V, T, (G, *))$ is WBP.
– $(G, *)$ operates transitiv on R via $* : G \times R \to R$
– Every knot v carries rv U_v with values in R
Assumption: $(T(v), C_v),\ v \in V$ iid.
Object of interest $L_v * U_v$ or functions of it.

SETTING

Setting, on D

$$X \stackrel{\mathcal{D}}{=} \sum_i A_i X_i \circ B_i + C$$

A_i, C rv D_+-valued, B_i rvs $D_\uparrow$-valued
σ-field on D via Skorodhod metric, Billingsley
(isomorph to product space $(\mathbb{R}^I, \mathcal{B}^I)$ where $1 \in I$ dense in $[0,1]$.)
$G = \overline{D}_+ \times D_\uparrow$ with semi group operation

$$*((a,b),(a',b')) = (aa' \circ b, b' \circ b).$$

grave is function identically 0,
neutral element function identically 1
$R = D_+$ with pointwise multiplication
edge weight $(T(v), U_v) \sim ((A_i, B_i)_i, C)$
object of interest

$$R_n = \sum_{|v| \leq n} L_v * U_v.$$

Backward equation

$$R_{n+1} = \sum_{i \in \mathbb{N}} L_i * R_n(i) + L_\emptyset * U_\emptyset = \sum_{i \in \mathbb{N}} A_i R_n(i) \circ B_i + C$$

FIXED POINT

$$K(\mu) \stackrel{\mathcal{D}}{=} \sum_{i \in \mathbb{N}} A_i X_i \circ B_i + C.$$

Weighted Branching Process

$$R_n = \sum_{|v| \le n} L_v * U_v.$$

• $K^n(\delta_0) \sim R_n$.
$K^n(\delta_0) \to_n$ fixed point $\Leftrightarrow R_n \to_n R$.
• $R_n \nearrow R = \sum_{v \in V} L_v * U_v$.
$A(s) := \sum_i A_i(s)$ $\qquad \overline{f} = \sup_s |f(s)|$
Theorem *Assume* $E\overline{A} < 1$ *and* $E\overline{C} < \infty$. *Then*

$$E\overline{R_{n+1} - R_n} \le E^n(\overline{A})E(\overline{C})$$

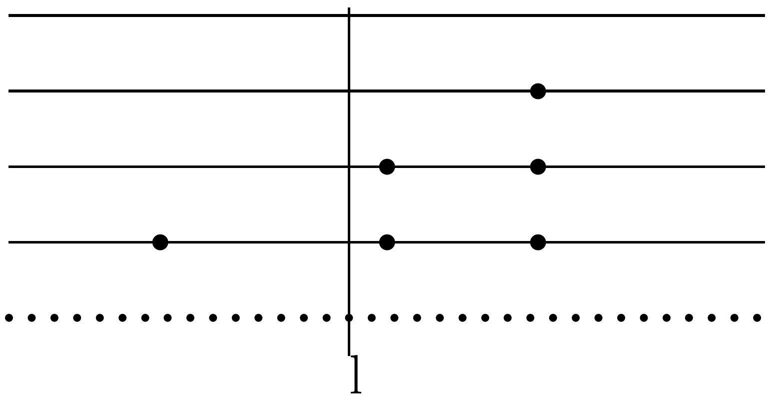
and R *takes values in* D.

The Supremum via fixed point of supremum-type

$$\overline{R} \le \overline{A} \sup_i \overline{R}(i) + \overline{C}.$$

Theorem *Assume* $\overline{Y} < 1$ *a.s.,* $\sum_i Y_i < \infty$ *and some exponential moments. Then* $\overline{R}$ *has exponential moments.*

MEDIAN-VERSION of FIND



– pivot is median of m randomly choosen
– costs $n - 1 \leq C_n \leq n + m^2$.
– Rank of median normalized $\frac{I_n}{n} \to_n U$ weak
  U median of m iid uniformly distributed
– Limit

$$(Z(t))_t \stackrel{\mathcal{D}}{=} \mathbb{1}_{t<U} U Z_1(\frac{t}{U}) + \mathbb{1}_{t \geq U}(1-U) Z_2(\frac{t-U}{1-U}) + 1$$

Corollary Y_n weakly to Z.

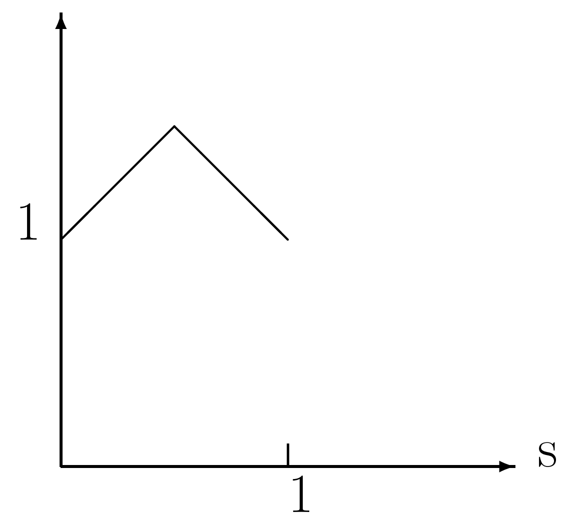
ADAPTED VERSION of FIND

– $m = m_n \to \infty$ slowly
– choose as pivot the $k^n(l)$ smallest out of m_n
– costs $n - 1 \leq C_n \leq n + (m_n)^2$.
– $\frac{k^n(l)}{m_n}$ close to $\frac{l}{n}$

Corollary Rösler
$Y_n \to Z$ weakly,

$$Z(s) = 1 + \min\{s, 1 - s\}$$

The very best possible!
Martinez, Panario and Viola '03



CONCLUSION

– (weaker) measure versus (stronger) process convergence

– Stochastic fixed points by contraction of

$$X \stackrel{\mathcal{D}}{=} \sum_i A_i X_i \circ B_i + C$$

Study of **Weighted Branching Processes** provides

– Fixed points of

$$X = \sum_i A_i X_i \circ B_i + C$$

– complete analysis of FIND

– Identified the very best version of FIND