On a discrete parking problem

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International Conference on the Analysis of Algorithms,
17.4.2008

Outline of the talk

A discrete parking problem

A discrete parking problem: Parking scheme

The parking scheme:

- Consider one-way street
- m parking lots are in a row
- n drivers wish to park in these lots
- Each driver has preferred parking lot to which he drives
- If parking lot is empty $\Rightarrow$ he parks there
- If not, he drives on and parks in the next free parking lot if there is one
- If all remaining parking lots are occupied $\Rightarrow$ leaves without parking

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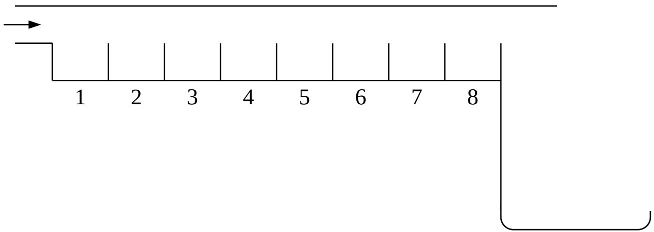
Example: 8 parking lots, 8 cars

Parking sequence: 3, 6, 3, 8, 6, 7, 4, 5



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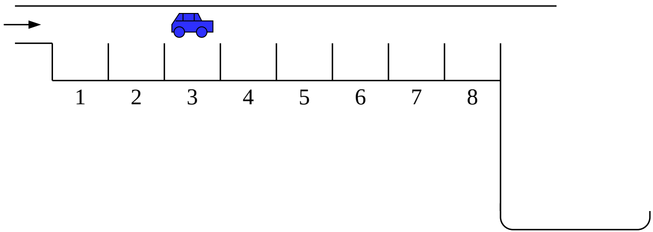

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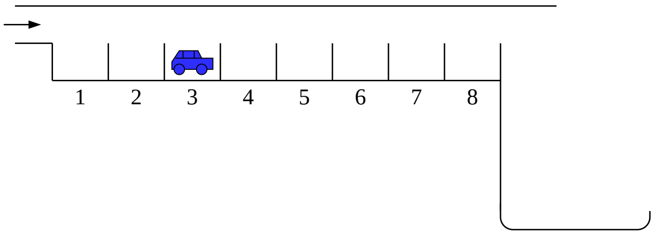

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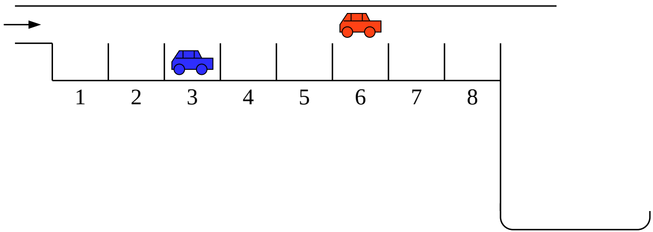

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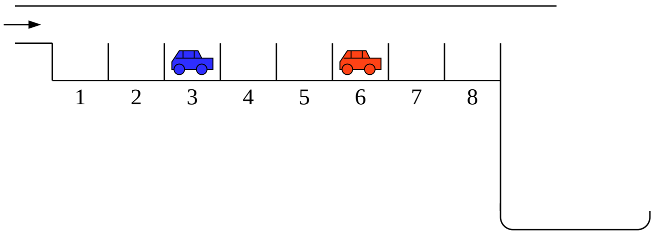

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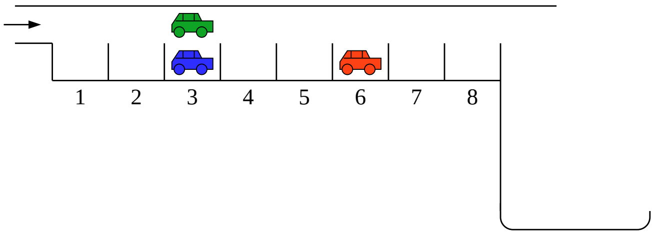

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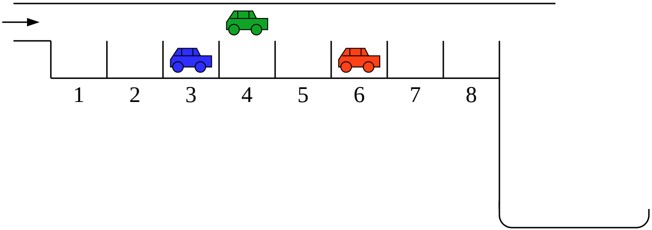

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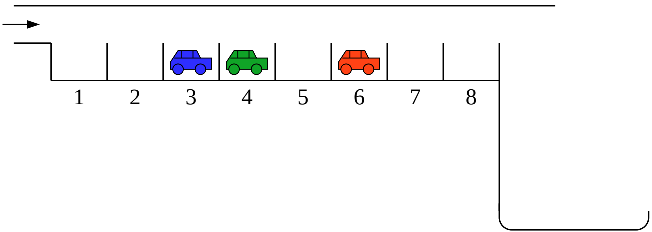

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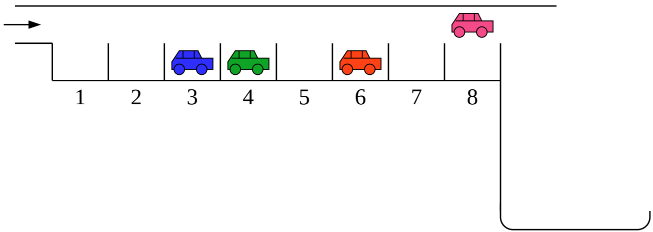

