Area under lattice paths associated with certain urn models.

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Outline of the talk

1. **Pólya-Eggenberger urn models**
   - Diminishing urn models
2. **Diminishing urn models: Area**
   - Diminishing urn models: Results
3. **Analysis**
4. **Further discussion**

Pólya-Eggenberger urn models

Pólya-Eggenberger urn models: Definition

Two types of balls: Urn contains n white balls and m black balls.
The evolution of the urn occurs in discrete time steps.

- At every step a ball is drawn at random from the urn.
- The color of the ball is inspected and then the ball is reinserted into the urn. According to the observed color of the ball, balls are added/removed due to the following rules:
- If a white ball has been drawn, a white balls and b black balls are put into the urn, and if a black ball has been drawn, c white balls and d black balls are put into the urn. The values $a, b, c, d \in \mathbb{Z}$ are fixed integers and the urn model is specified by the 2×2 ball replacement matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

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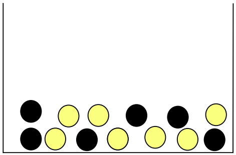
Pólya-Eggenberger urn models: Definition

Example

Ball replacement matrix $M = \left(\begin{smallmatrix} 2 & 1 \\ 1 & -1 \end{smallmatrix}\right)$
Intial configuration: $n = 7$ yellow (white) balls and $m = 6$ black ball



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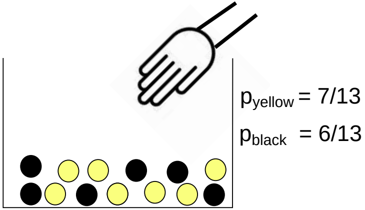

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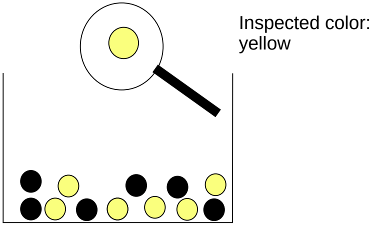

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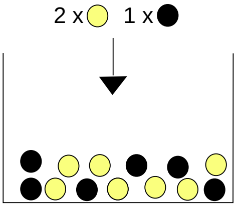

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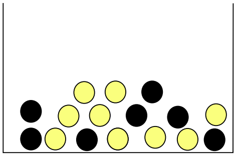

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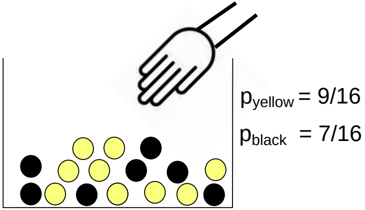

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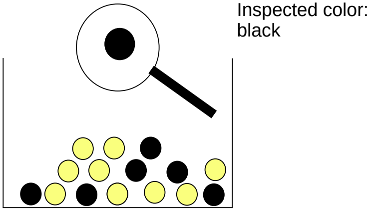

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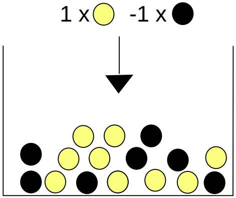

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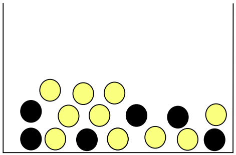

Pólya-Eggenberger urn models: Definition

An often posed question in this context is the **composition** of the urn after t draws:

"Starting with x_0 white and y_0 black balls, what is the distribution of (X_t, Y_t), where X_t, Y_t count the number of white, black balls after t draws?"

Huge literature on 2×2 concerning this question: Mahmoud 1998, 03; Flajolet, Gabarró, Pekari 05; Flajolet, Dumas, Puyhaubert 06; Pouyanne 05, 06; Janson 04, 06; and many others.

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Pólya-Eggenberger urn models: Weighted lattice path

We can associate to a given urn model certain weighted lattice paths.
Assume the urn contains n white and m black balls:

- we draw a white ball with probability $n/(n+m)$: Step $(m,n) \to (m+b, n+a)$ with weight $n/(n+m)$
- we draw a black ball with probability $m/(n+m)$: Step $(m,n) \to (m+d, n+c)$ with weight $m/(n+m)$

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$\Rightarrow$ The weight of a path after t successive draws consists of the product of the weight of every step.