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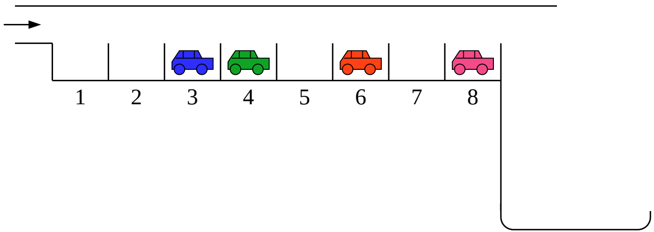

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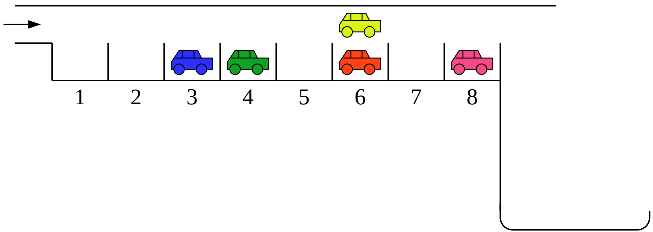

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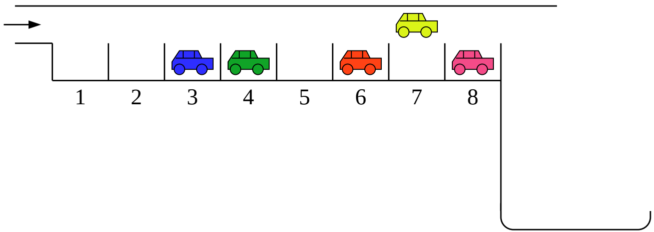

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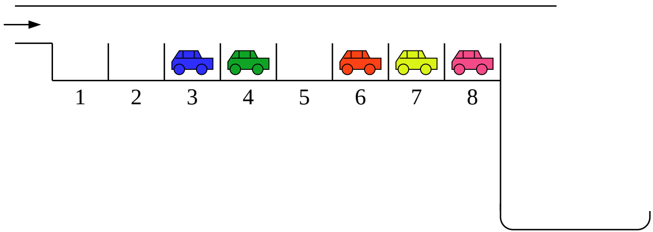

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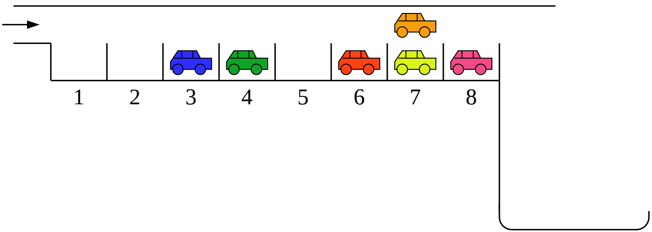

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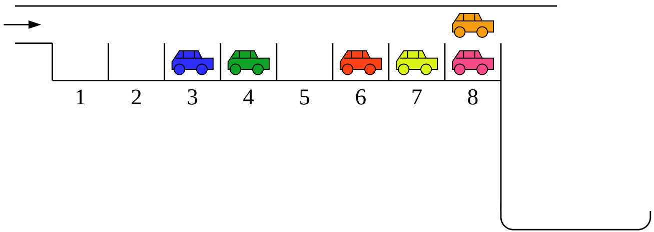

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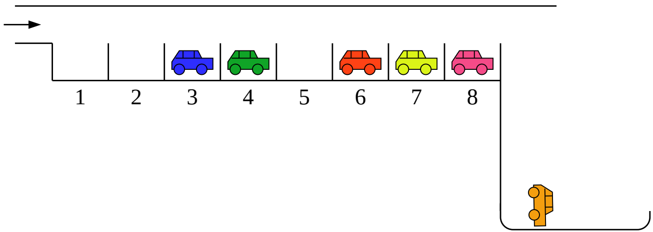

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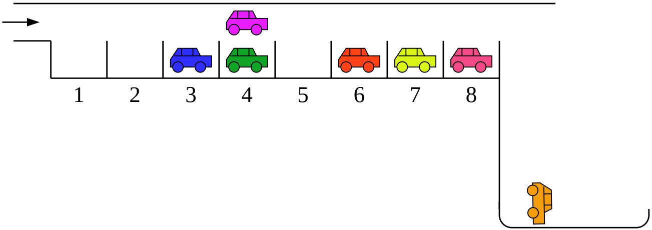

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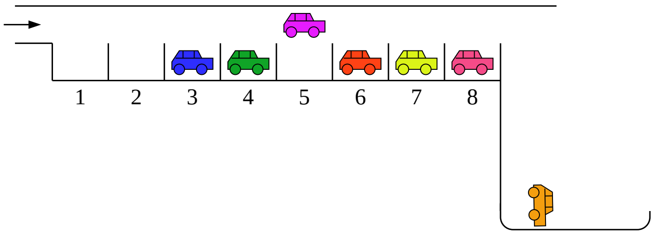

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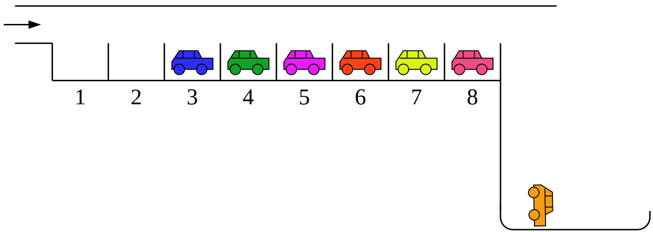

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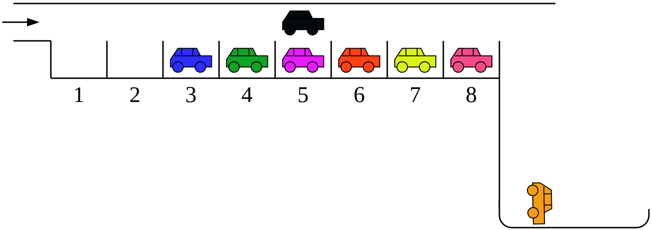

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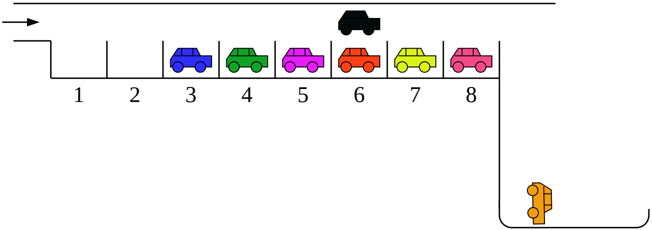

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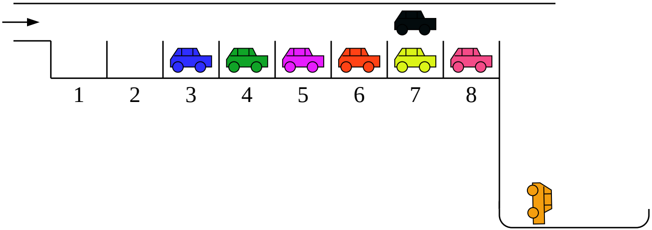

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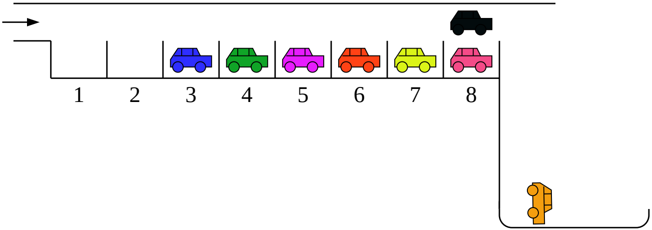

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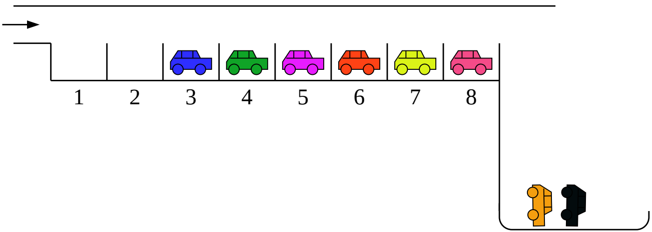

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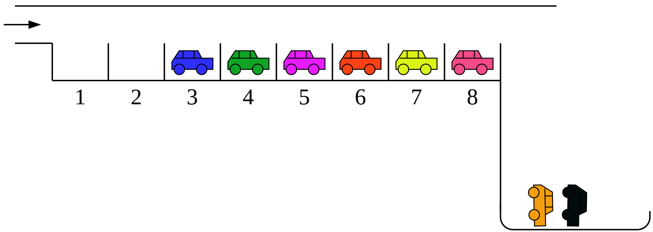

$\Rightarrow$ 2 cars are unsuccessful

A discrete parking problem: Unsuccessful cars

Number of unsuccessful cars:

Parking sequence $a_1, \ldots, a_n \in \{1, \ldots, m\}^n$
$\Rightarrow$ k unsuccessful cars $(\max(n-m, 0) \leq k \leq n-1)$

Formal description of $k = k(m; a_1, \ldots, a_n)$:

$$b_i := \#\ell : a_\ell \geq i$$

$$\Rightarrow k = \max_{1 \leq i \leq m+1} \{b_i + i\} - m - 1$$

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Parking functions: special instance $k = 0$
$\Rightarrow$ all cars can be parked

Introduced by Konheim and Weiss [1966]:
in analysis of linear probing hashing algorithm

- m places at a round table (≅ memory addresses)
- n guests arriving sequentially at certain places (≅ data elements)
- each guest goes clockwise to first empty place

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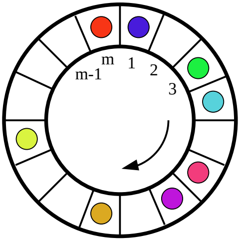

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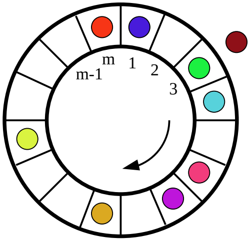

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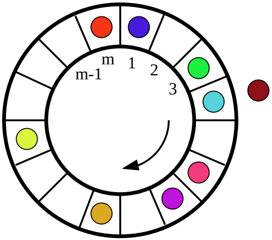

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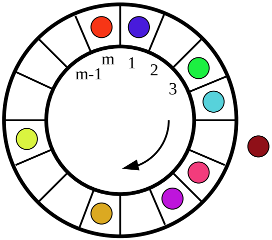

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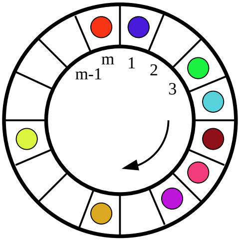

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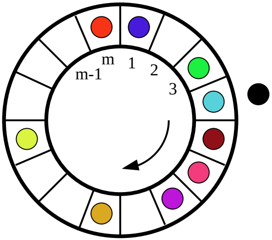

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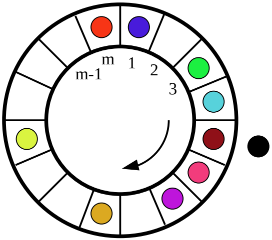

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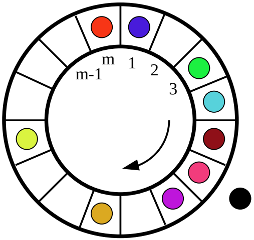

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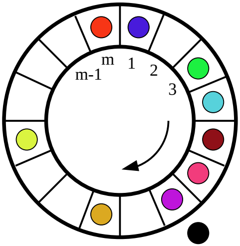

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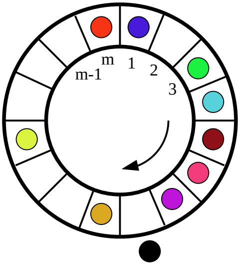