Pólya-Eggenberger urn models: Weighted lattice path

Example

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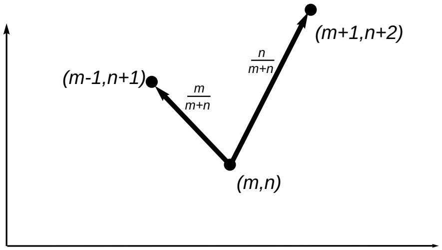
The set of steps of the urn can be visualized as follows



Pólya-Eggenberger urn models

Diminishing urn models

Diminishing urn models: Definition

We consider Pólya-Eggenberger urn models specified by a ball replacement matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where in addition there is a set of **absorbing states** $\mathcal{A} \subset \mathbb{N}_0 \times \mathbb{N}_0$. The urn evolves according to the matrix M in a state space $\mathcal{S}$, until an absorbing state $(i, j) \in \mathcal{A}$ is reached.

We always assume that M, $\mathcal{S}$ and $\mathcal{A}$ are chosen in a way that both the numbers of white and black balls are non-negative during the evolution of the urn, i.e. that the diminishing urn model well defined.

Diminishing urn models: Examples

Sampling without replacement

This simply urn model corresponds to sampling without replacement; we have ball replacement matrix $M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, with absorbing states $\mathcal{A} = \{(0, n) | n \in \mathbb{N}_0\}$.

What is the number of white balls when all black balls have been drawn?

(What is the composition of the urn after t draws?)

Diminishing urn models: Examples

The Pill's Problem, proposed by Knuth and McCarthy;

Hesterberg; Brennan and Prodinger, Panholzer and Kuba 07+, Hwang, Panholzer and Kuba 08+

In a bottle there are n small pills and m large pills. The large pill is equivalent to two small pills. Every day a person chooses a pill at random. If a small pill is chosen, it is eaten up, if a large pill is chosen it is broken into two halves, one half is eaten and the other half which is now considered to be a small pill is returned to the bottle. What is the number of small pills when all large pills have been consumed?

$\Rightarrow$ Ball replacement matrix $M = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$, with absorbing states $\mathcal{A} = \{(0, n) | n \in \mathbb{N}_0\}$.

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Diminishing urn models: Examples

OK-Corral urn

Williams, McIlroy; Flajolet, Huillet, Puyhaubert+; Kingman; Kingman, Volkov; Hwang, Panholzer, Kuba+

Two groups A and B of gunmen are fighting. Each group is selected uniformly at randon and kills then a member of the opposing group. How many survivors (say of group A) are there when the fight is over?

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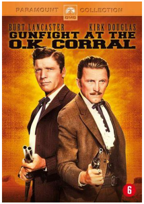

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Diminishing urn models: Examples

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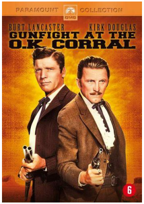

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Diminishing urn models: A class of urns

We will focus on the class of diminishing urns with ball replacement matrix given by

$$M = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix}, \quad a, d \in \mathbb{N}, c \in \mathbb{N}_0,$$

assuming that $c = p \cdot a$ with $p \in \mathbb{N}_0$, where $\mathbb{N}_0 := \{0, 1, 2, \dots\}$. The state space $\mathcal{S}$ is defined as $\mathcal{S} = \{d \cdot m, a \cdot n | n, m \in \mathbb{N}_0\}$ and the set of absorbing states $\mathcal{A} = \{0, a \cdot n | n, m \in \mathbb{N}_0\}$.