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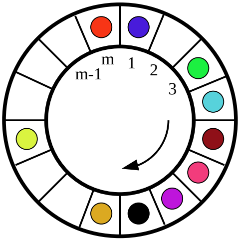

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Topic of active research in combinatorics

$\Rightarrow$ connections to many other objects:

- labelled trees, major functions, acyclic functions, Prüfer code, non-crossing partitions, hyperplane arrangements, priority queues, Tutte polynomial of graphs, linear probing hashing algorithm, invesions in trees

Generalizations:

- multiparking functions, G-parking functions, bucket parking functions

Authors working on parking functions, amongst others:

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A discrete parking problem: Enumeration results

Enumeration result for parking sequences:
Konheim and Weiss [1966]

$g(m, n)$: number of parking functions
for m parking lots and n cars

$$\boxed{g(m, n) = (m - n + 1)(m + 1)^{n-1}}$$

Questions for general parking sequences:
"Combinatorial question":

What is the number $g(m, n, k)$ of parking sequences $a_1, \ldots, a_n \in \{1, \ldots, m\}^n$ such that exactly k drivers are unsuccessful?

- Exact formulæ for $g(m, n, k)$?

A discrete parking problem: Enumeration results

Enumeration result for parking sequences:
Konheim and Weiss [1966]

$g(m,n)$: number of parking functions
for m parking lots and n cars

$$\boxed{g(m,n) = (m-n+1)(m+1)^{n-1}}$$

Questions for general parking sequences:
"Combinatorial question":

What is the number $g(m,n,k)$ of parking sequences $a_1,\dots,a_n \in \{1,\dots,m\}^n$ such that exactly k drivers are unsuccessful?

- Exact formulæ for $g(m,n,k)$?

A discrete parking problem: Enumeration results

"Probabilistic question":

> *What is the probability that for a randomly chosen parking sequence* $a_1, \ldots, a_n \in \{1, \ldots, m\}^n$ *exactly k drivers are unsuccessful ?*

r.v. $X_{m,n}$: counts number of unsuccessful cars for a randomly chosen parking sequence

- Probability distribution of $X_{m,n}$?
- Limiting distribution results (depending on growth of m, n) ?

A discrete parking problem: Enumeration results

Known results for $X_{m,n}$:

Gonnet and Munro [1984]:

- $X_{m,n}$ studied in analysis of algorithm "linear probing sort"

Exact and asymptotic results for expectation $\mathbb{E}(X_{m,n})$:

$$\mathbb{E}(X_{m,n}) = \frac{1}{2}\sum_{\ell=2}^{n} \frac{n^{\underline{\ell}}}{m^{\ell}}, \quad n \le m$$

$$\mathbb{E}(X_{m,m}) = \sqrt{\frac{\pi m}{8}} + \frac{2}{3} + \mathcal{O}\left(m^{-\frac{1}{2}}\right)$$

- Analysis uses "Poisson model"
- Transfer of results to "exact filling model"

A discrete parking problem: Enumeration results

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Results

Results: Exact enumeration formulæ

Exact enumeration results:

Cameron, Johannsen, Prellberg and Schweitzer [2007];
Panholzer [2007]

Number $g(m,n,k)$ of parking sequences for m parking lots and n drivers such that exactly k drivers are unsuccessful ($n \leq m+k$):

$$\boxed{\begin{aligned} g(m,n,k) = (m-n+k) \sum_{\ell=0}^{n-k} \binom{n}{\ell} (m-n+k+\ell)^{\ell-1} (n-k-\ell)^{n-\ell} \\ -(m-n+k+1) \sum_{\ell=0}^{n-k-1} \binom{n}{\ell} (m-n+k+1+\ell)^{\ell-1} (n-k-1-\ell)^{n-\ell} \end{aligned}}$$

Results: Exact enumeration formulaæ

Alternative expression: useful for k small

$$
\begin{aligned}
g(m,n,k) &= (m-n+k+1)(m+k+1)^{n-1} \\
&\quad - (m-n+k+1)\sum_{\ell=0}^{k-1}(-1)^{\ell}\binom{n}{\ell+1}(m+k-\ell)^{n-\ell-2}(k-\ell)^{\ell+1} \\
&\quad - (m-n+k)\sum_{\ell=0}^{k-1}(-1)^{\ell}\binom{n}{\ell}(m+k-\ell)^{n-\ell-1}(k-\ell)^{\ell}
\end{aligned}
$$

Results: Exact enumeration formulaæ

Alternative expression: useful for k small

$$g(m,n,k) = (m-n+k+1)(m+k+1)^{n-1} - (m-n+k+1)\sum_{\ell=0}^{k-1}(-1)^{\ell}\binom{n}{\ell+1}(m+k-\ell)^{n-\ell-2}(k-\ell)^{\ell+1} - (m-n+k)\sum_{\ell=0}^{k-1}(-1)^{\ell}\binom{n}{\ell}(m+k-\ell)^{n-\ell-1}(k-\ell)^{\ell}$$

Examples for small numbers k of unsuccessful cars:

$$g(m,n,0) = (m-n+1)(m+1)^{n-1}$$

$$g(m,n,1) = (m-n+2)(m+2)^{n-1} + (n^2-n-m^2-2m-1)(m+1)^{n-2}$$

$$g(m,n,2) = (m-n+3)(m+3)^{n-1} + (2n^2-mn-m^2-4n-4m-4)(m+2)^{n-2} + \frac{1}{2}n(-n^2-mn+2m^2+2n-5m+1)(m+1)^{n-3}$$

Results: Limiting distributions

Exact probability distribution of $X_{m,n}$:

$$\boxed{\mathbb{P}\{X_{m,n} = k\} = \frac{g(m,n,k)}{m^n}}$$

Limiting distribution results for $X_{m,n}$: Panholzer [2007]

Depending on growth of $m, n \Rightarrow$ nine different phases

m (parking lots) $\geq n$ (cars)

- $n \ll m$
- $n \sim \rho m, \quad 0 < \rho < 1$
- $\sqrt{m} \ll \Delta := m - n \ll m$
- $\Delta \sim \rho\sqrt{m}, \quad \rho > 0$
- $\Delta \ll \sqrt{m}$

m (parking lots) $< n$ (cars)

- $\Delta := n - m \ll \sqrt{n}$
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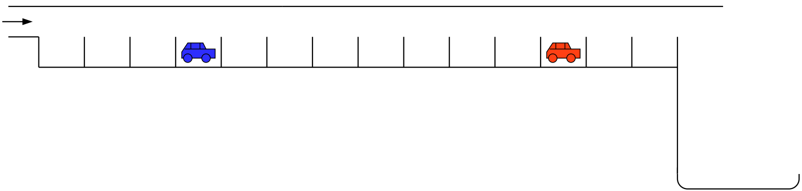
Results: Limiting distributions

Weak convergence of $X_{m,n}$ (m parking lots, n cars):

$n \ll m$: $\quad X_{m,n} \xrightarrow{(d)} X$

$$\boxed{\mathbb{P}\{X = 0\} = 1}$$

degenerate limit law



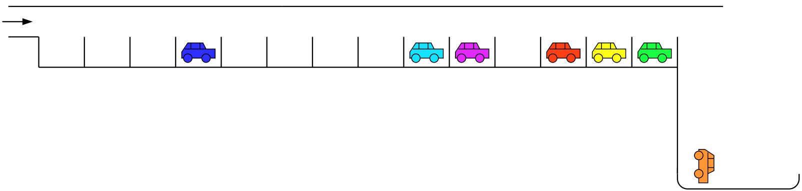
Results: Limiting distributions

Weak convergence of $X_{m,n}$ (m parking lots, n cars):

$n \sim \rho m, \;\; 0 < \rho < 1: \qquad X_{m,n} \xrightarrow{(d)} X_\rho$

$$\boxed{\mathbb{P}\{X_\rho \le k\} = (1-\rho)\sum_{\ell=0}^{k}(-1)^{k-\ell}\frac{(\ell+1)^{k-\ell}}{(k-\ell)!}\rho^{k-\ell}e^{(\ell+1)\rho}}$$

discrete limit law



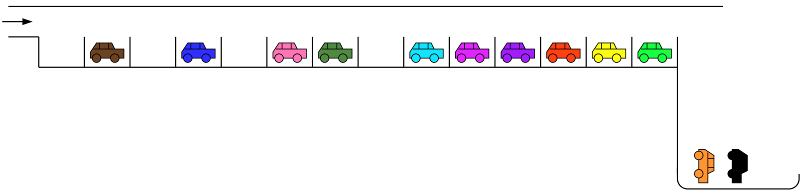
Results: Limiting distributions

Weak convergence of $X_{m,n}$ (m parking lots, n cars):

$\sqrt{m} \ll \Delta := m - n \ll m$: $\quad \frac{\Delta}{m} X_{m,n} \xrightarrow{(d)} X \overset{(d)}{=} \mathrm{EXP}(2)$

$$\boxed{\text{survival function: } \mathbb{P}\{X \geq x\} = e^{-2x}, \quad x \geq 0}$$

asymptotically exponential distributed



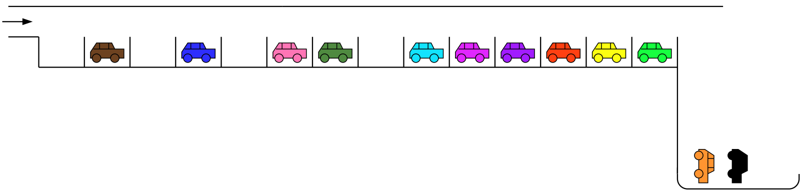
Results: Limiting distributions

Weak convergence of $X_{m,n}$ (m parking lots, n cars):

$\Delta := m - n \sim \rho\sqrt{m}, \;\; \rho > 0:$ $\quad \frac{1}{\sqrt{m}} X_{m,n} \xrightarrow{(d)} X_{rho} \stackrel{(d)}{=} \mathrm{LINEXP}(2, \rho)$

$$\boxed{\text{survival function: } \mathbb{P}\{X_\rho \geq x\} = e^{-2x(x+\rho)}, \quad x \geq 0}$$

asymptotically linear-exponential distributed



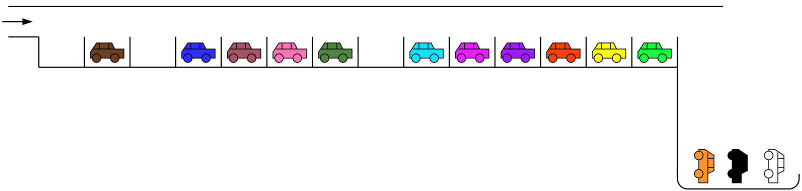
Results: Limiting distributions

Weak convergence of $X_{m,n}$ (m parking lots, n cars):

$0 \leq \Delta := m - n \ll \sqrt{m}$: $\quad \frac{1}{\sqrt{m}} X_{m,n} \xrightarrow{(d)} X \stackrel{(d)}{=} \text{RAYLEIGH}(2)$

$$\boxed{\text{survival function: } \mathbb{P}\{X \geq x\} = e^{-2x^2}, \quad x \geq 0}$$

asymptotically Rayleigh distributed



Results: Limiting distributions

Weak convergence of $X_{m,n}$ (m parking lots, n cars):

$0 \leq \Delta := n - m \ll \sqrt{n}$: $\quad \frac{X_{m,n+m-n}}{\sqrt{n}} \xrightarrow{(d)} X \stackrel{(d)}{=} \text{RAYLEIGH}(2)$