Diminishing urn models

- Distribution of the area

Diminishing urn models-Area: Problem statement

It is well known that the urn histories can be interpreted as weighted lattice path in $\mathbb{Z} \times \mathbb{Z}$. This naturally leads to the following question: For a given diminishing urn model with replacement matrix M, state space S and absorbing states $\mathcal{A}$, **what is the area of below the sample paths associated with the diminishing urn?**

The (discrete) area is measured as the number of points of $\mathcal{S} \subseteq \mathbb{N} \times \mathbb{N}$, which are below a certain sample path. Such questions relate the both widely studied topics of lattice path enumeration and Pólya-Eggenberger urn models.

Diminishing urn models-Area: Problem statement

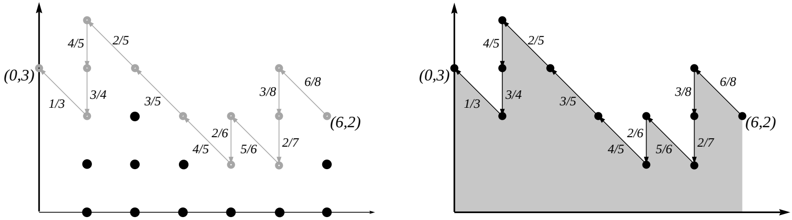

Figure: An example of a weighted path from $(6,2)$ to the absorbing state $(0,3)$ for the pills problem $M = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$ and the vertical absorbing axis $\mathcal{A} = \{(0,n) : n \geq 0\}$. The illustrated path has weight $\frac{6}{8}\frac{3}{8}\frac{2}{7}\frac{5}{6}\frac{2}{6}\frac{4}{5}\frac{3}{5}\frac{2}{5}\frac{4}{5}\frac{3}{4}\frac{1}{3} = \frac{3}{3500}$, discrete area 11 and continuous area 14.

Diminishing urn models-Area: Problem statement

The weighted lattice paths are generated according to ball replacement matrix $M = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix}$, with $a, d \in \mathbb{N}$ and $c = p \cdot a$, $p \in \mathbb{N}_0$.

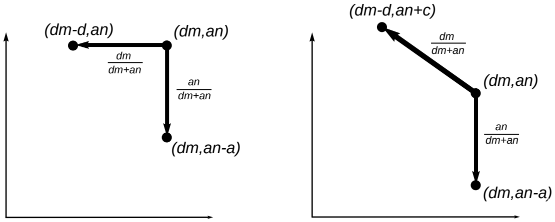

Figure: The steps associated with $M = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix}$ for $c = 0$ and $c > 0$.

Interlude: Lattice paths

A lattice path is the drawing in $\mathbb{Z} \times \mathbb{Z}$ of a sum of vectors from $\mathbb{Z} \times \mathbb{Z}$, where the vectors belong to a finite fixed set V, and where the origin of the path is usually taken as being the point $(0,0) \in \mathbb{Z} \times \mathbb{Z}$. If all vectors are in $\mathbb{N} \times \mathbb{Z}$, the path is called **directed** (the path is going "to the right").

The study of the area under lattice paths, measured either continuous, or discrete as the number of lattice points below the sample path, has a long history; We want to point out the connection between area under lattice paths and the area under a Browian excursion, see e.g. the works of Louchard.

Note that the standard probability model is very different. Any path, say of length n, is choosen with equal probability.

Diminishing urn models-Area: Problem statement

$A_{an,dm}$ satisfies the distributional equation

$$A_{an,dm} \stackrel{(d)}{=} \mathbb{I}_{n,m} A_{a(n-1),m} + (1 - \mathbb{I}_{n,m})(\tilde{A}_{a(n+p),d(m-1)} + n), \quad A_{an,0} = 0,$$

where $\mathbb{I}_{n,m}$ denotes the indicator variable of choosing a white ball,

$$\mathbb{P}\{\mathbb{I}_{n,m} = 1\} = \frac{an}{an + dm}, \quad \mathbb{P}\{\mathbb{I}_{n,m} = 0\} = \frac{dm}{an + dm},$$

with $\mathbb{I}_{n,m}$ being independent of the $A's$, and $\tilde{A}$ denotes a random variable with the same distribution as A.