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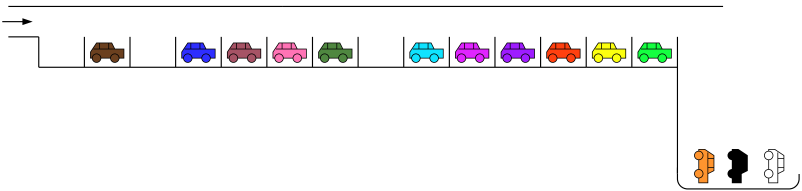

Results: Limiting distributions

Weak convergence of $X_{m,n}$ (m parking lots, n cars):

$\Delta := n - m \sim \rho\sqrt{n}, \;\; \rho > 0 :$ $\quad \frac{X_{m,n}+m-n}{\sqrt{n}} \xrightarrow{(d)} X_\rho \stackrel{(d)}{=} \mathrm{LINEXP}(2, \rho)$

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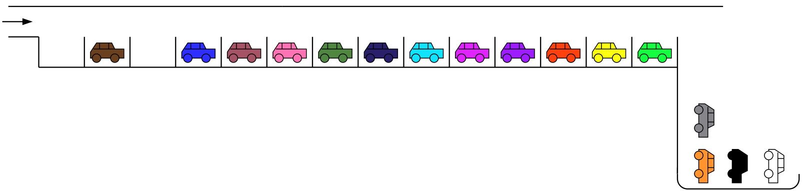

Results: Limiting distributions

Weak convergence of $X_{m,n}$ (m parking lots, n cars):

$$\sqrt{n} \ll \Delta := n - m \ll n : \qquad \frac{\Delta}{n}(X_{m,n} + m - n) \xrightarrow{(d)} X \stackrel{(d)}{=} \mathrm{EXP}(2)$$

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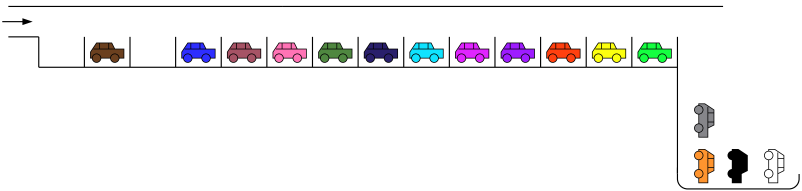

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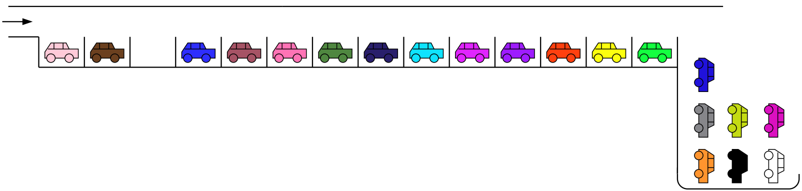
discrete limit law



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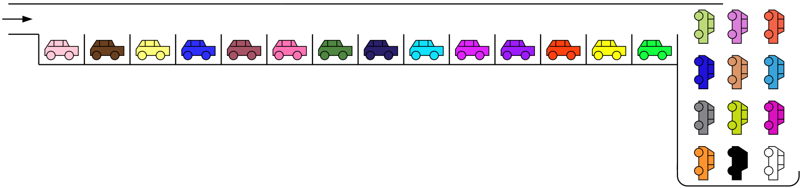

Analysis

Analysis: Outline

Outline of proof

Exact enumeration results:

- Recursive description of parameter
- Generating functions approach

Limiting distribution results:

- Asymptotic evaluation of distribution function
- Asymptotic evaluation of positive integer moments (Method of moments)

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- Asymptotic evaluation of positive integer moments (Method of moments)

Analysis: Exact enumeration results

Exact enumeration results

Quantity of interest:

- $g(m, n, k)$: number of sequences $\in \{1, \ldots, m\}^n$, such that exactly k cars are unsuccessful

Recursive description of $g(m, n, k)$:

Auxiliary quantities:

- $f(n) = (n+1)^{n-1}$: number of parking functions $\in \{1, \ldots, n\}^n$
- $s(m, k)$: number of sequences $\in \{1, \ldots, m\}^{m+k}$, such that all parking lots are occupied $\Leftrightarrow$ exactly k cars are unsuccessful

Analysis: Exact enumeration results

Exact enumeration results

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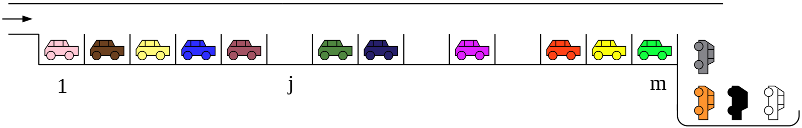
Analysis: Exact enumeration results

Case $n < m + k$: decomposition after first empty lot j:



$$\boxed{g(m,n,k) = \sum_{j=1}^{m} \binom{n}{j-1} f(j-1) g(m-j, n-j+1, k)}$$

Analysis: Exact enumeration results

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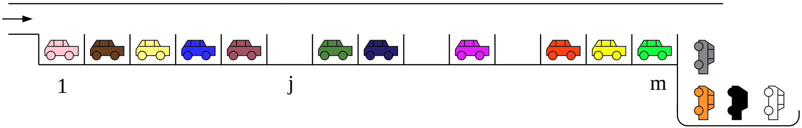
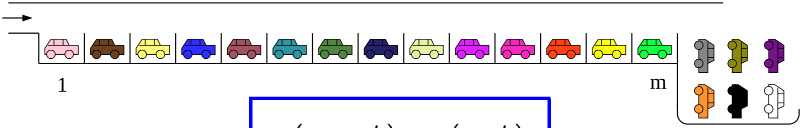


$$\boxed{g(m,n,k) = \sum_{j=1}^{m} \binom{n}{j-1} f(j-1)\, g(m-j, n-j+1, k)}$$

Case $n = m + k$: all parking lots are occupied:



$$\boxed{g(m,n,k) = s(m,k)}$$

Analysis: Exact enumeration results

Introducing suitable generating functions:

- $$G(z,u,v) := \sum_{m\ge 0}\sum_{n\ge 0}\sum_{k\ge 0} g(m,n,k)\frac{z^n}{n!}u^m v^k$$
- $$S(u,v) := \sum_{m\ge 0}\sum_{k\ge 0} s(m,k)\frac{u^m v^k}{(m+k)!}$$
- $$T(z) := \sum_{n\ge 1} n^{n-1}\frac{z^n}{n!} = \sum_{n\ge 1} f(n-1)\frac{z^n}{(n-1)!}$$

  $T(z)$: satisfies functional equation $T(z) = ze^{T(z)}$

Equation for generating functions:

$$\boxed{G(z,u,v) = \frac{S(zu,zv)}{1-\frac{T(zu)}{z}}}$$

Analysis: Exact enumeration results

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Analysis: Exact enumeration results

Evaluating at $v = 1$:

$$\frac{1}{1-ue^z} = \sum_{m\geq 0}\sum_{n\geq 0} m^n \frac{z^n}{n!} u^m = G(z,u,1) = \frac{S(zu,z)}{1-\frac{T(zu)}{z}}$$

$$\Rightarrow S(zu,z) = \frac{1-\frac{T(zu)}{z}}{1-ue^z}$$

Substituting $z \leftarrow zv$, $u \leftarrow \frac{u}{v}$:

$$S(zu,zv) = S(zv\cdot\frac{u}{v}, zv) = \frac{1-\frac{T(zu)}{zv}}{1-\frac{u}{v}e^{zv}}$$

Exact expression for generating function:

$$\boxed{G(z,u,v) = \frac{1-\frac{T(zu)}{zv}}{\left(1-\frac{T(zu)}{z}\right)\cdot\left(1-\frac{u}{v}e^{zv}\right)}}$$

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Analysis: Exact enumeration results

Extracting coefficients $\Rightarrow$ exact formula:

$$\boxed{\begin{aligned} g(m,n,k) &= (m-n+k)\sum_{\ell=0}^{n-k}\binom{n}{\ell}(m-n+k+\ell)^{\ell-1}(n-k-\ell)^{n-\ell} \\ &\quad -(m-n+k+1)\sum_{\ell=0}^{n-k-1}\binom{n}{\ell}(m-n+k+1+\ell)^{\ell-1}(n-k-1-\ell)^{n-\ell} \end{aligned}}$$

Exact distribution of $X_{m,n}$:

$$\mathbb{P}\{X_{m,n}=k\} = \frac{g(m,n,k)}{m^n}$$

Analysis: Exact enumeration results

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Analysis: Exact enumeration results

Abel's generalization of the binomial theorem:

$$(x+y)^n = \sum_{\ell=0}^{n} x(x-\ell z)^{\ell-1}(y+\ell z)^{n-\ell}$$

$\Rightarrow$ **alternative expression for $g(m,n,k)$:**

$$\boxed{\begin{aligned} g(m,n,k) &= (m-n+k+1)(m+k+1)^{n-1} \\ &\quad - (m-n+k+1)\sum_{\ell=0}^{k-1}(-1)^{\ell}\binom{n}{\ell+1}(m+k-\ell)^{n-\ell-2}(k-\ell)^{\ell+1} \\ &\quad - (m-n+k)\sum_{\ell=0}^{k-1}(-1)^{\ell}\binom{n}{\ell}(m+k-\ell)^{n-\ell-1}(k-\ell)^{\ell} \end{aligned}}$$

Analysis: Limiting distribution results (I)

Limiting distribution results for $X_{m,n}$ (I)

Special instance: m (parking lots) $= n$ (cars)

- complex-analytic techniques

Generating function of diagonal:

$$F(u,v) = \sum_{m\geq 0}\sum_{k\geq 0} m^m \mathbb{P}\{X_{m,m}=k\}\frac{u^m}{m!}v^k$$

Computed via contour integral:

$$F(u,v) = \frac{1}{2\pi i}\oint \frac{G(t,\frac{u}{t},v)}{t}dt = \frac{1}{2\pi i}\oint \frac{\left(t-\frac{T(u)}{v}\right)dt}{\left(t-T(u)\right)\cdot\left(t-\frac{u}{v}e^{tv}\right)}$$

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Analysis: Limiting distribution results (I)

Explicit formula:

- simple pole at $t = T(u)$
- computing residue

$$\boxed{F(u,v) = \frac{(v-1)T(u)}{vT(u) - ue^{T(u)v}}}$$

Method of moments:

$$\mathbb{E}(X_{m,m}^r) = \frac{m!}{m^m}[u^m]\left.\frac{\partial^r}{\partial v^r}F(u,v)\right|_{v=1}$$

Studying derivatives of $F(u,v)$ evaluated at $v=1$:

- local expansion around dominant singularity $u = \frac{1}{e}$
- Singularity analysis, Flajolet and Odlyzko [1990]

Analysis: Limiting distribution results (I)

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Studying derivatives of $F(u,v)$ evaluated at $v=1$:

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- Singularity analysis, Flajolet and Odlyzko [1990]

Analysis: Limiting distribution results (I)

Explicit formula:

- simple pole at $t = T(u)$
- computing residue

$$\boxed{F(u,v) = \frac{(v-1)T(u)}{vT(u) - ue^{T(u)v}}}$$

Method of moments:

$$\mathbb{E}(X_{m,m}^r) = \frac{m!}{m^m}[u^m]\,\frac{\partial^r}{\partial v^r}F(u,v)\Big|_{v=1}$$

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r-th moments converge to moments of Rayleigh r.v.:

$$\mathbb{E}\Big(\big(\frac{X_{m,m}}{\sqrt{m}}\big)^r\Big) \to 2^{-\frac{r}{2}}\,\Gamma\big(\frac{r}{2}+1\big)$$

Theorem of Fréchet and Shohat:

$$\boxed{\frac{X_{m,m}}{\sqrt{m}} \xrightarrow{(d)} \text{RAYLEIGH}(2)}$$

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Analysis: Limiting distribution results (II)

Limiting distribution results for $X_{m,n}$ (II)

Instance: m (parking lots) $>$ n (cars)

- Extension of previous approach

Generating function for $\Delta := m - n$:

$$F_\Delta(u,v) = \sum_{m\geq\Delta}\sum_{k\geq 0} m^{m-\Delta}\mathbb{P}\{X_{m,m-\Delta}=k\}\frac{u^m v^k}{(m-\Delta)!}$$

Computed via contour integral:

$$F_\Delta(u,v) = \frac{1}{2\pi i}\oint \frac{G(t,\frac{u}{t},v)t^\Delta}{t}dt = \frac{1}{2\pi i}\oint \frac{\left(t-\frac{T(u)}{v}\right)t^\Delta dt}{(t-T(u))\cdot\left(t-\frac{u}{v}e^{tv}\right)}$$

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Analysis: Limiting distribution results (II)

Explicit formula:

$$\boxed{F_{\Delta}(u,v) = \frac{(v-1)\big(T(u)\big)^{\Delta+1}}{vT(u) - ue^{T(u)v}}}$$

Exact formula for r-th factorial moments:

- Lagrange inversion formula

$$\boxed{\mathbb{E}\Big(\big(X_{m,m-\Delta}\big)^{\underline{r}}\Big) = \sum_{q=1}^{r} \gamma_{r,q} \sum_{\ell=r+q}^{m-\Delta} \binom{\ell-r-1}{q-1} \frac{(m-\Delta)^{\underline{\ell}}}{m^{\ell}}}$$

- $\gamma_{r,q}$: certain constants
- sums appearing related to Ramanujan's Q-function

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Analysis: Limiting distribution results (II)

Asymptotic evaluation:

- dissecting summation interval
- Stirling's formula
- tail exchange
- Euler's summation formula
  $\Rightarrow$ as. evaluation of sums via integrals

Method of moments:

suitably scaled r-th moments of $X_{m,m-\Delta}$ converge to moments of

- Rayleigh r.v.
- linear-exponential r.v.
- exponential r.v.

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Analysis: Limiting distribution results (III)

Limiting distribution results for $X_{m,n}$ (III)

Instance: m (parking lots) $< n$ (cars)

- consider $X_{m,n} + m - n$: number of empty parking lots
- Asymptotic evaluation of exact formula for survival function

Exact formula of survival function for $\Delta := n - m$:

$$\mathbb{P}\{X_{n-\Delta,n} - \Delta \geq k\} = \sum_{\ell=0}^{n-\Delta-k} \frac{k}{\ell+k} \binom{n}{\ell} \frac{(\ell+k)^{\ell}(n-\Delta-k-\ell)^{n-\ell}}{(n-\Delta)^n}$$

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Analysis: Limiting distribution results (III)

Asymptotic evaluation:

- dissecting summation interval
- Stirling's formula
- tail exchange
- inequalities, uniform estimates
- Euler's summation formula
  $\Rightarrow$ as. evaluation of sums via integrals

Analysis: Limiting distribution results (III)

Example: $\Delta \sim \rho\sqrt{n}, \ 0 < \rho < \infty, \quad k \sim x\sqrt{n}, \ 0 < x < \infty$

Pointwise convergence for all $0 < x < \infty$:

$$\mathbb{P}\{X_{n-\Delta,n} - \Delta \geq k\} \to \int_0^1 \frac{x e^{\frac{\rho^2}{2}}}{\sqrt{2\pi} t^{\frac{3}{2}} \sqrt{1-t}} e^{-\frac{x^2}{2t} - \frac{(x+\rho)^2}{2(1-t)}} dt$$

Evaluation of the integral:

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Characterization of the limiting distribution:

$$\boxed{\mathbb{P}\left\{\frac{X_{n-\Delta,n} - \Delta}{\sqrt{n}} \geq x\right\} \to e^{-2x(x+\rho)}}$$

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Outlook

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Possible further research directions

Refined analysis:

- Local limit laws
- case $n > m$: convergence of moments

Extensions to related problems:

- Analysis of "number of insertion steps"
- Bucket parking functions
- Multiparking functions

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Bucket parking scheme

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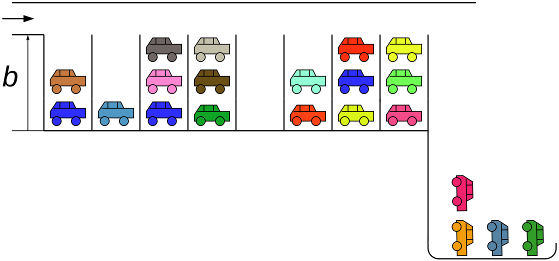
Blake and Konheim [1976]:

- Each parking lots can hold up to b cars
- Related to analysis of bucket hashing algorithms



Generating functions approach works:

$$G(z,u,v) = \frac{1}{1-\frac{u}{v^b}e^{zv}} \frac{(1-\frac{b}{zv}T(zu^{1/b}))\cdot(1-\frac{b}{zv}T(\omega zu^{1/b}))\cdots(1-\frac{b}{zv}T(\omega^{b-1}zu^{1/b}))}{(1-\frac{b}{z}T(zu^{1/b}))\cdot(1-\frac{b}{z}T(\omega zu^{1/b}))\cdots(1-\frac{b}{z}T(\omega^{b-1}zu^{1/b}))}$$