Diminishing urn models-Area: Continuous area

It is sufficient to study the discrete area, since the continuous area $C_{an,dm}$ and the discrete area $A_{an,dm}$ are related as follows.

$$A_{an,dm} \stackrel{(d)}{=} \frac{C_{an,dm} - \frac{mcd}{2}}{ad}.$$

Results

Diminishing urn models-Area: Results

Theorem

The limiting distributions of $A_{an,dm}$ can be classified according to the growth of m and n.

- *For arbitrary but fixed $m \in \mathbb{N}$ and $n \to \infty$:*

$$\frac{A_{an,dm}}{n} \to X_m, \quad X_m \stackrel{(d)}{=} B(\frac{dm}{a}, 1) \cdot (1 + \tilde{X}_{m-1}), \quad \text{for } m \geq 1,$$

  with $X_0 = 0$, where $B(\alpha, \beta)$ denotes a Beta-distributed random variable with parameters α and β, being independent of the X, and $\tilde{X}_{m-1}$ having the same recursive description as X_m.

Diminishing urn models-Area: Results

Theorem

- *For both n and m tending to infinity the centered and normalized random variable $A^*_{an,dm}$ is asymptotically gaussian distributed,*

$$A^*_{an,dm} := \frac{A_{an,dm} - \mathbb{E}(A_{an,dm})}{\sqrt{\mathbb{V}(A_{an,dm})}} \xrightarrow{(d)} \mathcal{N}(0,1),$$

where $\mathcal{N}(0,1)$ denotes the standard normal distribution. In the case of $c \neq 0$ this also holds for fixed n, and m tending to infinity.

Diminishing urn models-Area: Results

Theorem

- *In the case of $c = 0$, with n fixed, and m tending to infinity, the normalized random variable $A_{an,dm}/m$ converges to a random variable $W_n = W_n(a, d)$, which can be described by the distributional equation*

$$W_n \stackrel{(d)}{=} \tilde{W}_{n-1} B(\frac{an}{d}, 1) + n(1 - B(\frac{an}{d}, 1)), \quad \text{for}\, n \geq 1, \quad W_0 = 0,$$

where $B(\alpha, \beta)$ denotes a Beta-distributed random variable with parameters α and β, being independent of the W, and $\tilde{W}_{n-1}$ having the same recursive description as W_n.

Diminishing urn models-Area: Results

Remark

The distributional equations

$$X_m \stackrel{(d)}{=} B(\frac{dm}{a}, 1) \cdot (1 + \tilde{X}_{m-1}), \quad \text{for } m \geq 1, \quad X_0 = 0,$$

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can be iterated, leading to equivalent characterizations

$$X_m \stackrel{(d)}{=} \sum_{k=1}^{m} \prod_{l=0}^{k-1} B(\frac{d(m-l)}{a}, 1),$$

$$W_n \stackrel{(d)}{=} n - \sum_{k=1}^{n} \prod_{l=0}^{k-1} B(\frac{a(n-l)}{d}, 1).$$

Analysis

Analysis: A recurrence for the moments

Our analyis is based on a precise study of the moments of $A_{an,dm}$.

The s-th moment of $A_{an,dm}$, denoted by $e_{n,m}^{[s]} = \mathbb{E}(A_{an,dm}^s)$, satisfies

$$e_{n,m}^{[s]} = \frac{an}{an+dm} e_{n-1,m}^{[s]} + \frac{dm}{an+dm} \sum_{l=0}^{s} \binom{s}{l} n^l e_{n+p,m-1}^{[s-l]}, \qquad (1)$$

for $n \geq 0$ and $m \geq 1$.

Analysis: A recurrence for the moments

Proposition

The moments $e_{n,m}^{[s]} = \mathbb{E}(A_{an,dm}^s)$ *of the random variable* $A_{an,dm}$ *satisfy the expansion* $e_{n,m}^{[s]} = \sum_{l=0}^{s} \varphi_{s,l,m} n^l$. *For* $l = s$ *we have*

$$\varphi_{s,s,m} = \sum_{k=1}^{m} \frac{\binom{m}{k}}{\binom{m+\frac{as}{d}}{k}} \sum_{l=0}^{s-1} \binom{s}{l} \varphi_{l,l,m-k}.$$

Furthermore, for $1 \le l \le s-1$, *the values* $\varphi_{s,l,m}$ *are determined recursively as follows:*

$$\varphi_{s,l,m} = \frac{1}{al\binom{m+\frac{al}{d}}{m}} \sum_{k=1}^{m} \binom{k+\frac{al}{d}-1}{k} \psi_{s,l,k}, \tag{2}$$

Analysis: A recurrence for the moments

Proposition

Here, we have

$$\psi_{s,l,m} := a \sum_{k=l+1}^{s} \binom{k}{l-1} (-1)^{k-l-1} \varphi_{s,k,m} + dm \sum_{k=l+1}^{s} \binom{k}{l} p^{k-l} \varphi_{s,k,m-1}$$
$$+ dm \sum_{i=1}^{l} \binom{s}{i} \sum_{k=l-i}^{s-i} \binom{k}{l-i} \varphi_{s-i,k,m-1} p^{k-l+i}.$$

For $l = 0$ we have

$$\varphi_{s,0,m} = \sum_{k=1}^{m-1} \psi_{s,0,k}, \quad \textit{with} \quad \psi_{s,0,m} := \sum_{i=1}^{s} \varphi_{s,i,m} p^i.$$

The initial values are given by $\varphi_{s,l,0} = 0$, for $0 \le l \le s$, $s \ge 1$, and $\varphi_{0,0,m} = 1$, for $m \ge 0$.

Analysis: Expectation and Variance

Theorem

The expectation and the variance of the random variable $A_{an,dm}$ are given by the following formulæ.

$$\mathbb{E}(A_{an,dm}) = \frac{nm}{1+\frac{a}{d}} + \frac{cm(m-1)}{2a(1+\frac{a}{d})},$$

$$\mathbb{V}(A_{an,dm}) = \begin{cases} \frac{n^2m(\frac{a}{d})^2}{(1+\frac{a}{d})^2(1+\frac{2a}{d})} + \frac{nm^2\frac{a}{d}}{(1+\frac{a}{d})^2(2+\frac{a}{d})} + \frac{nm(\frac{a^2}{d^2}+\frac{a}{d}+1)\frac{a}{d}}{(1+\frac{a}{d})^2(1+\frac{2a}{d})(2+\frac{a}{d})}, \\ \frac{n^2m(\frac{a}{d})^2}{(1+\frac{a}{d})^2(1+\frac{2a}{d})} + n\varphi_{2,1,0} + \varphi_{2,0,m}, \end{cases}$$

for $c = 0$ and $c \neq 0$, respectively, where for $c \neq 0$ the quantities $\varphi_{2,1,m}$ and $\varphi_{2,0,m}$ are polynomials in m of degrees 3 and 4, respectively. The leading term with respect to m in $\varphi_{2,0,m}$ is given by $\frac{c^2m^4}{4a^2(1+\frac{a}{d})}$.

Analysis: The structure of the moments

A direct consequence of the recursive relation for the moments is the following.

Proposition

The values $\varphi_{s,l,m}$ are polynomials in m.
Case $c = 0$: for $s \geq 1$ and $1 \leq l \leq s$ the quantity $\varphi_{s,l,m}$ is a polynomial in m of degree s, $\varphi_{s,l,m} = \sum_{k=1}^{s} \vartheta_{s,l,k} m^k$; consequently

$$e_{n,m}^{[s]} = \sum_{l=1}^{s} \sum_{k=1}^{s} \vartheta_{s,l,k} n^l m^k.$$

Case $c \neq 0$: the quantity $\varphi_{s,l,m}$ is a polynomial in m of degree $2s - l$, $\varphi_{s,l,m} = \sum_{j=1}^{2s-l} \vartheta_{s,l,j} m^j$, for $s \geq 1$ and $0 \leq l \leq s$; consequently,

$$e_{n,m}^{[s]} = \sum_{l=0}^{s} \sum_{k=1}^{2s-l} \vartheta_{s,l,k} n^l m^k.$$

Analysis: Case m fixed

We obtain after normalization the expansion

$$\mathbb{E}\Big(\frac{A_{an,dm}^s}{n^s}\Big) = \frac{e_{n,m}^{[s]}}{n^s} = \varphi_{s,s,m}\Big(1 + \mathcal{O}(\frac{1}{n})\Big).$$

Hence the s-th moment of the scaled random variable $A_{an,dm}/n$ tends to $\varphi_{s,s,m}$, and by Carlemans criterion it follows that the moment sequence $(\varphi_{s,s,m})_{s\geq 1}$ describes a unique random variable X_m,

$$\frac{A_{an,dm}}{n} \xrightarrow{(d)} X_m, \quad \mathbb{E}(X_m^s) = \varphi_{s,s,m},$$

where

$$\varphi_{s,s,m} = \sum_{k=1}^{m} \frac{\binom{m}{k}}{\binom{m+\frac{as}{d}}{k}} \sum_{l=0}^{s-1} \binom{s}{l} \varphi_{l,l,m-k}.$$

Analysis: Case m fixed

In order to *identify* the limiting distribution we can either guess, or proceed differently. We rewrite the distributional equation for $A_{an,dm}$ as follows.

$$A_{an,dm} \stackrel{(d)}{=} \tilde{A}_{a(Y_{n,m}+p),d(m-1)} + Y_{n,m}, \quad \text{with} A_{an,0} = 0,$$

where

$$\mathbb{P}\{Y_{n,m} = k\} = \frac{\binom{k-1+\frac{dm}{a}}{k}}{\binom{n+\frac{dm}{a}}{n}}, \quad \text{for } 0 \le k \le n.$$

Note that the random variable $Y_{n,m}$ counts the contribution of the m-th column to $A_{an,dm}$. It can easily be checked that for fixed m, and n tending to infinity, the normalized random variable $Y_{n,m}/n$ tends to a Beta distributed random variable with parameters dm/a and 1.

Analysis: Case $c = 0$, and n fixed

We may proceed analogous and obtain

$$A_{an,dm} \stackrel{(d)}{=} \tilde{A}_{a(n-1),d(m-Z_{n,m})} + nZ_{n,m}, \quad \text{with} A_{an,0} = 0,$$

where

$$\mathbb{P}\{Z_{n,m} = m - k\} = \frac{\binom{k-1+\frac{an}{d}}{k}}{\binom{m+\frac{an}{d}}{m}}, \quad \text{for } 0 \le k \le m.$$

Note that the random variable $Z_{n,m}$ counts the contribution of the m-th row to $A_{an,dm}$.

Analysis: Case n, m tending to infinity

In the case of m, n tending to infinity, we have to center,
$\hat{A}_{an,dm} := A_{an,dm} - \mathbb{E}(A_{an,dm})$
We obtain the distributional equation

$$\hat{A}_{an,dm} \stackrel{(d)}{=} \mathbb{I}_{n,m}\Big(\hat{A}'_{a(n-1),m} - \frac{md}{a+d}\Big) + (1-\mathbb{I}_{n,m})\Big(\hat{A}'_{an,d(m-1)} + \frac{na}{a+d}\Big).$$

Analysis: Case n, m tending to infinity

The centered moments $\hat{e}^{[s]}_{n,m} := \mathbb{E}(\hat{A}^s_{an,dm})$ obey a recursive description, similar to the ordinary moments.

We obtain the expansion $\hat{e}^{[s]}_{n,m} = \sum_{l=0}^{s} \hat{\varphi}_{s,l,m} n^l$, where $\varphi_{s,l,m}$ satisfy certain recurrence relations.

Analysis: Case n, m tending to infinity

Lemma

The values $\hat{\varphi}_{s,l,m}$ are polynomials in m, with the degree bounded by

$$\deg \hat{\varphi}_{s,l,m} \leq \lfloor \frac{3s}{2} \rfloor - l.$$

For s even let $\gamma_{s,k} := \operatorname{lc} \hat{\varphi}_{s,k,m} = [m^{\frac{3s}{2}-k}]\hat{\varphi}_{s,k,m}$ denote the leading coefficient of $\hat{\varphi}_{s,k,m}$. Then $\gamma_{s,k}$ satisfies the recurrence relation

$$\gamma_{s,k} = \frac{1}{\frac{3ds}{2} + (a-d)k} \Biggl(\frac{cd(k+1)}{a} \gamma_{s,k+1} + a \binom{s}{2} \bigl(\frac{a}{a+d}\bigr)^2 \gamma_{s-2,k-1} + d \binom{s}{2} \bigl(\frac{a}{a+d}\bigr)^2 \gamma_{s-2,k-2} \Biggr),$$

for $0 \leq k \leq s$, with $\gamma_{s,k} = 0$ for $k < 0$ or $k > s$, and initial value $\gamma_{0,0} = 1$.

Analysis: Case n, m tending to infinity

Let $\tilde{\gamma}_{s,k} = \gamma_{s,s-k}$. The bivariate generating function $C(z,w) = \sum_{s\geq 0}\sum_{k\geq 0}\tilde{\gamma}_{s,k}\frac{z^s}{s!}w^k$ of the sequence $\tilde{\gamma}_{s,k}$ satisfies the first order partial differential equation

$$z\Big(a+\frac{d}{2}-\frac{cd}{a}w\Big)C_z(z,w)+w\Big(\frac{cd}{a}w-a+d\Big)C_w(z,w)$$
$$-\frac{adz^2}{2(a+d)^2}(a+dw)C(z,w)=0, \quad C(0,w)=1.$$

Analysis: Case n, m tending to infinity

The solution of the partial differential equation is given by

$$C(z,w) = \exp\left(\frac{z^2}{2}(\gamma_{2,2} + w\gamma_{2,1} + w^2\gamma_{2,0})\right),$$

where the values $\gamma_{2,2}$, $\gamma_{2,1}$ and $\gamma_{2,0}$ are given as follows.

$$\gamma_{2,2} = \frac{a^2 d}{(a+d)^2(2a+d)}, \quad \gamma_{2,1} = \frac{ad^2(2a+2c+d)}{(a+d)^2(2a+d)(a+2d)}$$
$$\gamma_{2,0} = \frac{cd^2(2a+2c+d)}{3(a+d)^2(2a+d)(a+2d)}.$$

Extracting coefficients leads to the required asymptotic expansion of the centered moments, which proves the normal limit law.

Further discussion

Further discussion:

Some remarks on

- Ordinary sampling without replacement: Case $c = 0$, $a = d = 1$.
- Biased model
- Other urn models

Further discussion: Integer partitions

The case $c = 0$, $a = d = 1$ corresponds to **ordinary sampling without replacement**. We obtain via generating functions approach the following result.

Theorem

The distribution of the random variable $A_{n,m}$ is given by

$$\mathbb{P}\{A_{n,m} = k\} = \frac{\lambda_{k,n,m}}{\binom{n+m}{n}}, \quad \text{with } 0 \le k \le nm.$$

Here $\lambda_{k,n,m}$ denote the number of integer partitions of k into n non-negative integers, all less or equal m.

$$\lambda_{k,n,m} = [z^n v^k] \frac{1}{\prod_{l=0}^{m}(1 - zv^l)}$$

Further discussion: Integer partitions

Obvious questions:

- Local Limit theorems
- Speed of convergence

Further discussion: Integer partitions

Note that our results imply for $\mathbb{P}\{A_{n,m} = k\} = \frac{\lambda_{k,n,m}}{\binom{n+m}{n}}$:

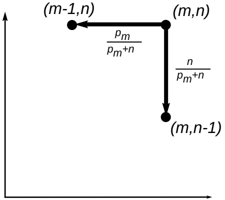
- Case fixed $m \in \mathbb{N}$ and $n \to \infty$: $\frac{A_{n,m}}{n} \to X_m$,
- Case $m, n \to \infty$: $\frac{A_{n,m} - \mathbb{E}(A_{n,m})}{\sqrt{\mathbb{V}(A_{n,m})}} \to \mathcal{N}(0,1)$,
- Case fixed $n \in \mathbb{N}$ and $m \to \infty$: $\frac{A_{n,m}}{n} \to W_n$.

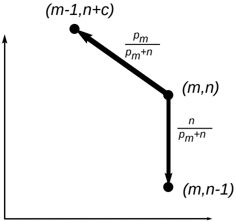
We expect that one should be able to obtain local limit theorems (much more difficult?) (Hwang, Hwang & Yeh 1997)

Further discussion: A biased urn model

We associate to the states of the urn a sequence P of postive real numbers $P := (p_m)_{m \in \mathbb{N}_0}$, with $p_0 = 0$ and $p_m \in \mathbb{R}^+$, where P is independent of n.

The probability of choosing a white ball is, for this class of biased diminishing urns, given by $n/(n + p_m)$, and the opposite probability of choosing a black ball by $p_m/(n + p_m)$. We can obtain again a recursive description of the moments structure of $A_{n,m} = A_{n,m}(P)$.



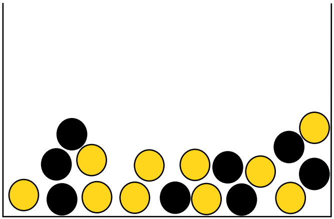


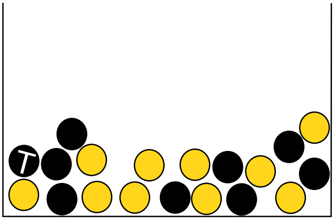
Further discussion: Different urn models

It would be interesting to discuss the distribution of the area associated with different classes of urn models.

For example: the O.K.Corral urn $M = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

in contrast to sampling without replacement urn $M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

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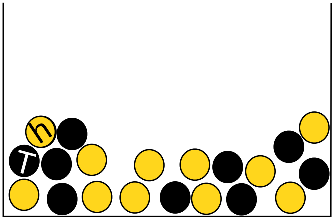
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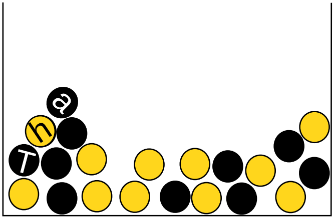
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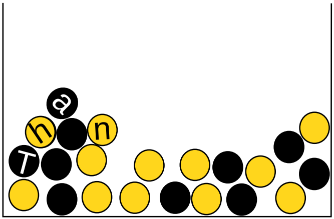
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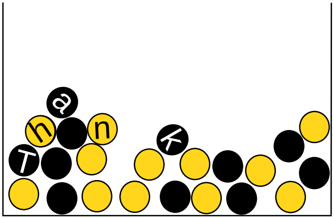
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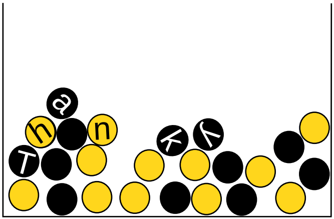

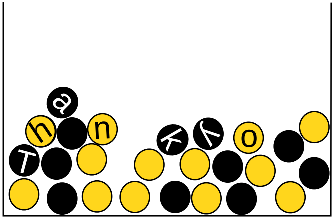
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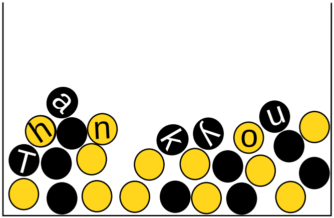
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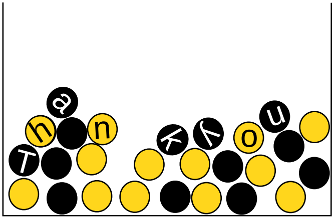

THANK YOU!